

Cycles in Graphs

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In this paper, we consider only finite simple graphs. A cycle C in G is called *separating* if deletion of its vertices results in a disconnected graph. A cycle is *non-separating* if it is not separating. Lovász[3] remarked that every 3-connected graph has a non-separating cycle. Thomassen and Toft[4] extended his result and proved that every 3-connected graph has a non-separating induced cycle. We prove in this paper that every 3-connected graph has many non-separating induced cycles.

When we prove the property of 3-connected graph, the idea of a contractible edge is very useful. In a 3-connected graph, an edge e is called *contractible* if contraction of e results in a 3-connected graph. An edge is *non-contractible* if it is not contractible. Obviously, an edge xy in G is non-contractible if and only if G has a cutset of order three which contains both x and y .

We denote the set of vertices and the set of edges of the graph G by $V(G)$ and $E(G)$, respectively. Let $x \in V(G)$. We denote the set of the vertices adjacent to x by $\Gamma_G(x)$, the degree of x by $d_G(x)$ and the order of G by $|G|$. Other notations may be found in [2].

For a 3-connected graph G and $x \in V(G)$, we write

$$\Gamma^{(1)}(x) := \{y \in \Gamma_G(x) \mid xy \text{ is a contractible edge}\}$$

$$\Gamma^{(2)}(x) := \{y \in \Gamma_G(x) \mid xy \text{ is a non-contractible edge}\}$$

and define $U_i (i=0,1,2,3), W_i (i \geq 0)$, subset of $V(G)$, by

$$U_i := \{x \in V(G) \mid d_G(x)=3, |\Gamma^{(1)}(x)| = i\} \quad (i=0,1,2,3)$$

$$W_i := \{x \in V(G) \mid d_G(x) \geq 4, |\Gamma^{(1)}(x)| = i\} \quad (i \geq 0)$$

Ando, Enomoto and Saito[1] proved the following theorems.

Theorem A *Let G be a 3-connected graph of order at least five and $x \in V(G)$. If $d_G(x)=3$, say $\Gamma_G(x)=\{a,b,c\}$, and both xb and xc are non-contractible, then $d_G(b)=d_G(c)=3$, and b and c are adjacent.*

Theorem B $U_0 = \emptyset$ if $|G| \geq 5$.

Theorem C *Suppose $|G| \geq 5$. If $x \in W_0$ then $|\Gamma^{(2)}(x) \cap U_2| \geq 3$. If $x \in W_1$ then $|\Gamma^{(2)}(x) \cap U_2| \geq 2$.*

Theorem B is easily deduced from Theorem A.

The following theorem is a slight generalization of the result of Ando, Enomoto and Saito([1, Theorem 3]).

Theorem 1. *Let G be a 3-connected graph of order at least five, $x \in V(G)$ and $X \subset V(G)$. Suppose $\Gamma^{(1)}(x) \subset X$ and $d_G(x) \geq 4$. Moreover, suppose there exists a least cutset S , which contains x , such that $G-S$ has a connected component disjoint from X . Then $(\Gamma^{(2)}(x) - X) \cap U_2 \neq \emptyset$.*

Proof. Define

$$C_x := \{S; \text{a least cutset such that } x \in S$$

and $G-S$ has a connected component A which is disjoint from $X\}$.

By the assumption $C_x \neq \emptyset$. For each $S \in C_x$, let A_S be the smallest component of $G-S$ such that A_S is disjoint from X . Moreover, we choose $S \in C_x$ such that $|A_S|$ is minimum. Let $B_S = V(G) - A_S - S$.

Since $A_S \cap X = \emptyset$, there exists a vertex y in $A_S \cap \Gamma_G(x)$ such that xy is a non-contractible edge. Let T be a least cutset containing the edge xy , C be one of the connected components of $G - T$ and $D = V(G) - T - C$. Let

$$X_1 := (S \cap C) \cup (S \cap T) \cup (A_S \cap T)$$

and

$$X_2 := (S \cap C) \cup (S \cap T) \cup (B_S \cap T).$$

First we claim that $S \cap C \neq \emptyset$. Assume $S \cap C = \emptyset$. Since $y \in A_S \cap T$, $|X_2| \leq 2$. If $B_S \cap C \neq \emptyset$ then X_2 is a cutset of order at most two. This contradicts the assumption on connectivity of G . Therefore, $B_S \cap C = \emptyset$ and it follows $A_S \cap C \neq \emptyset$ since $C \neq \emptyset$. Hence X_1 is a cutset and since $X_1 \subset T$, $|X_1| \leq 3$. This contradicts either the connectivity of G or the minimality of A_S .

