

Periodic L_p estimates by $\mathcal{R}$ -boundedness and applications to the Navier-Stokes equations

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1 Introduction

This is a joint work with Thomas Eiter (Weierstrass inst. Applied Analysis and Stochastics, Berlin) and Mads Kyed (Hochschule Flensburg) and this manuscript was written based on Eiter, Kyed and Shibata [7].

Let Ω be a C^2 exterior domain in $\mathbb{R}^3$ and Γ the boundary of Ω . At beginning, I consider the Navier Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \mathbf{p} = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}|_\Gamma = 0. \quad (1)$$

Here, $\mathbf{F}$ is a 2π periodic external force, that is $\mathbf{F}(\cdot, t + 2\pi) = \mathbf{F}(\cdot, t)$ for any $t \in \mathbb{R}$ and $\mathbb{T} = \mathbb{R} \setminus 2\pi\mathbb{Z}$. There are a lot of studies of periodic solutions to the Navier-Stokes equations (1), cf. [9, 14, 21, 22]. See [7] for more reference. By far the most popular method that emerged is based on a representation formula that arises from the principle that a solution to the initial value problem tends to a periodic orbit as $t \rightarrow \infty$ regardless of the initial value. Equivalently formulated, a solution to the initial-value problem with time-periodic right-hand side tends to a periodic orbit as $t \rightarrow \infty$. Since the Stokes operator generates a C^0 analytic semigroup, which is denoted by $\{T(t)\}_{t \geq 0}$ here, the solution is represented as

$$u(t) = \int_{-\infty}^t T(t-s)F(s) ds - \int_{-\infty}^t T(t-s)P(\mathbf{u} \cdot \nabla \mathbf{u})(\tau) ds, \quad (2)$$

where P denotes the solenoidal projection defined by (11) of Sect 2, below. Then, it is easy to verify that this integral expression indeed leads to a periodic solution of the same period as F . The challenge with the method based on (2) is to construct a framework of Banach spaces such that the integral expression is well defined. Since F is time-periodic and therefore non-decaying, this clearly requires suitable decay properties of the semigroup $\{T(t)\}_{t \geq 0}$, which is basically guaranteed by L_p - L_q decay estimates due to Iwashita [13].

In this note, I would like to propose a completely different method based on the $\mathcal{R}$ -solver and operator valued de Leewe theorem. In this section, I would like to explain our method created in [6, 7] in the abstract framework. Let X, Y, Z be three Banach spaces such that $X \subset Z \subset Y$ and let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(X, Z) \cap \mathcal{L}(Z, Y)$. Here, $\mathcal{L}(E, F)$ denotes the set of all bounded linear operators from E into F . Let $\mathbb{T} = \mathbb{R} \setminus 2\pi\mathbb{Z}$ be the 2π torus and we consider time periodic problem

$$\partial_t u - Au = F, \quad Bu = G \quad \text{for } t \in \mathbb{T}, \quad (3)$$

where $F(t+2\pi) = F(t)$ and $G(t+2\pi) = G(t)$ for $t \in \mathbb{R}$. Our approach is to use $\mathcal{R}$ -solvers associated with the corresponding resolvent problem:

$$\lambda v - Av = f, \quad Bv = g. \quad (4)$$

Here, λ is a complex number ranging on

$$\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \pi - \epsilon, \quad |\lambda| \geq \lambda_0\}$$

for some $\lambda_0 > 0$ and $\epsilon \in (0, \pi/2)$. The situation here is the case where the following generalized resolvent estimate holds:

$$|\lambda| \|v\|_Y + \|v\|_X \leq C(\|f\|_Y + \|g\|_Z + \|\lambda^{1/2} g\|_Y)$$

when $Z = (Y, X)_{1/2}$. This is an abstract version of the Agranovich and M. I. Vishik type estimate for parameter elliptic problems (cf. [1], [2], [3], [4]).

To introduce the $\mathcal{R}$ -solver ($\mathcal{R}$ bounded solution operator) of problem (4), we start with the definition of $\mathcal{R}$ -bounded family.

Definition 1. Let E and F be two Banach spaces. We say that an operator family $\mathcal{T} \subset \mathcal{L}(E, F)$ is $\mathcal{R}$ bounded if there exist constants $C > 0$ and $q \in [1, \infty)$ such that for any integer n , $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset E$, the inequality:

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j f_j \right\|_F^q du \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(u) f_j \right\|_E^q du$$

is valid, where the Rademacher functions r_k , $k \in \mathbb{N}$, are given by $r_k : [0, 1] \rightarrow \{-1, 1\}$; $t \mapsto \text{sign}(\sin 2^k \pi t)$.

The smallest such C is called $\mathcal{R}$ bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(E, F)} \mathcal{T}$.

Next, we introduce an operator valued Fourier multiplier. Let $m(\xi)$ be an $L_\infty(\mathbb{R} \setminus \{0\})$ function with value in $\mathcal{L}(E, F)$. We set

$$T_m f = \mathcal{F}_{\mathbb{R}}^{-1} [m(\xi) \mathcal{F}_{\mathbb{R}}[f](\xi)] \quad f \in \mathcal{S}(\mathbb{R}, E),$$

where $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}}^{-1}$ denote respective Fourier transformation and inverse Fourier transformation on $\mathbb{R}$. The following theorem concerned with the boundedness of the operator T_m was proved by Weis [20].

