

Hamilton-Jacobi equations and Euclidean Sobolev inequality

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1 Introduction

The result of this note is a special case of [3], and the readers should refer to it for more detailed results and their proofs.

Let Ω be a bounded and Lebesgue measurable set in $\mathbb{R}^n$. Let $0 < \alpha < \beta < \infty$. Then, as is well-known, the following inequality holds:

$$(1.1) \quad |\Omega|^{-1/\alpha} \|f\|_{\alpha, \Omega} \leq |\Omega|^{-1/\beta} \|f\|_{\beta, \Omega} \leq \|f\|_{\infty, \Omega}, \quad f \in L^\infty(\Omega)$$

where $|\Omega|$ is the Lebesgue measure of Ω and $\|\cdot\|_{\beta, \Omega}$ is the $L^\beta(\Omega)$ -norm ($0 < \beta < \infty$) with respect to the Lebesgue measure in $\mathbb{R}^n$. Furthermore, this inequality is optimal in the sense that all inequalities in (1.1) are reduced to equalities when f is a constant function on Ω . This inequality shows a norm-monotone property of $\{|\Omega|^{-1/\beta} \|f\|_{\beta, \Omega}\}_{0 < \beta < \infty}$.

However, as far as we know, there is no inequality corresponding to (1.1) when a bounded and Lebesgue measurable set Ω in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. A reason for it is that when $\Omega = \mathbb{R}^n$, we have $|\Omega|^{-1/\beta} = 0$ for all $0 < \beta < \infty$.

The goal of this note is to provide an inequality corresponding to (1.1) when a bounded and Lebesgue measurable set Ω in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. This inequality is obtained by using the Euclidean logarithmic Sobolev inequality and Hamilton-Jacobi equations. We use the inequalities obtained by [4, 5], and minimize this inequality with respect to some parameter, and finally get the desired inequality by letting another parameter tend to ∞ .

2 Preliminaries

In this section, we collect some results of [4, 5]. For $p \geq 1$, we denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions f on $\mathbb{R}^n$ such that f and $|Df|$ are in $L^p(\mathbb{R}^n)$. Throughout this note, the integral without its domain is understood as the one over $\mathbb{R}^n$.

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Lemma 2.1 *Let $p \geq 1$. Then, we have the following Euclidean logarithmic Sobolev inequality:*

$$(2.1) \quad \int |f|^p \log |f|^p dx \leq \frac{n}{p} \log \left(L_p \int |Df|^p dx \right) \quad \text{for } f \in W^{1,p}(\mathbb{R}^n) \text{ with } \int |f|^p dx = 1.$$

Here,

$$(2.2) \quad L_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left(\frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(n \frac{p-1}{p} + 1\right)} \right)^{p/n},$$

and this is the best possible constant satisfying (2.1).

We denote by $\|\cdot\|_\alpha$ the $L^\alpha(\mathbb{R}^n)$ -norm with respect to the Lebesgue measure in $\mathbb{R}^n$.

Lemma 2.2 *Let $p > 1$. For $f \in \text{Lip}(\mathbb{R}^n)$, let $u \in \text{Lip}(\mathbb{R}^n \times [0, \infty))$ be a viscosity subsolution of the Hamilton-Jacobi equation*

$$(2.3) \quad u_t(x, t) + \frac{1}{p} |Du(x, t)|^p = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad u = f \quad \text{on } \mathbb{R}^n \times \{0\}.$$

If there is a constant $\alpha > 0$ such that $e^f \in L^\alpha(\mathbb{R}^n)$, then $e^{u(\cdot, t)} \in L^\beta(\mathbb{R}^n)$ for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$. Furthermore, we have

$$(2.4) \quad \|e^{u(\cdot, t)}\|_\beta \leq \|e^f\|_\alpha \left(\frac{n L_p e^{p-1} (\beta - \alpha)}{p^p t} \right)^{\frac{n}{p} \frac{\beta - \alpha}{\alpha \beta}} \frac{\alpha^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})}}{\beta^{\frac{n}{\alpha \beta} (\frac{\beta}{p} + \frac{\alpha}{q})}}, \quad t > 0,$$

where $q > 1$ is the exponent conjugate of p , i.e., $(1/p) + (1/q) = 1$.

3 A result

Let $\theta > 0$. For $\alpha > 0$, we set

$$(3.1) \quad \mathcal{L}_{\alpha, \theta} = \left\{ f \in \text{Lip}(\mathbb{R}^n) : \text{Lip}(f) \leq \theta, e^f \in L^\alpha(\mathbb{R}^n) \right\},$$

where $\text{Lip}(f)$ is the Lipschitz constant of f , i.e., $\text{Lip}(f) = \sup_{x \neq y} |f(x) - f(y)|/|x - y|$. Let us denote by ω_{n-1} the surface area of the unit ball in $\mathbb{R}^n$. We set

$$(3.2) \quad k_n = \left(\frac{1}{\omega_{n-1} (n-1)!} \right)^{1/n}.$$

Now, we state our result of this note and give a sketch of its proof.

