

ASYMPTOTIC MULTIPLIER IDEAL SHEAVES ASSOCIATED TO POTENTIAL TRIPLES

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In birational geometry, understanding the intertwined relation between the singularities and positivity given by the divisors is inevitable. For instance, the recent advancement of the birational geometry including the minimal model program and the settlement of the BAB conjecture are undoubtedly the products of the deep results that unveil how singularities and positivity of divisors affect the geometry of the pairs. In this paper, we elaborate on how the local and global geometric properties of divisors can be studied in a common framework that we describe below.

For a pair (X, Δ) (i.e., X is a normal projective variety and Δ is an effective $\mathbb{R}$ -divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier), it is well known that the multiplier ideal sheaf $\mathcal{J}(X, \Delta) \subseteq \mathcal{O}_X$ detects the singularities of the pair (X, Δ) . More precisely, a pair (X, Δ) is klt if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ and if the strict inclusion $\mathcal{J}(X, \Delta) \subsetneq \mathcal{O}_X$ holds, then $\mathcal{J}(X, \Delta)$ precisely defines the non-klt locus $\text{Nklt}(X, \Delta)$ of (X, Δ) . On the other hand, for a smooth projective variety X and a big Cartier divisor D on X , the asymptotic multiplier ideal sheaves $\mathcal{J}(X, \lfloor mD \rfloor)$ ($m \in \mathbb{Z}_{>0}$) detect the nefness of D . More precisely, D is nef if and only if $\mathcal{J}(X, \lfloor mD \rfloor) = \mathcal{O}_X$ for all positive integers m and the union of the closed sets defined by $\mathcal{J}(X, \lfloor mD \rfloor)$ for all $m > 0$ coincides with the non-nef locus $\text{NNef}(D)$ of D . However, it is not known clearly what information a single ideal sheaf $\mathcal{J}(X, \lfloor D \rfloor)$ carries. We first construct and study the ideal sheaf which reflects both the aforementioned local and global properties simultaneously.

We first define the following notion. A *potential triple* (X, Δ, D) consists of:

- a normal projective variety X ,
- an effective divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier,
- a pseudoeffective $\mathbb{R}$ -Cartier divisor D on X .

We often simply call (X, Δ, D) a *triple*. We note that the notion of a triple is more general than that of a generalized pair since D in a triple (X, Δ, D) is only a pseudoeffective $\mathbb{R}$ -divisor, not necessarily a b-nef divisor. (See Remark 8.) We can check that many of the notions defined for pairs and generalized pairs also extend to triples. We also note that the notion of triples has already been treated in many of the previous papers where the pairs (X, Δ) together with graded linear systems are studied in relation with log canonical thresholds and K -stability.

Definition 1.

- (1) The *log discrepancy* $a(E; X, \Delta, D)$ of a triple (X, Δ, D) along a prime divisor E over X is defined as

$$a(E; X, \Delta, D) := A_{X, \Delta}(E) - \sigma_E(D)$$

where $A_{X, \Delta}(E)$ is the log discrepancy of the pair (X, Δ) along E and $\sigma_E(D)$ is the asymptotic divisorial valuation of D along E .

- (2) The triple (X, Δ, D) is said to be *potentially klt* if $\inf_E a(E; X, \Delta, D) > 0$ where $\inf$ is taken over all prime divisors E over X . The triple (X, Δ, D) is said to be *weakly potentially klt* if $a(E; X, \Delta, D) > 0$ for any prime divisor E over X . A triple (X, Δ, D) is said to be *potentially lc* if $\inf_E a(E; X, \Delta, D) \geq 0$. As usual, we write *pklt*, *plc* for potentially klt, potentially lc, respectively.

Let (X, Δ, D) be a triple where D is an effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X . Let p be a positive integer and $f : Y \rightarrow X$ be a resolution of the linear system $|pD|$ for which there exists a decomposition $f^*[pD] = |M_p| + F_p$ into the free part $|M_p|$ and the fixed part F_p with the simple normal crossing support. The family $\{f_*\mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta) + \frac{1}{p}F_p \rfloor) \mid p \in \mathbb{Z}_{>0}\}$ of the ideal sheaves contains the unique maximal element. We denote it by $\mathcal{J}(X, \Delta, \|D\|)$ and call it the asymptotic multiplier ideal sheaf associated to the triple (X, Δ, D) . Note that the ideal sheaves $\mathcal{J}(X, \Delta)$ and $\mathcal{J}(X, \|D\|)$ appear as special cases of $\mathcal{J}(X, \Delta, \|D\|)$ for any semiample divisor D and $\Delta = 0$, respectively. The following is the first main result.