The similar argument leads us to $S \cap D \neq \emptyset$. Therefore, we know that $|S \cap C| = |S \cap T| = |S \cap D| = 1$.

Next we claim $B_S \cap T \neq \emptyset$. Assume $B_S \cap T = \emptyset$. Without loss of generality, we may assume that $B_S \cap C \neq \emptyset$, and it follows that X_2 is a cutset of order at most two. This is a contradiction.

Therefore, we know that $|A_S \cap T| = |S \cap T| = |B_S \cap T| = 1$, and $|X_1| = 3$. Since A_S is minimal, $A_S \cap C = \emptyset$ and $A_S \cap D = \emptyset$. Hence we have $|A_S| = 1$ and $A_S = \{y\}$. Then $\Gamma_G(y) = S$ and $d_G(y) = 3$.

Let $a \in S - \{x\}$. If ya is a non-contractible edge, then $d_G(x) = 3$ by Theorem A. This contradicts the assumption that $d_G(x) \geq 4$. Hence ay is a contractible edge. This implies $y \in U_2$. ■

For $X \subset V(G)$ and $F \subset E(G)$, we say X covers F in case that any edge of F is incident with at least one of the vertices of X .

Theorem 2. *Let G be a 3-connected graph of order at least six. Then, the set of the contractible edges of G cannot be covered with any set of two*

vertices.

Proof. We assume that there is a 3-connected graph G of order at least six such that a set of contractible edges is covered with two vertices, say u and v . From Theorem B, $V(G)$ can be written as

$$V(G) = U_1 \cup U_2 \cup W_0 \cup W_1 \cup W_2 \cup \{u, v\}.$$

since $W_i = U_i = \emptyset$ if $i \geq 3$.

Claim 1: $W_i - \{u, v\} = \emptyset$ for $i = 0, 1$.

Suppose $W_i - \{u, v\} \neq \emptyset$, say $x \in W_i - \{u, v\}$. Let X be $(\Gamma^{(2)}(x) \cap U_2) \cup \{u, v\}$. By Theorem C, there exists $a \in (\Gamma^{(2)}(x) \cap U_2) - \{u, v\}$. (If $x \in W_1$ and $\Gamma^{(2)}(x) \cap U_2 = \{u, v\}$, then xu and xv are non-contractible, which contradicts the assumption that $x \in W_1$). If $a \in (\Gamma^{(2)}(x) \cap U_2) - \{u, v\}$, then $\Gamma_G(a) = \{x, u, v\}$. Hence $S = \{x, u, v\}$ is a least cutset which contains x and $G - S$ has a connected component disjoint from X , if $\Gamma_G(x) \not\subset X$. Assume $\Gamma_G(x) \not\subset X$. Then by applying Theorem C, we have $(\Gamma^{(2)}(x) - X) \cap U_2 \neq \emptyset$. This contradicts the fact that $\Gamma^{(2)}(x) \cap U_2 \subset X$. It follows $\Gamma_G(x) \subset X$.

Next assume that $V(G) \neq \{x\} \cup X$, then $\{u, v\}$ is a cutset of G , which is impossible because G is 3-connected.

Thus G is the graph such that

$$V(G) = \{x\} \cup (\Gamma^{(2)}(x) \cap U_2) \cup \{u, v\} \quad (\text{disjoint})$$

$$E(G) \supset \{xy, yu, yv \mid y \in \Gamma^{(2)}(x) \cap U_2\}$$

and for every $y \in \Gamma^{(2)}(x) \cap U_2$, xy is a non-contractible edge and yu and yv are contractible edges. In this graph, however, for each vertex $y \in \Gamma^{(2)}(x) \cap U_2$, $G - \{x, y\}$ is 2-connected since $|G| \geq 6$. This contradicts the fact that an edge xy is non-contractible. Hence the claim follows.

Now we have

$$V(G) = U_1 \cup U_2 \cup W_2 \cup \{u, v\} \tag{1}$$

Claim 2: Let $x \in W_2$, then $d_G(x)=4$, say $\Gamma_G(x)=\{a,b,u,v\}$, and $\{x,a,b\}$ is a cutset separating u and v .

Let X be $\{u,v\}$ ($=\Gamma^{(1)}(x)$). If there exists a least cutset S such that $x \in S$ and $G-S$ has a connected component disjoint from X , then by Theorem C, $(\Gamma^{(2)}(x)-X) \cap U_2 \neq \emptyset$. This means that $\{x,u,v\}$ is a cutset, which is a contradiction since edges xu and xv are contractible. Therefore, every least cutset S which contains x , say $\{x,a,b\}$, separates u and v . Let A be a connected component of $G-S$ which contains u , and $B=V(G)-S-A$. If there is a vertex $y \in (A-\{u\}) \cap \Gamma_G(x)$, then by (1) $y \in U_1 \cup U_2 \cup W_2$. However y and v cannot be adjacent, so $y \in U_1$. Since an edge xy is non-contractible, we have $d_G(x)=3$ by Theorem A. This is a contradiction since $x \in W_2$. Since $d_G(x) \geq 4$, $\Gamma_G(x)=\{a,b,u,v\}$, and the claim follows.