Theorem 2. *Let E and F be two UMD Banach spaces. Let $m(\xi) \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(E, F))$ and assume that*

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(E,F)}(\{m(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) &\leq r_b \\ \mathcal{R}_{\mathcal{L}(E,F)}(\{\xi m'(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) &\leq r_b\end{aligned}$$

with some constant $r_b > 0$. Then, for any $p \in (1, \infty)$, $T_m \in \mathcal{L}(L_p(\mathbb{R}, E), L_p(\mathbb{R}, F))$ and

$$\|T_m f\|_{L_p(\mathbb{R}, F)} \leq C_p r_b \|f\|_{L_p(\mathbb{R}, E)}$$

with some constant C_p depending solely on p .

In view of Theorem 2, we introduce the following definition.

Definition 3 ($\mathcal{R}$ -solver). For every $\lambda = \gamma + i\tau \in \Sigma_{\epsilon, \lambda_0}$, there exists a map

$$\mathcal{S}(\lambda) : Y \times Y \times Z \rightarrow X \quad (F_1, F_2, F_3) \mapsto \mathcal{S}(\lambda)(F_1, F_2, F_3) \in X$$

such that

- (i) $v = \mathcal{S}(\lambda)(f, \lambda^{1/2}g, g)$ is a solution of problem (4),
- (ii) $\mathcal{S}(\lambda)$ satisfies

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(Y \times Y \times Z, X)}(\{(\tau \partial_\tau)^\ell \mathcal{S}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b, \\ \mathcal{R}_{\mathcal{L}(Y \times Y \times Z, Y)}(\{(\tau \partial_\tau)^\ell (\lambda \mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b,\end{aligned}$$

for $\ell = 0, 1$ with some constant r_b .

$\mathcal{S}(\lambda)$ is called an $\mathcal{R}$ -solver or $\mathcal{R}$ -bounded solution operator of problem (4).

Before considering periodic problem (3), we consider

$$\partial_t \tilde{u} - A\tilde{u} = \tilde{F}, \quad B\tilde{u} = \tilde{G} \quad \text{for } t \in \mathbb{R}. \quad (5)$$

Let $\varphi(t) \in C^\infty(\mathbb{R})$ which equals 1 for $|t| \geq \lambda_0 + 1$ and 0 for $|t| \leq \lambda_0 + 1/2$. From (ii) of Definition 3 $\varphi(\tau)\mathcal{S}(i\tau)$ is $\mathcal{R}$ bounded, that is

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(Y \times Y \times Z, X)}(\{(\tau \partial_\tau)^\ell \varphi(\tau)\mathcal{S}(i\tau) \mid \tau \in \mathbb{R}\}) &\leq \tilde{r}_b, \\ \mathcal{R}_{\mathcal{L}(Y \times Y \times Z, Y)}(\{(\tau \partial_\tau)^\ell (i\tau \varphi(\tau)\mathcal{S}(i\tau)) \mid \tau \in \mathbb{R}\}) &\leq \tilde{r}_b,\end{aligned}$$

for $\ell = 0, 1$ with some constant $\tilde{r}_b$. Let

$$\mathbf{S}(\tilde{F}, \tilde{G}) = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi(\tau)\mathcal{S}(i\tau)\mathcal{F}_{\mathbb{R}}[(\tilde{F}, \Lambda^{1/2}\tilde{G}, \tilde{G})](\tau)]$$

where $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}}^{-1}$ denote Fourier transformation and Fourier inverse transformation on $\mathbb{R}$, and, $\Lambda^{1/2}\tilde{G} = \mathcal{F}_{\mathbb{R}}^{-1}[(i\tau)^{1/2}\mathcal{F}_{\mathbb{R}}[\tilde{G}](\tau)]$. Writing $H_{\varphi} = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi(\tau)\mathcal{F}_{\mathbb{R}}[H](\tau)]$ for $H = \tilde{F}$ and $\tilde{G}$, we see easily that $w = \mathbf{S}(\tilde{F}, \tilde{G})$ satisfies equations:

$$\partial_t w - A\tilde{w} = \tilde{F}_{\varphi}, \quad Bw = \tilde{G}_{\varphi} \quad \text{for } t \in \mathbb{R}.$$

Moreover, by Theorem 2, for any $p \in (1, \infty)$ there exists a constant C_p depending on p such that

$$\|\partial_t \mathbf{S}(\tilde{F}, \tilde{G})\|_{L_p(\mathbb{R}, Y)} + \|\mathbf{S}(\tilde{F}, \tilde{G})\|_{L_p(\mathbb{R}, X)} \leq C_p \tilde{b}(\|\tilde{F}_{\varphi}\|_{L_p(\mathbb{R}, Y)} + \|\Lambda^{1/2}\tilde{G}_{\varphi}\|_{L_p(\mathbb{R}, Y)} + \|\tilde{G}_{\varphi}\|_{L_p(\mathbb{R}, Z)}). \quad (6)$$

