

Notes on bilinear lattice bump Fourier multipliers

By

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Abstract

We consider the bilinear Fourier multiplier operator with the multiplier written as a linear combination of a fixed bump function. For those operators we prove two transference theorems, one in amalgam spaces and the other in Wiener amalgam spaces.

§ 1. Introduction

For $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the *bilinear Fourier multiplier operator* T_σ is defined by

$$T_\sigma(f_1, f_2)(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} \sigma(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) d\xi_1 d\xi_2,$$
$$x \in \mathbb{R}^n, \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^n),$$

where $\widehat{f_1}, \widehat{f_2}$ denote the Fourier transforms.

Let X_1, X_2 , and Y be function spaces on $\mathbb{R}^n$ equipped with quasi-norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$, and $\|\cdot\|_Y$, respectively. If there exists a constant $C \in [0, \infty)$ such that

$$\|T_\sigma(f_1, f_2)\|_Y \leq C \|f_1\|_{X_1} \|f_2\|_{X_2}, \quad f_1 \in \mathcal{S} \cap X_1, \quad f_2 \in \mathcal{S} \cap X_2,$$

then we denote the smallest possible C by $\|T_\sigma\|_{X_1 \times X_2 \rightarrow Y}$. If there exists no such finite constant C , then we define $\|T_\sigma\|_{X_1 \times X_2 \rightarrow Y} = \infty$. We shall simply call $\|T_\sigma\|_{X_1 \times X_2 \rightarrow Y}$ the operator norm of T_σ in $X_1 \times X_2 \rightarrow Y$.

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The bilinear Fourier multiplier operator was introduced by Coifman–Meyer [3, 4, 5, 6] and there have been many works. In the present article, we shall be interested in the multiplier of the following special form. For $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$ and $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we define

$$\sigma_{a,\Phi}(\xi_1, \xi_2) = \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) \Phi(\xi_1 - \mu_1, \xi_2 - \mu_2), \quad \xi_1, \xi_2 \in \mathbb{R}^n.$$

For notational convenience, we write the corresponding bilinear operator as

$$T_{a,\Phi} = T_{\sigma_{a,\Phi}}.$$

The multiplier $\sigma_{a,\Phi}$ can be considered as a test case of more general bilinear Fourier multipliers. It was discussed in different kinds of context in several papers. In the papers [14, 15], the authors used the estimates for operators of the form $T_{a,\Phi}$ as key tools to prove boundedness of bilinear singular integrals with rough kernels. A study wholly focusing on $\sigma_{a,\Phi}$ was given recently by Briánková–Grafakos–He–Honzíková [2], where the authors call $\sigma_{a,\Phi}$ the *lattice bump multiplier*. The main result of [2] gives estimate for the operator norm of $T_{a,\Phi}$ in $L^{p_1} \times L^{p_2} \rightarrow L^p$, $1/p = 1/p_1 + 1/p_2$, in terms of $\|a\|_{\ell^\infty}$ and the cardinality of $\text{supp } a$ (see Theorem 1.2 and Remark 1.1 of [2]), which generalize the estimates given in [14, 15]. In [20, 21, 22], the present authors considered bilinear Fourier multipliers σ satisfying the estimates

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \sigma(\xi_1, \xi_2)| \leq C_{\alpha,\beta} W(\xi_1, \xi_2)$$

with a fixed nonnegative function W , and gave some sufficient conditions on W for T_σ to be bounded in L^2 -based amalgam spaces and in Wiener amalgam spaces. The results of [20, 21, 22] imply the estimates for the operator norm of $T_{a,\Phi}$ in terms of the absolute values $|a(\mu_1, \mu_2)|$, which cover the estimate of [2].