Now we consider $G-\{u,v\}$. Since G is 3-connected, $G-\{u,v\}$ is connected and $\delta(G-\{u,v\}) \leq 2$ by (1) and Claim 2. Hence $G-\{u,v\}$ is a path or a cycle.

First assume that $G-\{u,v\}$ is a path, say $G-\{u,v\} \simeq P_n$. Since $|G| \geq 6$, $n \geq 4$. Let x be a internal vertex of P_n . Then $\{x,u,v\}$ is a cutset of G . On the other hand, xu or xv is contractible since $x \in U_1 \cup U_2 \cup W_2$. This is a contradiction. Therefore, $G-\{u,v\}$ is a cycle, say $G-\{u,v\} \simeq C_n$. In this case $U_2-\{u,v\} = \emptyset$, and hence $V(G)=U_1 \cup W_2 \cup \{u,v\}$. First we claim $U_1-\{u,v\} = \emptyset$. Assume the contrary and let $x \in U_1-\{u,v\}$. By Theorem A, two neighbors of x in $V(G)-\{u,v\}$ are adjacent. This is impossible since $|G| \geq 6$.

Thus $V(G)=W_2 \cup \{u,v\}$. However, for each $x \in V(G)-\{u,v\}$, $\Gamma^{(2)}(x) \cup \{x\}$ is not a cutset since $|G| \geq 6$. This contradicts Claim 2. This is a final contradiction and the proof is complete. ■

Theorem 3. *Let G be a 3-connected graph and e be an edge of G . Then there exists a non-separating induced cycle which contains e .*

Proof. We prove the theorem by induction on $|G|$. When $|G| \leq 5$, we can easily check the result. Now we can assume that $|G| \geq 6$, and that all 3-connected graph of order less than $|G|$ has a non-separating induced cycle which contains a specified edge.

Assume G has no non-separating induced cycle which contains a specified edge e . Let a and b be the endvertices of e . By Theorem 2, G has a contractible edge, say xy , which is not incident with a or b . Let G' be the graph obtained from G by contraction of xy . By induction hypothesis, G' has a non-separating induced cycle C' which contains e . If the contracted vertex z is not on C' , then C' is also a non-separating induced cycle in G which contains e . Therefore, we can assume that z lies on C' . Let u, v be the vertices adjacent to z on C' and P' be the path obtained by $C' - z$. In G , u and v are adjacent to x or y . Now two cases occur.

Case 1: At least one of $\{x, y\}$ is adjacent to both u and v .

Without loss of generality, we have $\Gamma_G(x) \supset \{u, v\}$. If y is adjacent to the vertex of $V(G) - V(C')$, then $P' \cup \{x\}$ with edges xu and xv is a non-separating cycle in G , a contradiction. Therefore y can be adjacent only to u, v and x , since C' is an induced cycle in G' . This implies $\Gamma_G(y) = \{u, v, x\}$ since the minimum degree of G is at least three. Applying the same argument to y , we have $\Gamma_G(x) = \{u, v, y\}$. Then, degree of z is two in G' , which contradicts the fact that xy is a contractible edge.

Case 2: The vertex x is adjacent to one of u, v and y to the other.

We can assume that x is adjacent to u (and not to v) and y to v (and not to u). Then, $P' \cup \{x, y\}$ with edges ux , xy and yv is a non-separating induced cycle in G .

This completes the proof of the theorem. ■

Corollary 4. *Let G be a 3-connected graph and x and y be vertices of G . Then there exists an induced x,y -path P such that $G-V(P)$ is connected.*

Proof. If x and y are adjacent, then the edge xy is a desired path. Otherwise, in the graph obtained from G by adding an edge xy , there exists a non-separating induced cycle which contains xy , by Theorem 3. This cycle induces a non-separating induced path in G . ■

The above corollary leads us to the following conjecture.

Conjecture. *For a given integer k ($k \geq 1$), there exists a minimum number n_k such that every n_k -connected graph G satisfies the following property (*).*

(*) *For every pair of distinct vertices x,y of G , there exists an induced x,y -path P such that $G-V(P)$ is k -connected.*

Theorem 3 indicates that $n_1=3$.

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