Theorem 2. *Let (X, Δ, D) be a triple with a big $\mathbb{Q}$ -divisor D on X . Then the following are equivalent:*

- (1) *the triple (X, Δ, D) is pklt.*
- (2) *$\mathcal{J}(X, \Delta, \|D\|) = \mathcal{O}_X$.*
- (3) *there exists an effective divisor $D' \sim_{\mathbb{R}} D$ such that $(X, \Delta + D')$ is klt.*

The effective divisor D' in (3) is called a *klt complement* of the triple (X, Δ, D) . The proof of Theorem 2 depends on the results from the valuation theory which is previously developed in the papers [JM] by Jonsson-Mustařa and [Xu] by Xu. This replaces the proof of [CP, Proposition 4.4].

An interesting application of Theorem 2 is the following characterization of Fano type varieties.

Corollary 3. *Let X be a $\mathbb{Q}$ -factorial normal projective variety. Then X is a Fano type variety if and only if there exists an effective $\mathbb{Q}$ -divisor Δ on X such that $-(K_X + \Delta)$ is big and (X, Δ) is pklt.*

Note that Corollary 3 also gives a “boundary free” characterization of Fano type varieties. That is, a variety X is of Fano type if and only if $-K_X$ is big and $(X, \Delta = 0)$ is pklt.

Next we consider the case $\mathcal{J}(X, \Delta, \|D\|) \subsetneq \mathcal{O}_X$. The *potentially non-klt locus* $\text{pNklt}(X, \Delta, D)$ of a triple (X, Δ, D) is defined as the union of the centers $c_X(E)$ of the prime divisors E over X such that $a(E; X, \Delta, D) \leq 0$. It is known that if D is nef, then $\sigma_E(D) = 0$ for all E . Therefore, if D is nef, then $\text{pNklt}(X, \Delta, D)$ coincides with the non-klt locus $\text{Nklt}(X, \Delta)$ of the pair (X, Δ) .

Let $\text{pNklt}(X, \Delta, D)$ be the union of the centers $c_X(E)$ of prime divisors E over X with $a(E; X, \Delta, D) \leq 0$. Then it is easy to check that $\text{pNklt}(X, \Delta, D)$ is contained in $\mathcal{Z}(\mathcal{J}(X, \Delta, \|D\|))$:

$$\text{pNklt}(X, \Delta, D) \subseteq \mathcal{Z}(\mathcal{J}(X, \Delta, \|D\|)).$$

Theorem 4. *Let (X, Δ, D) be a triple with a big $\mathbb{Q}$ -divisor D on X . Suppose that (X, Δ) is klt and D admits a birational Zariski decomposition. Then we have*

$$\text{pNklt}(X, \Delta, D) = \mathcal{Z}(\mathcal{J}(X, \Delta, \|D\|)).$$

As an application of Theorem 2, we can prove that for any pklt triple (X, Δ, D) , we can run the $(K_X + \Delta + D)$ -MMP with scaling. This MMP can be viewed as a triple analog of the existence of MMP for a projective klt pair (X, Δ) (cf. [BCHM, Corollary 1.4.1]).

Theorem 5. *Let (X, Δ, D) be a pklt triple with a pseudoeffective $\mathbb{Q}$ -divisor D . Then we can run the $(K_X + \Delta + D)$ -MMP with scaling of an ample divisor.*

We also prove the Nadel type vanishing theorem for $\mathcal{J}(X, \Delta, \|D\|)$.

Theorem 6. *Let (X, Δ, D) be a triple with effective $\mathbb{Q}$ -divisors Δ and D on X . If L is a Cartier divisor such that $L \sim_{\mathbb{Q}} K_X + \Delta + D$, then*

$$H^q(X, \mathcal{O}_X(L) \otimes \mathcal{J}(X, \Delta, \|D\|)) = 0 \quad \text{for all } q > \dim X - \kappa(D).$$

In this theorem, if (X, Δ, D) is a pklt triple and D is big, then we obtain the following vanishing.