We now consider time periodic equations (3). Let $\mathcal{F}_{\mathbb{T}}$ and $\mathcal{F}_{\mathbb{T}}^{-1}$ denote Fourier transformation and Fourier inverse transformation on $\mathbb{T}$, that is

$$\mathcal{F}_{\mathbb{T}}[f](k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt, \quad \mathcal{F}_{\mathbb{T}}^{-1}[(a_k)_{k \in \mathbb{Z}}](t) = \sum_{k \in \mathbb{Z}} e^{ikt} a_k,$$

where $\mathbb{Z}$ is the set of all integers and $(a_k)_{k \in \mathbb{Z}}$ denotes a sequence. Let $L_p(\mathbb{T}, X)$ be the set of all $L_p(\mathbb{R})$ function f valued with X such that $f(t + 2\pi) = f(t)$ for any $t \in \mathbb{R}$ and let $W_p^1(\mathbb{T}, X) = \{f \in L_p(\mathbb{T}, X) \mid \partial_t f \in L_p(\mathbb{T}, X)\}$. Set

$$\|f\|_{L_p(\mathbb{T}, X)} = \left\{ \int_0^{2\pi} \|f(t)\|_X^p dt \right\}^{1/p}.$$

To treat time periodic problem, we use the operator valued de Leewe transference theorem stated as follows:

Theorem 4. *Let E and F be Banach spaces and let $p \in (1, \infty)$. Let $m(\xi)$ be an $L_{\infty}(\mathbb{R} \setminus \{0\})$ function valued in $\mathcal{L}(E, F)$ and T_m is a bounded linear operator from $L_p(\mathbb{R}, E)$ to $L_p(\mathbb{R}, F)$. Suppose that for all $f \in E$ the point $k \in \mathbb{Z}^d$ is a Lebesgue point of $\xi \mapsto m(\xi)x$, and set $m_k f = m(ik)f$. Then, $(m_k)_{k \in \mathbb{Z}^d}$ is a Fourier multiplier from $L_p(\mathbb{T}^d, E)$ to $L_p(\mathbb{T}^d, F)$, and in fact,*

$$\|\tilde{T}_{(m_k)_{k \in \mathbb{Z}^d}}\|_{\mathcal{L}(L_p(\mathbb{T}^d, E), L_p(\mathbb{T}^d, F))} \leq \|T_m\|_{\mathcal{L}(L_p(\mathbb{R}^d, E), L_p(\mathbb{R}^d, F))}.$$

Here,

$$\tilde{T}_{(m_k)_{k \in \mathbb{Z}^d}}[f] = \mathcal{F}_{\mathbb{T}}^{-1}[m_k \mathcal{F}_{\mathbb{T}}[f](k)]_{k \in \mathbb{Z}^d}(t) = \sum_{k \in \mathbb{Z}^d} e^{ikt} m_k \mathcal{F}_{\mathbb{T}}[f](k).$$

Proof. For the proof, refer Proposition 5.7.1 in [12]. □

Now, we consider (3), and set

$$u_1 = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(\tau)\mathcal{S}(i\tau)\mathcal{F}_{\mathbb{T}}[(F, \Lambda^{\alpha}G, G)(\tau)]].$$

Then, $\mathbf{u}_1$ satisfies periodic problem:

$$\partial_t u_1 - Au_1 = F_\varphi \quad Bu_1 = G_\varphi \quad \text{for } t \in \mathbb{T}.$$

Here, $H_\varphi = \mathcal{F}_\mathbb{T}^{-1}[\varphi(\tau)\mathcal{F}_\mathbb{T}[H](i\tau)]$ for $H \in \{F, G\}$. Moreover, combining (6) and Theorem 4, we have

$$\|\partial_t u_1\|_{L_p(\mathbb{T}, Y)} + \|u_1\|_{L_p(\mathbb{T}, X)} \leq C(\|F_\varphi\|_{L_p(\mathbb{T}, Y)} + \|\Lambda^{1/2}G_\varphi\|_{L_p(\mathbb{T}, Y)} + \|G_\varphi\|_{L_p(\mathbb{T}, Z)}).$$

Thus, problem (3) is reduced to show the existence of finite number of solutions v_k of equations:

$$ikv_k - Av_k = \mathcal{F}_\mathbb{T}[F](ik), \quad Bv_k = \mathcal{F}_\mathbb{T}[G](ik).$$

And then,

$$u = u_1 + \sum_{|k| \leq \lambda_0 + 1/2} v_k \tag{7}$$

is a solution of (3). This is our strategy to solve periodic evolution problem (3).