In the present article, we shall not consider any particular estimates of the operator norm of $T_{a,\Phi}$ but we shall consider some *transference theorems* for $T_{a,\Phi}$. The transference theorem was first given by de Leeuw [7], who proved that, under certain condition on the multiplier $m(\xi)$ on $\mathbb{R}$, if the Fourier multiplier operator T_m is bounded in $L^p(\mathbb{R})$, $p \in [1, \infty]$, then the periodic Fourier multiplier operators $T_{m(\epsilon \cdot)}^{\text{period}}$, $\epsilon \in (0, \infty)$, are uniformly bounded in $L^p(\mathbb{T})$, where T_m and $T_{m(\epsilon \cdot)}^{\text{period}}$ are defined by

$$T_m f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} m(\xi) \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}, \quad f \in \mathcal{S}(\mathbb{R}),$$

and

$$T_{m(\epsilon \cdot)}^{\text{period}} F(x) = \sum_{\mu \in \mathbb{Z}} e^{2\pi i x \mu} m(\epsilon \mu) \widehat{F}(\mu), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad F \in C^\infty(\mathbb{T})$$

($\widehat{F}(\mu)$ denotes the Fourier coefficient of F). The converse to this theorem was given by Igari [17, Theorem 2] and Stein–Weiss [24, Theorems 3.18 in Chapter VII]. Transference theorems were also given in several different settings; see [23, 18, 25, 1, 8, 19]. Transference theorems for bilinear Fourier multipliers were given by Fan–Sato [9].

The purpose of the present article is to give two transference theorems for the bilinear operators $T_{a,\Phi}$. We shall introduce two other operators associated to $T_{a,\Phi}$. One is the bilinear Fourier multiplier operator T_a^{period} that acts on periodic functions and the other is the bilinear operator S_a that acts on sequence spaces. Under certain conditions on Φ , we shall prove that $T_{a,\Phi}$ is bounded in amalgam spaces if and only if T_a^{period} is bounded in corresponding L^p spaces, and $T_{a,\Phi}$ is bounded in Wiener amalgam spaces if and only if S_a is bounded in corresponding ℓ^q spaces. Precise statements will be given in Theorems 1 and 3.

Most of the techniques used in the present article are in fact well-known in the theory of transference theorems. To put it clearly, our arguments are modifications of those given in [20, 21, 22].

Throughout this article, we use the following notations: $\langle z \rangle = (1 + |z|^2)^{1/2}$ for $z \in \mathbb{R}^n$; $Q = (-1/2, 1/2]^n$ is the unit cube centered at the origin; $KQ = (-K/2, K/2]^n$ for $K \in (0, \infty)$; the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is denoted by $\widehat{f}$ or by $\mathcal{F}f$; the inverse Fourier transform is denoted by $\mathcal{F}^{-1}$; for $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator is defined by

$$m(D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} m(\xi) \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

§ 2. The case of amalgam spaces

In this section, we shall give our first main theorem, which concerns the operator norm of $T_{a,\Phi}$ in amalgam spaces.

We begin with the definition of amalgam spaces. For $p, q \in (0, \infty]$, the *amalgam space* (L^p, ℓ^q) is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(L^p, \ell^q)} = \left\| \left\| \mathbf{1}_Q(x - k) f(x) \right\|_{L_x^p(\mathbb{R}^n)} \right\|_{\ell_k^q(\mathbb{Z}^n)} = \left\{ \sum_{k \in \mathbb{Z}^n} \left(\int_{k+Q} |f(x)|^p dx \right)^{q/p} \right\}^{1/q} < \infty,$$

where the representations of $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\ell^q}$ need the usual modifications if $p = \infty$ or $q = \infty$. For properties of amalgam spaces, see Holland [16] or Fournier–Stewart [12].