Corollary 7. *Let (X, Δ, D) be a triple with a big $\mathbb{Q}$ -divisor D on X . Suppose that (X, Δ, D) is pklt and L a line bundle on X such that $L \sim_{\mathbb{Q}} K_X + \Delta + D$. Then,*

$$H^q(X, \mathcal{O}_X(L)) = 0$$

for any $q > 0$.

The multiplier ideal sheaves of metrics with minimal singularities are the analytic counterpart of the asymptotic multiplier ideal sheaves. Let X be a smooth projective variety and L a big line bundle on X . For a singular metric $h_{\min}$ of L with minimal singularities and nonnegative curvature current, let $\mathcal{J}(h_{\min})$ be the associated multiplier ideal sheaf. Then we have $\mathcal{J}(h_{\min}) = \mathcal{J}(X, \|L\|)$.

We also note in Remark 8 that the generalized pairs in [BZ] can be naturally treated as potential triples and our results recover some of the results on vanishing of cohomologies for the generalized pairs at least in the generalized klt case or the usual klt case.

Remark 8. We note that the notion of potential triples (X, Δ, D) is more general than that of generalized pairs $(X, B + M)$. This can be seen as follows.

Note first that a generalized pair $(X, B + M)$ with an $\mathbb{R}$ -Cartier divisor M can be naturally considered as a triple (X, B, M) . Furthermore, we claim that if a generalized pair $(X, B + M)$ has klt singularities (ϵ -lc singularities), then the triple (X, B, M) is pklt (ϵ -plc, respectively). Let $f : X' \rightarrow X$ be a log resolution of (X, B) and M' the nef divisor on X' such that $f_*M' = M$. There exists a divisor B' on X' such that $K_{X'} + B' + M' = f^*(K_X + B + M)$. We may assume that a prime divisor E over X is a prime divisor on X' . Note then that by definition, we have $a(E; X, B, M) \geq 1 - \text{mult}_E(B')$ and this proves the claim. However, the converse does not hold in general. First of all, not all potential triples can be considered as generalized pairs. Such examples can be easily found as follows: Let D be a pseudoeffective $\mathbb{R}$ -Cartier divisor having the Zariski decomposition $D = P + N$ with $N \neq 0$. Then (X, Δ, N) is a potential triple by definition. However, (X, Δ, N) is not a generalized pair. Suppose that (X, Δ, N) is a generalized pair. Then $N = f_*(P')$ for some birational morphism $f : Y \rightarrow X$ and a nef divisor P' which is non-zero and effective on the support $\text{Supp } f_*^{-1}N$. Since $f^*D = f^*P + f^*N$ is also the Zariski decomposition and $f^*D = f^*P + P' + E$, we have $f^*P \geq f^*P + P'$. Thus we have $0 \geq P'$ and this is a contradiction.

Example 9. The following example also shows that for a potential triple (X, Δ, D) which can be also viewed as a generalized pair $(X, \Delta + D)$, the notion of pklt for the triple (X, Δ, D) is also strictly more general than the notion of klt for the generalized pair $(X, \Delta + D)$.

Suppose that a generalized pair $(X, \Delta + D)$ defines a potential triple (X, Δ, D) which is pklt. The following example shows that the generalized pair $(X, \Delta + D)$ can have non-klt singularities. Let $\pi : X \rightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at a point $x \in \mathbb{P}^2$, and H a line passing through the point x . Denote by H' the strict transform of H . Then rH' is nef for all positive real numbers r , and it defines a generalized pair $(\mathbb{P}^2, 0 + rH)$. Let E be the exceptional divisor on X . Then we have

$$K_X + (r - 1)E + rH' = \pi^*(K_{\mathbb{P}^2} + rH).$$

Hence, for $r > 2$, the generalized pair $(\mathbb{P}^2, 0 + rH)$ is not lc. However, the potential triple $(\mathbb{P}^2, 0, rH)$ is obviously pklt.

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