2 A time periodic problem for the Stokes equations in exterior domains

Detailed arguments below are referred to Eiter, Kyed and Shibata [7]. Let Ω be a uniform C^2 bounded domain or exterior domain in $\mathbb{R}^N$ ($N \geq 2$) and let Γ be the boundary of Ω . Let $L_q(\Omega)$ be the set of all Lebesgue measurable functions f defined on Ω such that

$$\|f\|_{L_q(\Omega)} = \left\{ \int_\Omega |f(x)|^q dx \right\}^{1/q} < \infty$$

for $1 \leq q < \infty$. And let

$$W_q^m(\Omega) = \{f \in L_q(\Omega) \mid D_x^\alpha f(x) \in L_q(\Omega) \text{ } (|\alpha| \leq m)\}, \quad \|f\|_{W_q^m(\Omega)} = \sum_{|\alpha| \leq m} \|D_x^\alpha f\|_{L_q(\Omega)}.$$

We consider time periodic problem for the Stokes equations:

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathbf{p} = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}|_\Gamma = 0. \tag{8}$$

The corresponding resolvent equations read for the following equations:

$$\lambda \mathbf{v} - \Delta \mathbf{v} + \nabla \mathbf{q} = \mathbf{F}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0. \tag{9}$$

First, we introduce the Helmholtz decomposition. Let

$$\hat{H}_q^1(\Omega) = \{u \in L_{q, \text{loc}}(\Omega) \mid \nabla u \in L_q(\Omega)^N\}.$$

We consider the weak Neumann problem:

$$(\nabla u, \nabla \varphi) = (\mathbf{f}, \nabla \varphi) \quad \forall \varphi \in \hat{H}_q^1(\Omega). \quad (10)$$

Here, $(f, g) = \int_{\Omega} f(x)g(x) dx$. From [10, 18] we know that for any $\mathbf{f} \in L_q(\Omega)^N$ problem (10) admits a unique solution $u \in \hat{H}_q^1(\Omega)$ satisfying the estimate: $\|\nabla u\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$.

Let

$$J_q(\Omega) = \{\mathbf{u} \in L_q(\Omega)^N \mid (\mathbf{u}, \nabla \varphi) = 0 \quad \forall \varphi \in \hat{H}_q^1(\Omega)\},$$

which is called the solenoidal space. For $\mathbf{u} \in W_q^1(\Omega)$, what $\operatorname{div} \mathbf{u} = 0$ in Ω is equivalent to that $\mathbf{u} \in J_q(\Omega)$.

For any $\mathbf{f} \in L_q(\Omega)$, let $u \in \hat{H}_q^1(\Omega)$ be a unique solution of (10) and set $\mathbf{g} = \mathbf{f} - \nabla u$. We see that $\mathbf{g} \in J_q(\Omega)$. $\mathbf{f} = \mathbf{g} + \nabla u$ is called the Helmholtz decomposition. Let $P : L_q(\Omega) \rightarrow J_q(\Omega)$ be the solenoidal projection defined by

$$\mathbf{g} = P\mathbf{f}. \quad (11)$$

According to Shibata [17], we know that for any $\epsilon \in (0, \pi/2)$, there exists a positive constant λ_0 and $\mathcal{R}$ -solvers

$$\mathcal{T}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(L_q(\Omega)^N, H_q^2(\Omega)^N)), \quad \mathcal{P}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(L_q(\Omega)^N, \hat{H}_q^1(\Omega)))$$

such that for any $\mathbf{f} \in L_q(\Omega)^N$, $\mathbf{v} = \mathcal{T}(\lambda)\mathbf{f}$ and $\mathbf{q} = \mathcal{P}(\lambda)\mathbf{f}$ are unique solutions of equations (9) and

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(L_q(\Omega)^N, H_q^2(\Omega)^N)}(\{(\tau \partial_{\tau})^{\ell} \mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau \partial_{\tau})^{\ell} (\lambda \mathcal{T}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau \partial_{\tau})^{\ell} (\nabla \mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq r_b \quad (\ell = 0, 1), \end{aligned} \quad (12)$$

Let $\varphi(\tau)$ be a $C^\infty(\mathbb{R})$ function which equals to 1 for $|\tau| \geq \lambda_0 + 1$ and 0 for $|\tau| \leq \lambda_0 + 1/2$, and set

$$\mathbf{u}_{\varphi} = \mathcal{F}_{\mathbb{T}}^{-1}[\mathcal{T}(i\tau)\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)], \quad \mathbf{p}_{\varphi} = \mathcal{F}_{\mathbb{T}}^{-1}[\mathcal{P}(i\tau)\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)].$$

Then, $\mathbf{u}_{\varphi}$ and $\mathbf{p}_{\varphi}$ satisfy equations:

$$\partial_t \mathbf{u}_{\varphi} - \Delta \mathbf{u}_{\varphi} + \nabla \mathbf{p}_{\varphi} = \mathbf{f}_{\varphi}, \quad \operatorname{div} \mathbf{u}_{\varphi} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_{\varphi}|_{\Gamma} = 0, \quad (13)$$

where $\mathbf{f}_{\varphi} = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)]$, and the estimate:

$$\|\partial_t \mathbf{u}_{\varphi}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}_{\varphi}\|_{L_p(\mathbb{T}, W_q^2(\Omega))} \leq C\|\mathbf{f}_{\varphi}\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C\|\mathbf{f}\|_{L_p(\mathbb{T}, L_q(\Omega))} \quad (14)$$

provided that $\mathbf{f} \in L_p(\mathbb{T}, L_q(\Omega)^N)$ and $1 < p, q < \infty$. Thus, we have to consider the low frequency part. Let k be an element of $\mathbb{Z}$ such that $|k| \leq \lambda_0 + 1/2$. Let $\mathbf{u}_k \in W_q^2(\Omega)^N$ and $\mathbf{p}_k \in \hat{H}_q^1(\Omega)$ be unique solutions of the resolvent problem:

$$ik\mathbf{u}_k - \Delta \mathbf{u}_k + \nabla \mathbf{p}_k = \mathcal{F}_{\mathbb{T}}[\mathbf{f}](k), \quad \operatorname{div} \mathbf{u}_k = 0 \quad \text{in } \Omega, \quad \mathbf{u}_k|_{\Gamma} = 0. \quad (15)$$