For a complex valued L^1 -function F on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, we define its Fourier coefficient by

$$\widehat{F}(\mu) = \int_{\mathbb{T}^n} F(x) e^{-2\pi i \mu \cdot x} dx, \quad \mu \in \mathbb{Z}^n.$$

(Although we use the same notation $\widehat{\cdot}$ to denote both the Fourier coefficient and the Fourier transform, we shall use capital letters to denote functions on $\mathbb{T}^n$, which will help the reader to distinguish the Fourier coefficient from the Fourier transform.) For $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$, we define the operator T_a^{period} by

$$T_a^{\text{period}}(F_1, F_2)(x) = \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} e^{2\pi i x \cdot (\mu_1 + \mu_2)} a(\mu_1, \mu_2) \widehat{F_1}(\mu_1) \widehat{F_2}(\mu_2),$$

$$x \in \mathbb{T}^n, \quad F_1, F_2 \in C^\infty(\mathbb{T}^n).$$

For any $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$, the operator T_a^{period} is a bilinear mapping from $C^\infty(\mathbb{T}^n) \times C^\infty(\mathbb{T}^n)$ to $C^\infty(\mathbb{T}^n)$. For $p_1, p_2, p \in (0, \infty]$, we define

$$\|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} = \sup \left\{ \frac{\|T_a^{\text{period}}(F_1, F_2)\|_{L^p(\mathbb{T}^n)}}{\|F_1\|_{L^{p_1}(\mathbb{T}^n)} \|F_2\|_{L^{p_2}(\mathbb{T}^n)}} \mid F_1, F_2 \in C^\infty(\mathbb{T}^n) \setminus \{0\} \right\}.$$

Finally, to give our theorems, we need some condition that assures the map $a \mapsto \sigma_{a, \Phi}$ to be injective. For this we introduce the following: we say that a function $\Phi \in C_0^\infty(\mathbb{R}^d)$ satisfies the *condition (B)* if there exists a point $\xi^0 \in \mathbb{R}^d$ such that

$$\xi^0 \notin \bigcup_{\mu \in \mathbb{Z}^d \setminus \{0\}} \text{supp } \Phi(\cdot - \mu) \quad \text{and} \quad \Phi(\xi^0) \neq 0.$$

Now the following is the first main theorem of this article.

Theorem 1. *Let $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the condition (B) and let $p_1, p_2, p, q_1, q_2, q \in (0, \infty]$ satisfy $1/q_1 + 1/q_2 \geq 1/q$. Then there exists a constant $c \in (0, \infty)$ depending only on $n, p_1, p_2, p, q_1, q_2, q$, and Φ , such that*

$$c^{-1} \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq \|T_{a, \Phi}\|_{(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)} \leq c \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p}$$

for all $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$.

Before we give the proof of this theorem, we give some remarks.

Remark 2. (1) The amalgam space (L^p, ℓ^q) coincides with the Lebesgue space L^p if $p = q$. Hence the following assertion is a special case of Theorem 1: If $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the condition (B) and if $p_1, p_2, p \in (0, \infty]$ satisfy $1/p_1 + 1/p_2 \geq 1/p$, then there exists a constant $c \in (0, \infty)$ depending only on n, p_1, p_2, p , and Φ , such that

$$c^{-1} \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq \|T_{a, \Phi}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq c \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p}$$

for all $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$, where the spaces L^{p_1}, L^{p_2}, L^p in the quasi-norms of T_a^{period} and $T_{a, \Phi}$ are the spaces on $\mathbb{T}^n$ and on $\mathbb{R}^n$, respectively.

(2) The latter inequality

$$\|T_{a,\Phi}\|_{(L^{p_1},\ell^{q_1})\times(L^{p_2},\ell^{q_2})\rightarrow(L^p,\ell^q)} \leq c\|T_a^{\text{period}}\|_{L^{p_1}\times L^{p_2}\rightarrow L^p}$$

in the conclusion of Theorem 1 holds for all $\Phi \in C_0^\infty(\mathbb{R}^n)$, without the condition (B). This will be seen from the proof to be given below.

(3) The assumption $1/q_1 + 1/q_2 \geq 1/q$ in Theorem 1 gives no essential restriction. In fact, T_σ with a nontrivial $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ has a finite operator norm in $(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)$ only if $1/q_1 + 1/q_2 \geq 1/q$. For a proof of this fact, see Lemma 6 in Appendix.

