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Dedekind sums and special values of L-functions

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1. The purpose of this note is to supply the author's paper [2] with a remark and an example. Let K be an imaginary quadratic field with class number one (This assumption on the class number is used only in the argument after the formula (9).) and F be an quadratic extention of K. Consider a Grössencharacter ψ of F defined modulo an ideal $\mathfrak{m}$ of F such that

$$(1) \qquad \psi((\mu)) = \overline{N_{F/K}(\mu)}$$

for every integer μ of F with $\mu \equiv 1 \pmod{\mathfrak{m}}$ and let

$$L(s,\psi) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \psi(\mathfrak{a})N(\mathfrak{a})^{-s} \qquad (\,\mathrm{Re}(s) > 3/2\,)$$

where $\mathfrak{a}$ runs over all integral ideals of F which is prime to $\mathfrak{m}$. In the following we shall give an expression of $L(1,\psi)$ in terms of Dedekind sums of Sczech [4] and give an example of calculation of the value $L(1,\psi)$. If we have an elliptic curve E defined over F which has complex multiplication by an order of K, the complex conjugate $\overline{\psi}_E$ of the Grössencharacter ψ_E of F associated with E is of the type mentioned above. Since the Hasse-Weil zeta function of E over F is equal to

$L(s,\psi_E)\cdot L(s,\overline{\psi}_E)$, our study on the value $L(1,\psi)$ will have some interest in view of the well-known conjecture of Birch and Swinnerton-Dyer.

2. First let us review some of the results of [4] and [2]. Let $\mathcal{O} = \mathcal{O}_K$ be the ring of integers of K and D the discriminant of K. For a non-negative integer k and two complex numbers p and q, put

$$G(s,k,p,q) = \sum_{\substack{m \in \mathcal{O} \\ m+q \neq 0}} e\left(\frac{m\overline{p}}{\sqrt{D}}\right) \left(\overline{m+q}\right)^k \, | m+q |^{-2s-k}$$

for $\mathrm{Re}(s)>1$, where $e(z) = \exp(2\pi i(z+\overline{z}))$ and $\mathrm{Im}(\sqrt{D}) > 0$. This can be continued analytically to the whole s-plane . Functions $G_k(p) := G(k/2,k,0,p)$ and $G(p) := \frac{2\pi i}{\sqrt{D}} G(0,2,0,p)$ are periodic with respect to the lattice $\mathcal{O}$ and can be expressed by means of the Weierstrass elliptic functions (cf. [4]). For two integers a and c in $\mathcal{O}$ with $c \neq 0$ and two complex numbers p and q, define

$$D(a,c;p,q) = -\frac{1}{c} \sum_{r \in L/cL} G_1\left(\frac{a(r+p)}{c} + q\right) G_1\left(\frac{r+p}{c}\right).$$

Furthermore for every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathcal{O})$, let

$$\Phi(A)(p,q) = -\left(\overline{\frac{a}{c}}\right) G(p) - \left(\overline{\frac{d}{c}}\right) G(p^*) - \frac{a}{c} G_0(p)G_2(q)$$
$$- \frac{d}{c} G_0(p^*)G_2(q^*) - D(a,c;p,q)$$

if $c \neq 0$, and

$$\Phi(A)(p,q) = -\overline{\left(\frac{b}{d}\right)}\, G(p) - \frac{b}{d}\, G_0(p)G_2(q)$$

if $c = 0$. Here $(p^*, q^*) := (p,q)A = (\, ap+cq,\ bp+dq\,)$. In particular, we have

(2) $$\Phi\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}(p,q) = -\overline{d}\, G(q) - d\, G_0(q)G_2(p-dq) - G_1(p)G_1(q)$$

and

(3) $$\Phi\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}(p,q) = 0$$

for every unit ξ contained in K. One of the main results of [4] is the formula

(4) $$\Phi(AB)(p,q) = \Phi(A)(p,q) + \Phi(B)((p,q)A)$$

which holds for arbitrary matrices A and B in $SL(2,\mathcal{O})$.

Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathcal{O})$ with $(\, a+d\,)^2 \neq 0, 1, 4$ and $p,\ q \in K$ which satisfy

(5) $$(p,q)A \equiv (p,q) \mod \mathcal{O}^2.$$

Then we can consider the L-series $L(A,s;p,q)$ as follows. There are two distinct complex numbers α and α' such that

$$A\begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}$$

and $\varepsilon = c\alpha + d$ is a unit of the quadratic extension field $F := K(\alpha)$ of K which satisfies $N_{F/K}(\varepsilon) = 1$. We assume $|\varepsilon| > 1$ by changing α and α' if necessary. Let, for $\mathrm{Re}(s) > 3/2$,

(6) $$L(A,s;p,q) = \sum_{\mu \in (\mathcal{M}_{p,q} - \{0\})/\langle\varepsilon\rangle} \overline{N_{F/K}(\mu)}\, N(\mu)^{-s}$$

where

$$\mathcal{M}_{p,q} := (\mathcal{O}+p)\alpha + \mathcal{O}+q.$$

Note that $\varepsilon M_{p,q} = M_{p,q}$ by the assumption (5). The L-series (6) has an analytic continuation to the whole s-plane and the following formula is proved in [2]:

(7) $$(\alpha-\alpha')L(A,s;p,q) = \Phi(A).$$

3. Returning to the situation of section 1, take an complete set of representatives $\{ \mathfrak{a}_1, \cdots, \mathfrak{a}_h \}$ of the absolute ideal class group of F such that $(\mathfrak{a}_j, \mathfrak{m}) = 1$ and a unit ε with $|\varepsilon| > 1$, $N_{F/K}(\varepsilon) = 1$ and $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$. Then,

$$L(s,\psi) = \sum_{j=1}^{h} \sum_{(\mu)\subset \mathfrak{a}_j^{-1}} \psi(\mu\mathfrak{a}_j)N(\mu\mathfrak{a}_j)^{-s}$$

$$= [O_F^{\times}:\langle\varepsilon\rangle]^{-1} \sum_{j=1}^{h} \psi(\mathfrak{a}_j)N(\mathfrak{a}_j)^{-s} \sum_{\mu \in \mathfrak{a}_j^{-1}/\langle\varepsilon\rangle} \psi((\mu))N(\mu)^{-s}$$

where $O_F^{\times}$ denotes the group of all units of F. Define, for every $\mu \in F$ which is prime to $\mathfrak{m}$, the root of unity $\chi(\mu)$ by

(8) $$\psi((\mu)) = \chi(\mu)\ \overline{N_{F/K}(\mu)}.$$

If μ is congruent maltiplicatively to 1 modulo $\mathfrak{m}$, then $\chi(\mu) = 1$. It follows that

$$\sum_{\mu \in \mathfrak{a}_j^{-1}/\langle\varepsilon\rangle} \psi((\mu))N(\mu)^{-s}$$

(9) $$= \sum_{\substack{\nu \in \mathfrak{a}_j^{-1}/\mathfrak{m}\mathfrak{a}_j^{-1} \\ (\nu,\mathfrak{m})=1}} \chi(\nu) \sum_{\mu \in (\nu+\mathfrak{m}\mathfrak{a}_j^{-1})/\langle\varepsilon\rangle} \overline{N_{F/K}(\mu)}\ N(\mu)^{-s}.$$

Because the class number of K is one, there exist β_j and γ_j in F such that

$$\mathfrak{m}\mathfrak{a}_j^{-1} = O_K\beta_j + O_K\gamma_j \qquad (j = 1,\cdots,h).$$

Put $\alpha_j = \beta_j/\gamma_j$ and define the matrix A_j in $SL(2, \mathcal{O}_K)$ by

$$\varepsilon \begin{pmatrix} \alpha_j \\ 1 \end{pmatrix} = A_j \begin{pmatrix} \alpha_j \\ 1 \end{pmatrix}.$$

For each ν in $\mathfrak{a}_j^{-1}$, denote by $p_j(\nu)$ and $q_j(\nu)$ the elements of K such that

$$\nu = p_j(\nu)\beta_j + q_j(\nu)\gamma_j.$$

Then the summation over μ in (9) is equal to

$$\overline{N_{F/K}(\gamma_j)}\ N(\gamma_j)^{-s}\ L(A_j, s; p_j(\nu), q_j(\nu))$$

and we have

(10) $\qquad L(1, \psi)$

$$= [\mathcal{O}_F^{\times} : <\varepsilon>]^{-1} \sum_{j=1}^{h} \psi(\mathfrak{a}_j) N(\mathfrak{a}_j)^{-1} N_{F/K}(\gamma_j)^{-1} (\alpha_j - \alpha_j')^{-1}$$

$$\times \sum_{\substack{\nu \in \mathfrak{a}_j^{-1}/\mathfrak{m}\mathfrak{a}_j^{-1} \\ (\nu, \mathfrak{m})=1}} \chi(\nu) \Phi(A_j)(p_j(\nu), q_j(\nu)).$$