2.1 Ω is a bounded domain

We know that when Ω is bounded, $\mathbf{u}_k \in W_q^2(\Omega)$ and $\mathbf{p}_k \in \hat{H}_q^1(\Omega)$ exist uniquely and they satisfy the estimate:

$$\|\mathbf{u}_k\|_{W_q^2(\Omega)} + \|\nabla \mathbf{p}_k\|_{L_q(\Omega)} \leq C \|\mathcal{F}_{\mathbb{T}}[\mathbf{f}](k)\|_{W_q^2(\Omega)} \leq C \int_0^{2\pi} \|\mathbf{f}(\cdot, t)\|_{L_q(\Omega)} dt. \quad (16)$$

Combining (14) and (16), we have the following theorem.

Theorem 5. *Let Ω be a bounded uniformly C^2 domain in $\mathbb{R}^N$ ($N \geq 2$). Let $1 < p, q < \infty$. Then, for any $\mathbf{f} \in L_p(\mathbb{T}, L_q(\Omega)^N)$, problem (8) admits unique solutions $\mathbf{u}$ and $\mathbf{p}$ with*

$$\mathbf{u} \in W_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, W_q^2(\Omega)^N), \quad \mathbf{p} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)),$$

which satisfy the estimate:

$$\|\partial_t \mathbf{u}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}\|_{L_p(\mathbb{T}, W_q^2(\Omega))} + \|\nabla \mathbf{p}\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C \|\mathbf{f}\|_{L_p(\mathbb{T}, L_q(\Omega))}.$$

2.2 Ω is a three dimensional exterior domain

We now continue the argument in the case where Ω is an exterior domain. We assume that Ω is a three dimensional exterior domain. Let $b > 0$ be a large number such that $\Omega^c \subset B_b = \{x \in \mathbb{R}^3 \mid |x| < b\}$. We will discuss the unique existence theorem of 2π -periodic solutions of (8).

We divide periodic function $\mathbf{f}$ into two parts as $\mathbf{f} = \mathbf{f}_{\perp} + \mathbf{f}_S$, where we have set

$$\mathbf{f}_S(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{f}(x, t) dt, \quad \mathbf{f}_{\perp}(x, t) = \mathbf{f}(x, t) - \mathbf{f}_S(x).$$

Obviously, $\int_0^{2\pi} \mathbf{f}_{\perp}(\cdot, t) dt = 0$. We call that $\mathbf{f}_{\perp}$ is the oscillatory part of $\mathbf{f}$ and $\mathbf{f}_S$ the stationary part of $\mathbf{f}$.

We know that when $k \neq 0$ and $|k| \leq \lambda_0 + 1/2$, problem (15) admits unique solutions $\mathbf{u}_k \in W_q^2(\Omega)^N$ and $\mathbf{p}_k \in \hat{H}_q^1(\Omega)$ which satisfy the estimate (16). Thus, setting

$$\mathbf{u}_{\perp} = \mathbf{u}_{\varphi} + \sum_{1 \leq |k| \leq \lambda_0 + 1/2} \mathbf{u}_k, \quad \mathbf{p}_{\perp} = \mathbf{p}_{\varphi} + \sum_{1 \leq |k| \leq \lambda_0 + 1/2} \mathbf{p}_k,$$

we see that

$$\mathbf{u}_{\perp} \in L_p(\mathbb{T}, W_q^2(\Omega)^N) \cap W_p^1(\mathbb{T}, L_q(\Omega)^N), \quad \mathbf{p}_{\perp} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$$

and $\mathbf{u}_{\perp}$ and $\mathbf{p}_{\perp}$ satisfy the equations:

$$\partial_t \mathbf{u}_{\perp} - \Delta \mathbf{u}_{\perp} + \nabla \mathbf{p}_{\perp} = \mathbf{f}_{\perp}, \quad \operatorname{div} \mathbf{u}_{\perp} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_{\perp}|_{\Gamma} = 0.$$

Moreover, we have

$$\|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T}, W_q^2(\Omega))} + \|\nabla \mathbf{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C \|\mathbf{f}\|_{L_p(\mathbb{R}, W_q^2(\Omega))}.$$