Now we shall proceed to the proof of Theorem 1. The proof is a modification of the arguments given in [20, 21]. We shall divide the proof into two parts, proof of the latter inequality and proof of the former inequality. In the proofs, $a(\mu_1, \mu_2)$ denotes an arbitrary sequence in $\ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$. We use the letter c to denote positive constants with the same properties as c of the theorem. Notice that c in different places may not be the same constant.

Proof of the latter inequality of Theorem 1. Here we shall prove the inequality

$$(2.1) \quad \|T_{a,\Phi}\|_{(L^{p_1},\ell^{q_1})\times(L^{p_2},\ell^{q_2})\rightarrow(L^p,\ell^q)} \leq c\|T_a^{\text{period}}\|_{L^{p_1}\times L^{p_2}\rightarrow L^p}.$$

Here we don't need the condition (B).

First we follow the methods of Coifman–Meyer [4, 5] to write $T_{a,\Phi}$ as a superposition of simple operators of product forms.

Take a number $K \in (0, \infty)$ that satisfies $\text{supp } \Phi \subset 2^{-1}KQ \times 2^{-1}KQ$ and take a function ϕ such that

$$\phi \in C_0^\infty(\mathbb{R}^n), \quad \phi(\xi) = 1 \text{ on } 2^{-1}KQ, \quad \text{supp } \phi \subset KQ.$$

Since $\text{supp } \Phi \subset 2^{-1}KQ \times 2^{-1}KQ$ we use the Fourier series expansion on $KQ \times KQ$ to write Φ as

$$\Phi(\xi_1, \xi_2) = \sum_{k_1, k_2 \in \mathbb{Z}^n} b(k_1, k_2) e^{2\pi i K^{-1}(\xi_1 \cdot k_1 + \xi_2 \cdot k_2)}, \quad (\xi_1, \xi_2) \in KQ \times KQ,$$

where $\{b(k_1, k_2)\}$ is a rapidly decreasing sequence. Multiplying this by $\phi(\xi_1)\phi(\xi_2)$, we have

$$\begin{aligned} \Phi(\xi_1, \xi_2) &= \sum_{k_1, k_2 \in \mathbb{Z}^n} b(k_1, k_2) e^{2\pi i K^{-1}(\xi_1 \cdot k_1 + \xi_2 \cdot k_2)} \phi(\xi_1) \phi(\xi_2) \\ (2.2) \quad &= \sum_{k_1, k_2 \in \mathbb{Z}^n} b(k_1, k_2) (\phi_{k_1} \otimes \phi_{k_2})(\xi_1, \xi_2), \end{aligned}$$

where

$$\begin{aligned} (\phi_{k_1} \otimes \phi_{k_2})(\xi_1, \xi_2) &= \phi_{k_1}(\xi_1)\phi_{k_2}(\xi_2), \\ \phi_{k_j}(\xi_j) &= e^{2\pi i K^{-1}\xi_j \cdot k_j} \phi(\xi_j), \quad j = 1, 2. \end{aligned}$$

Thus

$$\begin{aligned} \sigma_{a,\Phi}(\xi_1, \xi_2) &= \sum_{k_1, k_2} \sum_{\mu_1, \mu_2} a(\mu_1, \mu_2) b(k_1, k_2) (\phi_{k_1} \otimes \phi_{k_2})(\xi_1 - \mu_1, \xi_2 - \mu_2) \\ (2.3) \quad &= \sum_{k_1, k_2} b(k_1, k_2) \sigma_{a, \phi_{k_1} \otimes \phi_{k_2}}(\xi_1, \xi_2). \end{aligned}$$

Since the sequence $\{b(k_1, k_2)\}$ is rapidly decreasing, in order to prove (2.1) it is sufficient to prove the estimate

$$(2.4) \quad \|T_{a, \phi_{k_1} \otimes \phi_{k_2}}\|_{(L^{p_1, \ell^{q_1}}) \times (L^{p_2, \ell^{q_2}}) \rightarrow (L^p, \ell^q)} \leq c \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p};$$

recall that c should not depend on k_1, k_2 .