Theorem 3.1 *Let $\alpha, \theta > 0$. For $f \in \mathcal{L}_{\alpha, \theta}$, we have the following inequality:*

$$(3.3) \quad \|e^f\|_\infty \leq \|e^f\|_\beta (k_n \theta \beta)^{n/\beta} \leq \|e^f\|_\alpha (k_n \theta \alpha)^{n/\alpha}, \quad \alpha \leq \beta \leq \infty.$$

Inequality (3.3) is optimal in the sense that equality holds when $f(x) = C - \theta|x|$ for some constant $C \in \mathbb{R}$.

Remark. Note that $\lim_{\beta \rightarrow \infty} (k_n \theta \beta)^{n/\beta} = 1$. Hence, the family $\{\|e^f\|_\beta (k_n \theta \beta)^{n/\beta}\}_{\alpha < \beta < \infty}$ interpolates continuously and monotonically between $\|e^f\|_\alpha (k_n \theta \alpha)^{n/\alpha}$ and $\|e^f\|_\infty$.

Sketch of Proof. Let $f \in \mathcal{L}_{\alpha, \theta}$. Then, the function $v(x, t) = f(x) - (\theta^p t/p)$ is a subsolution of (2.3), so that $v \leq u$ on $\mathbb{R}^n \times [0, \infty)$ by [7]. By Lemma 2.2, we have, for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$,

$$(3.4) \quad \|e^f\|_\beta \leq \|e^f\|_\alpha e^{\theta^p t/p} t^{-\frac{n}{p} \frac{\beta - \alpha}{\alpha \beta}} \times \left(\frac{n L_p e^{p-1} (\beta - \alpha)}{p^p} \right)^{\frac{n}{p} \frac{\beta - \alpha}{\alpha \beta}} \frac{\alpha^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})}}{\beta^{\frac{n}{\alpha \beta} (\frac{\beta}{p} + \frac{\alpha}{q})}}, \quad t > 0,$$

where $q > 1$ is the exponent conjugate of p , i.e., $(1/p) + (1/q) = 1$. By minimizing the right-hand side of (3.4) with respect to the t -variable, we have

$$(3.5) \quad \begin{aligned} \|e^f\|_\beta &\leq \|e^f\|_\alpha \left(\frac{\theta^p e}{n^{\frac{\beta - \alpha}{\alpha \beta}}} \right)^{\frac{n}{p} \frac{\beta - \alpha}{\alpha \beta}} \times \left(\frac{n L_p e^{p-1} (\beta - \alpha)}{p^p} \right)^{\frac{n}{p} \frac{\beta - \alpha}{\alpha \beta}} \frac{\alpha^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})}}{\beta^{\frac{n}{\alpha \beta} (\frac{\beta}{p} + \frac{\alpha}{q})}} \\ &= \|e^f\|_\alpha \left(\frac{\theta e L_p^{1/p}}{p} \right)^{\frac{n}{\alpha} - \frac{n}{\beta}} \times \alpha^{\frac{n}{\alpha}} \beta^{-\frac{n}{\beta}}. \end{aligned}$$

Hence, we obtain

$$(3.6) \quad \|e^f\|_\beta (k_p^{(n)} \theta \beta)^{n/\beta} \leq \|e^f\|_\alpha (k_p^{(n)} \theta \alpha)^{n/\alpha},$$

where

$$(3.7) \quad \begin{aligned} k_p^{(n)} &= \frac{e L_p^{1/p}}{p} \\ &= \left(\frac{n}{eq} \right)^{1/q} \left[\Gamma \left(\frac{n}{q} + 1 \right) \right]^{-1/n} \frac{e}{n \sqrt{\pi}} \left[\Gamma \left(\frac{n}{2} + 1 \right) \right]^{1/n}. \end{aligned}$$

Now, letting p tend to ∞ in (3.7), i.e., letting q tend to 1 in (3.7), we conclude that

$$\begin{aligned} \lim_{p \rightarrow \infty} k_p^{(n)} &= \lim_{q \rightarrow 1} \left(\frac{n}{eq} \right)^{1/q} \left[\Gamma \left(\frac{n}{q} + 1 \right) \right]^{-1/n} \frac{e}{n \sqrt{\pi}} \left[\Gamma \left(\frac{n}{2} + 1 \right) \right]^{1/n} \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{(n!)^{1/n}} \left[\Gamma \left(\frac{n}{2} + 1 \right) \right]^{1/n} = \left(\frac{1}{\omega_{n-1} (n-1)!} \right)^{1/n} = k_n. \end{aligned}$$

The proof is completed. $\square$

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