Let $\mathbf{u}_S$ and $\mathbf{p}_S$ be the stationary part of $\mathbf{u}$ and $\mathbf{p}$, and then $\mathbf{u}_S$ and $\mathbf{p}_S$ satisfy the stationary equations:

$$-\Delta \mathbf{u}_S + \nabla \mathbf{p}_S = \mathbf{f}_S, \quad \operatorname{div} \mathbf{u}_S = 0 \quad \text{in } \Omega, \quad \mathbf{u}_S|_\Gamma = 0. \quad (17)$$

Using the fundamental solutions:

$$U_{ij}(x) = -\frac{1}{8\pi} \left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3} \right), \quad q_j(x) = \frac{1}{3\pi} \frac{x_j}{|x|^3},$$

of the Stokes equations in $\mathbb{R}^3$ and the cut off technique, we have

Lemma 6. *Let Ω be a uniformly C^2 exterior domain in $\mathbb{R}^3$, and let $3 < q < \infty$. Let $< g >_\ell = \sup_{x \in \Omega} (1 + |x|)^\ell |g(x)|$, and*

$$L_{q,3b}(\Omega) = \{g \in L_q(\Omega) \mid g(x) = 0 \text{ for } |x| > 3b\}.$$

If $\mathbf{f}_S = \operatorname{div} \mathbf{F} + \mathbf{g}$ such that

$$< \operatorname{div} \mathbf{F} >_3 + < \mathbf{F} >_2 < \infty, \quad \mathbf{g} \in L_{q,3b}(\Omega),$$

then problem (17) admits unique solutions $\mathbf{u}_S \in W_q^2(\Omega)^3$ and $\mathbf{p}_S \in W_q^1(\Omega)$ satisfying the estimate:

$$\begin{aligned} & \|\mathbf{u}_S\|_{W_q^2(\Omega)} + < \mathbf{u}_S >_1 + < \nabla \mathbf{u}_S >_2 + \|\mathbf{p}_S\|_{W_q^1(\Omega)} + < \mathbf{p}_S >_2 \\ & \leq C(< \operatorname{div} \mathbf{F}_S >_3 + < \mathbf{F}_S >_2 + \|\mathbf{g}\|_{L_q(\Omega)}). \end{aligned}$$

It is important to investigate the asymptotic behaviour of oscillatory parts $\mathbf{u}_\perp$, especially to solve the Navier-Stokes equations. For this purpose we use the following lemma which shows the asymptotic behaviours of the fundamental solution $\Gamma_\perp$ of the resolvent equations:

$$ik\mathbf{v} - \Delta \mathbf{v} + \nabla q = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \mathbb{R}^3.$$

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Lemma 7 (Eiter and Kyed [5]). *Let*

$$\Gamma_\perp = \mathcal{F}_{\mathbb{R}^3 \times \mathbb{T}}^{-1} \left[\frac{1 - \delta_{\mathbb{Z}}}{|\xi|^2 + ik} \left(\mathbf{I} - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right].$$

Then, $\Gamma_\perp \in L_q(\mathbb{R}^3 \times \mathbb{T})$ for $q \in (1, 5/3)$ and $\nabla \Gamma_\perp \in L_q(\mathbb{R}^3 \times \mathbb{T})^3$ for $q \in (1, 5/4)$. And for any multi-index $\alpha \in \mathbb{N}_0^3$, $\delta > 0$ and $r \in [1, \infty)$,

$$\|D_x^\alpha \Gamma_\perp(x, \cdot)\|_{L_p(\mathbb{T})} \leq \frac{C_{\alpha, \delta}}{|x|^{3+|\alpha|}} \quad (|x| > \delta).$$

We consider again equations:

$$\partial_t \mathbf{u}_\perp - \Delta \mathbf{u}_\perp + \nabla \mathbf{p}_\perp = \mathbf{f}_\perp, \quad \operatorname{div} \mathbf{u}_\perp = 0 \quad \text{in } \Omega, \quad \mathbf{u}_\perp|_\Gamma = 0. \quad (18)$$

To state the asymptotic behaviour of $\mathbf{u}_\perp$, we introduce the norm

$$\langle f \rangle_{p,\ell} = \sup_{x \in \Omega} \|f(x, \cdot)\|_{L_p(\mathbb{T})} (1 + |x|)^\ell.$$

Using the fundamental solutions to the oscillatory part $\Gamma_\perp$ and the cut off technique, we have the following lemma.

Lemma 8. *Let $3 < q < \infty$ and $\ell \in (0, 3]$. Assume that $\mathbf{f}_\perp = \operatorname{div} \mathbf{F}_\perp + \mathbf{g}_\perp$ such that*

$$\begin{aligned} \int_0^{2\pi} \mathbf{F}_\perp(x, t) dt &= 0, \quad \langle \mathbf{F}_\perp \rangle_{p,\ell} + \langle \operatorname{div} \mathbf{F}_\perp \rangle_{p,\ell+1} < \infty, \\ \int_0^{2\pi} \mathbf{g}_\perp(x, t) dt &= 0, \quad \mathbf{g}_\perp \in L_p(\mathbb{T}, L_{q,3b}(\Omega)^3). \end{aligned}$$

Then, $\mathbf{u}_\perp$ satisfies the estimate:

$$\langle \mathbf{u}_\perp \rangle_{p,\ell} + \langle \nabla \mathbf{u}_\perp \rangle_{p,\ell+1} \leq C(\langle \operatorname{div} \mathbf{F}_\perp \rangle_{p,\ell+1} + \langle \mathbf{F}_\perp \rangle_{p,\ell} + \|\mathbf{g}\|_{L_p(\mathbb{T}, L_q(\Omega))}).$$

Summing up, we have obtained the following theorem.