Now let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$. To calculate the (L^p, ℓ^q) -quasi-norm of a function, it is convenient to write the variables of $\mathbb{R}^n$ as $x + \rho$ with $x \in Q$ and $\rho \in \mathbb{Z}^n$. Thus let $x \in Q$ and $\rho \in \mathbb{Z}^n$. We have

$$\begin{aligned} &T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)(x + \rho) \\ &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \iint_{\xi_1, \xi_2 \in \mathbb{R}^n} a(\mu_1, \mu_2) e^{2\pi i(x+\rho) \cdot (\xi_1 + \xi_2)} \\ &\quad \times \phi_{k_1}(\xi_1 - \mu_1) \phi_{k_2}(\xi_2 - \mu_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) d\xi_1 d\xi_2 \\ &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \iint_{\xi_1, \xi_2 \in \mathbb{R}^n} a(\mu_1, \mu_2) e^{2\pi i(x+\rho) \cdot (\xi_1 + \mu_1 + \xi_2 + \mu_2)} \\ &\quad \times \phi_{k_1}(\xi_1) \phi_{k_2}(\xi_2) \widehat{f_1}(\xi_1 + \mu_1) \widehat{f_2}(\xi_2 + \mu_2) d\xi_1 d\xi_2 \\ &= (*). \end{aligned}$$

Here notice that $e^{2\pi i \rho \cdot (\mu_1 + \mu_2)} = 1$ since $\rho \cdot (\mu_1 + \mu_2)$ are integers. We write

$$\begin{aligned} e^{2\pi i(x+\rho) \cdot (\xi_1 + \mu_1 + \xi_2 + \mu_2)} &= e^{2\pi i x \cdot (\xi_1 + \xi_2)} e^{2\pi i x \cdot (\mu_1 + \mu_2)} e^{2\pi i \rho \cdot (\xi_1 + \xi_2)} \\ &= e^{2\pi i x \cdot (\mu_1 + \mu_2)} e^{2\pi i \rho \cdot \xi_1} e^{2\pi i \rho \cdot \xi_2} \sum_{\alpha} \frac{1}{\alpha!} (2\pi i)^{|\alpha|} x^\alpha \xi_1^\alpha \sum_{\beta} \frac{1}{\beta!} (2\pi i)^{|\beta|} x^\beta \xi_2^\beta, \end{aligned}$$

where the sums are taken over all multi-indices α and β . Thus

$$\begin{aligned} (*) &= \sum_{\mu_1, \mu_2} \sum_{\alpha, \beta} a(\mu_1, \mu_2) \frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} x^{\alpha+\beta} e^{2\pi i x \cdot (\mu_1 + \mu_2)} \\ &\quad \times \left(\int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_1} \phi_{k_1}(\xi_1) \xi_1^\alpha \widehat{f_1}(\xi_1 + \mu_1) d\xi_1 \right) \left(\int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_2} \phi_{k_2}(\xi_2) \xi_2^\beta \widehat{f_2}(\xi_2 + \mu_2) d\xi_2 \right) \\ &= (**). \end{aligned}$$

We define $F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta} \in C^\infty(\mathbb{T}^n)$ so that their Fourier coefficients are given by

$$\begin{aligned} (F_{k_1, \rho, \alpha})^\wedge(\mu) &= \int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_1} \phi_{k_1}(\xi_1) \xi_1^\alpha \widehat{f_1}(\xi_1 + \mu) d\xi_1, \quad \mu \in \mathbb{Z}^n, \\ (G_{k_2, \rho, \beta})^\wedge(\mu) &= \int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_2} \phi_{k_2}(\xi_2) \xi_2^\beta \widehat{f_2}(\xi_2 + \mu) d\xi_2, \quad \mu \in \mathbb{Z}^n. \end{aligned}$$