4. An example. Let $K = \mathbb{Q}(i)$ and $F = K(\rho) = \mathbb{Q}(\zeta_{12})$ where $i = \zeta_4$, $\rho = \zeta_3$ with $\zeta_n = \exp(2\pi i/n)$. The class number of F is one and the group $\mathcal{O}_F^{\times}$ is generated by ζ_{12} and $\varepsilon_0 = 1 + \zeta_{12}$ (cf. Kuroda [3]). Consider the elliptic curve

$$E : \ Y^2 = X^4 - \varepsilon_0^{\,2}$$

defined over F which has complex multiplication by $\mathcal{O}_K$. Denote by $\psi = \psi_E$ the Grössencharacter of F associated with E. Then, for every integral ideal $\mathfrak{a}$ of F prime to 2, we have

$$\psi(\mathfrak{a}) = \left(\frac{\varepsilon_0}{\mathfrak{a}}\right) \theta(N_{F/K}(\mathfrak{a}))$$

from Davenport and Hasse [1] and Weil [5]. Here $\left(\frac{\varepsilon_0}{\mathfrak{a}}\right)$ means the quadratic residue symbol in F and, for every integral ideal $\mathfrak{b}$ of K prime to 2, $\theta(\mathfrak{b})$ denotes the generator of $\mathfrak{b}$ which is congruent to 1 modulo $(1+i)^3$. The Grössencharacter $\overline{\psi}$ satisfies the condition (1) with $\mathfrak{m} = 4O_F$. Taking $\varepsilon = -(2+\sqrt{3})^2 = \rho\varepsilon_0^{\,4}$, $\mathfrak{a}_1 = O_F$ and $\beta_1 = 4\alpha$ ($\alpha := \zeta_6$), $\gamma_1 = 4$ in the notation of section 3, we see, from (10),

$$L(1,\overline{\psi}) = \frac{-i}{2^8\cdot 3\sqrt{3}} \sum_{\nu\in(O_F/(4))^\times} \overline{\chi(\nu)}\ \Phi(A)\left(\frac{p(\nu)}{4}, \frac{q(\nu)}{4}\right),$$

where

$$A := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1+i \end{pmatrix}^4,$$

the character χ of $(O_F/(4))^\times$ is determined by the relation

$$\psi((\mu)) = \chi(\mu) N_{F/K}(\mu) \qquad (\mu \in O_F,\ (\mu,2)=1),$$

and integers $p(\nu)$ and $q(\nu)$ of K are defined by

$$\nu = p(\nu)\alpha + q(\nu)$$

for every $\nu \in O_F$. Decomposing A, for example, as

$$A = -\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} C^2$$

with

$$C = i\begin{pmatrix} 1 & i \\ -i & 1+i \end{pmatrix}^2 = \begin{pmatrix} 2i & -2-i \\ 2+i & -2+i \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

the values $\Phi(A)(\frac{p(\nu)}{4}, \frac{q(\nu)}{4})$ can be evaluated by (2), (3), (4) and the 4-division values of G_1, G_2 and G which are listed in the table 1 below. The calculation shows

$$L(1, \overline{\psi}_E) = \frac{1}{2\sqrt{6}} \Omega^2 ,$$

where Ω is the positive number such that $O_K\Omega$ is the period lattice of the Weierstrass p-function satisfying $p'^2 = 4p^3 - 4p$.

Table 1

α	$G_1(\alpha)\Omega^{-1}$	$G_2(\alpha)\Omega^{-2}$	$G(\alpha)\Omega^{-2}$
0	0	0	0
$\frac{1+i}{2}$	0	0	0
$\frac{1}{2}$	0	1	$\frac{1}{2}$
$\frac{1+i}{4}$	$\frac{-1+i}{2}$	$-i$	$-\frac{i}{4}$
$\frac{1}{4}$	$\frac{-1-\sqrt{2}}{2}$	$1+\sqrt{2}$	$\frac{1+2\sqrt{2}}{8}$
$\frac{1+2i}{4}$	$\frac{1-\sqrt{2}}{2}$	$1-\sqrt{2}$	$\frac{1-2\sqrt{2}}{8}$

References

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