Theorem 9. *Let $2 < p < \infty$ and $3 < q < \infty$, and $\ell \in (0, 3]$. For all $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ with $\mathbf{f}_S = \operatorname{Div} \mathbf{G}_S + \mathbf{g}_S$ and $\mathbf{f}_\perp = \operatorname{div} \mathbf{G}_\perp + \mathbf{g}_\perp$ such that $\mathbf{g}_S \in L_{q,3b}(\Omega)^3$, $\mathbf{g}_\perp \in L_p(\mathbb{T}, L_{q,3b}(\Omega)^3)$ and*

$$\langle \mathbf{G}_S \rangle_2 + \langle \operatorname{div} \mathbf{G}_S \rangle_3 + \langle \mathbf{G}_\perp \rangle_{p,\ell} + \langle \operatorname{div} \mathbf{G}_\perp \rangle_{p,\ell+1} < \infty$$

problem (8) admits unique solutions $\mathbf{u}$ and $\mathbf{p}$ with

$$\mathbf{u} \in W_p^1(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, W_q^2(\Omega)^3), \quad \mathbf{p} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$$

satisfying the estimate:

$$\begin{aligned} &\|\mathbf{u}_S\|_{W_q^2(\Omega)} + \langle \mathbf{u}_S \rangle_1 + \langle \nabla \mathbf{u}_S \rangle_2 + \|\mathbf{p}_S\|_{W_q^1(\Omega)} + \langle \mathbf{p}_S \rangle_2 + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T}, W_q^2(\Omega))} \\ &+ \|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \langle \mathbf{u}_\perp \rangle_{p,\ell} + \langle \nabla \mathbf{u}_\perp \rangle_{p,\ell+1} + \|\nabla \mathbf{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \\ &\leq C(\langle \operatorname{div} \mathbf{G}_S \rangle_3 + \langle \mathbf{G}_S \rangle_2 + \langle \operatorname{div} \mathbf{G}_\perp \rangle_{p,\ell+1} + \langle \mathbf{G}_\perp \rangle_{p,\ell} \\ &\quad + \|\mathbf{g}_S\|_{L_q(\Omega)} + \|\mathbf{g}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}). \end{aligned}$$

3 Time periodic solutions to the Navier-Stokes equations

Since we already know Theorem 9, which tells us the unique existence of time periodic solutions to the Stokes equations based on the maximal regularity for the high-frequency

part and the space decay properties of solutions, we can show the unique existence of strong solutions stated as follows, which is completely different approach from [9, 14, 21, 22].

Theorem 10. *Let $2 < p < \infty$ and $3 < q < \infty$. Assume that $\mathbf{F} = \mathbf{F}_S + \mathbf{F}_\perp$ with $\mathbf{F}_S = \operatorname{div} \mathbf{G}_S$ and $\mathbf{F}_\perp = \operatorname{div} \mathbf{G}_\perp$. Then, there exists a small constant $\epsilon > 0$ such that if $\mathbf{F}$ satisfy the smallness condition: $\langle \mathbf{F}_S \rangle_3 + \langle \mathbf{G}_S \rangle_2 + \langle \mathbf{F}_\perp \rangle_{p,2} + \langle \mathbf{G}_\perp \rangle_{p,1} < \epsilon^2$, then problem (1) admits unique solutions $\mathbf{u} = \mathbf{u}_S + \mathbf{u}_\perp$ and $\mathbf{p} = \mathbf{p}_S + \mathbf{p}_\perp$ with*

$$\begin{aligned} \mathbf{u}_S &\in W_q^2(\Omega)^3, \quad \mathbf{u}_\perp \in L_p(\mathbb{T}, W_q^2(\Omega)^3) \cap W_p^1(\mathbb{T}, L_q(\Omega)^3), \\ \mathbf{p}_S &\in W_q^1(\Omega), \quad \mathbf{p}_\perp \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)) \end{aligned}$$

satisfying the estimate:

$$\begin{aligned} &\langle \mathbf{u}_S \rangle_1 + \langle \nabla \mathbf{u}_S \rangle_2 + \|\mathbf{u}_S\|_{W_q^2(\Omega)} + \langle \mathbf{p}_S \rangle_2 + \|\mathbf{p}_S\|_{W_q^1(\Omega)} + \langle \mathbf{u}_\perp \rangle_{p,1} \\ &+ \langle \nabla \mathbf{u}_\perp \rangle_{p,2} + \|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T}, W_q^2(\Omega))} + \|\nabla \mathbf{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} < \epsilon. \end{aligned}$$

Proof. To move $\mathbf{u} \cdot \nabla \mathbf{u}$ to the right hand side and using the Banach fixed point theorem based on Theorem 9, we can prove Theorem 10 immediately. $\square$