Then

$$\begin{aligned} (**) &= \sum_{\mu_1, \mu_2} \sum_{\alpha, \beta} a(\mu_1, \mu_2) \frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} x^{\alpha+\beta} e^{2\pi i x \cdot (\mu_1 + \mu_2)} (F_{k_1, \rho, \alpha})^\wedge(\mu_1) (G_{k_2, \rho, \beta})^\wedge(\mu_2) \\ &= \sum_{\alpha, \beta} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} x^{\alpha+\beta} T_a^{\text{period}}(F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta})(x). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)(x + \rho) \\ &= \sum_{\alpha, \beta} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} x^{\alpha+\beta} T_a^{\text{period}}(F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta})(x), \quad x \in Q, \quad \rho \in \mathbb{Z}^n. \end{aligned}$$

From the last formula, we have

$$\begin{aligned} &\|T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)\|_{(L^p, \ell^q)} = \left\| \|T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)(x + \rho)\|_{L_x^p(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \\ &= \left\| \left\| \sum_{\alpha, \beta} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} x^{\alpha+\beta} T_a^{\text{period}}(F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta})(x) \right\|_{L_x^p(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \\ &\leq \left\{ \sum_{\alpha, \beta} \left(\frac{(2\pi i)^{|\alpha|}}{\alpha!} \frac{(2\pi i)^{|\beta|}}{\beta!} \left\| T_a^{\text{period}}(F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta})(x) \right\|_{L_x^p(Q)} \right) \right\}^{\epsilon} 1/\epsilon \\ &= (***) \end{aligned}$$

with $\epsilon = \min\{p, q, 1\}$. We set $1/q_1 + 1/q_2 = 1/s$. Our assumption implies $1/s \geq 1/q$ and hence the embedding $\ell^s \hookrightarrow \ell^q$ holds. Thus, the definition of $\|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p}$, the embedding $\ell^s \hookrightarrow \ell^q$, and Hölder's inequality with exponents $1/q_1 + 1/q_2 = 1/s$ yield

$$\begin{aligned} &\left\| \|T_a^{\text{period}}(F_{k_1, \rho, \alpha}, G_{k_2, \rho, \beta})(x)\|_{L_x^p(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \\ &\leq \left\| \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \|F_{k_1, \rho, \alpha}\|_{L^{p_1}(Q)} \|G_{k_2, \rho, \beta}\|_{L^{p_2}(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \\ &\leq \left\| \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \|F_{k_1, \rho, \alpha}\|_{L^{p_1}(Q)} \|G_{k_2, \rho, \beta}\|_{L^{p_2}(Q)} \right\|_{\ell_\rho^s(\mathbb{Z}^n)} \\ &\leq \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \left\| \|F_{k_1, \rho, \alpha}\|_{L^{p_1}(Q)} \right\|_{\ell_\rho^{q_1}(\mathbb{Z}^n)} \left\| \|G_{k_2, \rho, \beta}\|_{L^{p_2}(Q)} \right\|_{\ell_\rho^{q_2}(\mathbb{Z}^n)}. \end{aligned}$$

Hence

$$\begin{aligned}
(*) \leq & \left\{ \sum_{\alpha, \beta} \left(\frac{(2\pi)^{|\alpha|}}{\alpha!} \frac{(2\pi)^{|\beta|}}{\beta!} \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \right. \right. \\
& \times \left. \left. \left\| \|F_{k_1, \rho, \alpha}\|_{L^{p_1}(Q)}\|_{\ell_{\rho}^{q_1}(\mathbb{Z}^n)} \left\| \|G_{k_2, \rho, \alpha}\|_{L^{p_2}(Q)}\|_{\ell_{\rho}^{q_2}(\mathbb{Z}^n)} \right\| \right)^{\epsilon} \right\}^{1/\epsilon}.
\end{aligned}$$