We now consider the Navier-Stokes equations in a periodically moving exterior domain. Let $\phi \in C^0(\mathbb{T}, C^3(\Omega)^N) \cap C^1(\mathbb{T}, C^1(\Omega)^N)$ with

$$\|\phi\|_{C^0(\mathbb{T}, C^3(\Omega))} + \|\partial_t \phi\|_{C^0(\mathbb{T}, C^1(\Omega))} \leq \epsilon^2 \quad (19)$$

and Ω_t and Γ_t are given by

$$\Omega_t = \{x = y + \phi(y, t) \mid y \in \Omega\}, \quad \Gamma_t = \{x = y + \phi(y, t) \mid y \in \Gamma\} \quad (t \in \mathbb{R}).$$

Consider the Navier-Stokes equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \mathbf{p} = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_t, \quad \mathbf{u}|_{\Gamma_t} = 0. \quad (20)$$

When ϵ is small enough and Ω is a bounded domain, Farwig, Kozono, Tsuda, and Wegmann [8, 11] proved the global well-posedness. Eiter, Kyes and Shibata [7] also proved the global well-posedness by using a perturbation method based on Theorem 5. The method in [7] is completely different from [8, 11].

Moreover, in [7], the case where Ω_t is an exterior domain of $\mathbb{R}^3$ is treated. If we assume that ϵ in (19) is small enough, we have the inverse transform: $y = x + \psi(x, t)$, and we reduce equations (20) to the following equations:

$$\partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla \mathbf{q} = \mathbf{G} + \mathcal{L}(\mathbf{w}, \mathbf{q}) + \mathcal{N}(\mathbf{w}), \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{w}|_\Gamma = 0 \quad (21)$$

with a fixed domain Ω . Here, $\mathcal{L}$ is a linear operator of the form:

$$\mathcal{L}(\mathbf{w}, \mathbf{q}) = a(x, t) \partial_t \mathbf{w} + \sum_{|\alpha| \leq 2} b_\alpha(x, t) D_x^\alpha \mathbf{w} + c(x, t) \nabla \mathbf{q}$$

with $\|(a, b_\alpha, c)\|_{L_\infty(\Omega \times \mathbb{T})} \leq C\epsilon^2$, and $\mathcal{N}(\mathbf{w})$ is a nonlinear term satisfying the estimate $|\mathcal{N}(\mathbf{w})| \leq C|\mathbf{w}||\nabla \mathbf{w}|$. Using the standard iteration argument based on Theorem 9, we have the following theorem.

Theorem 11. *Let $2 < p < \infty$ and $3 < q < \infty$. Assume that $2/p + 3/q < 2$. Assume that $\mathbf{G} = \mathbf{G}_S + \mathbf{G}_\perp$ with $\mathbf{G}_S = \operatorname{div} \mathbf{H}_S$ and $\mathbf{G}_\perp = \operatorname{div} \mathbf{H}_\perp$. Then, there exists a small constant $\epsilon > 0$ such that if ϕ and $\mathbf{G}$ satisfy the smallness condition:*

$$\begin{aligned} \|\phi\|_{C^0(\mathbb{T}, C^3(\Omega))} + \|\partial_t \phi\|_{C^0(\mathbb{T}, C^1(\Omega))} &\leq \epsilon^2, \\ \|\mathbf{G}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \langle \mathbf{G}_\perp \rangle_{p,2} + \langle \mathbf{H}_\perp \rangle_{p,1} + \langle \mathbf{G}_S \rangle_3 + \langle \mathbf{H}_S \rangle_2 &< \epsilon^2 \end{aligned}$$

then problem (21) admits unique solutions $\mathbf{w} = \mathbf{w}_S + \mathbf{w}_\perp$ and $\mathbf{q} = \mathbf{q}_S + \mathbf{q}_\perp$ with

$$\mathbf{w}_\perp \in H_p^1(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H_q^2(\Omega)^3), \mathbf{w}_S \in H_q^2(\Omega)^3, \mathbf{q}_\perp \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)), \mathbf{q}_S \in H_q^1(\Omega)$$

satisfying the estimate:

$$\begin{aligned} \langle \mathbf{w}_\perp \rangle_{p,1} + \langle \nabla \mathbf{w}_\perp \rangle_{p,2} + \|\mathbf{w}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\partial_t \mathbf{w}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\nabla \mathbf{q}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \\ + \langle \mathbf{w}_S \rangle_1 + \langle \nabla \mathbf{w}_S \rangle_2 + \|\mathbf{w}\|_{H_q^2(\Omega)} + \|\mathbf{q}_S\|_{H_q^1(\Omega)} \leq \epsilon. \end{aligned}$$

Here,

$$\langle g_\perp \rangle_{p,\ell} = \sup_{x \in \Omega} \|g_\perp(x, \cdot)\|_{L_p(\mathbb{T})} (1 + |x|)^\ell, \quad \langle g_S \rangle_\ell = \sup_{x \in \Omega} |g_S(x)| (1 + |x|)^\ell.$$

This theorem gives us the unique existence of periodic solutions of equations (20) for small ϵ when Ω_t is an exterior domain of $\mathbb{R}^3$.

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