Thus, if we prove the estimates

$$(2.5) \quad \left\| \|F_{k_1, \rho, \alpha}\|_{L^{p_1}(Q)}\|_{\ell_{\rho}^{q_1}(\mathbb{Z}^n)} \right\| \leq c \langle \alpha \rangle^N (1+K)^{|\alpha|} \|f_1\|_{(L^{p_1}, \ell^{q_1})},$$

$$(2.6) \quad \left\| \|G_{k_2, \rho, \beta}\|_{L^{p_2}(Q)}\|_{\ell_{\rho}^{q_2}(\mathbb{Z}^n)} \right\| \leq c \langle \beta \rangle^N (1+K)^{|\beta|} \|f_2\|_{(L^{p_2}, \ell^{q_2})}$$

with N depending only on n, p_1, p_2, q_1, q_2 , then we obtain

$$\begin{aligned}
& \|T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)\|_{(L^p, \ell^q)} \\
& \leq c \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \|f_1\|_{(L^{p_1}, \ell^{q_1})} \|f_2\|_{(L^{p_2}, \ell^{q_2})} \\
& \quad \times \left\{ \sum_{\alpha, \beta} \left(\frac{(2\pi)^{|\alpha|}}{\alpha!} \frac{(2\pi)^{|\beta|}}{\beta!} \langle \alpha \rangle^N (1+K)^{|\alpha|} \langle \beta \rangle^N (1+K)^{|\beta|} \right)^{\epsilon} \right\}^{1/\epsilon} \\
& = c \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \|f_1\|_{(L^{p_1}, \ell^{q_1})} \|f_2\|_{(L^{p_2}, \ell^{q_2})},
\end{aligned}$$

which is the desired estimate (2.4).

Thus our task is to prove (2.5) and (2.6). By symmetry, it is sufficient to prove one of them. We shall prove (2.5). Here we use the Poisson summation formula

$$\sum_{\mu \in \mathbb{Z}^n} e^{2\pi i \mu \cdot x} \widehat{f_1}(\xi_1 + \mu) = \sum_{\nu \in \mathbb{Z}^n} e^{-2\pi i \xi_1 \cdot (x + \nu)} f_1(x + \nu)$$

(for this formula, see for example [24, Chapter VII, Section 2] or [13, Section 3.2.3]).

Using this formula, we can write $F_{k_1, \rho, \alpha}(x)$ as

$$\begin{aligned}
F_{k_1, \rho, \alpha}(x) &= \sum_{\mu \in \mathbb{Z}^n} (F_{k_1, \rho, \alpha})^{\wedge}(\mu) e^{2\pi i \mu \cdot x} \\
&= \sum_{\mu \in \mathbb{Z}^n} e^{2\pi i \mu \cdot x} \int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_1} \phi_{k_1}(\xi_1) \xi_1^{\alpha} \widehat{f_1}(\xi_1 + \mu) d\xi_1 \\
&= \sum_{\nu \in \mathbb{Z}^n} \int_{\mathbb{R}^n} e^{2\pi i \rho \cdot \xi_1} \phi_{k_1}(\xi_1) \xi_1^{\alpha} e^{-2\pi i \xi_1 \cdot (x + \nu)} f_1(x + \nu) d\xi_1 \\
&= \sum_{\nu \in \mathbb{Z}^n} \mathcal{F}^{-1}(\phi_{k_1}(\xi_1) \xi_1^{\alpha})(\rho - x - \nu) f_1(x + \nu).
\end{aligned}$$

For $x \in Q$ and for each $N \in \mathbb{N}$, we have

$$\begin{aligned}
& |\mathcal{F}^{-1}(\phi_{k_1}(\xi_1)\xi_1^\alpha)(\rho - x - \nu)| = |\mathcal{F}^{-1}(e^{2\pi i K^{-1}k_1 \cdot \xi_1} \phi(\xi_1)\xi_1^\alpha)(\rho - x - \nu)| \\
& = |\mathcal{F}^{-1}(\phi(\xi_1)\xi_1^\alpha)(K^{-1}k_1 + \rho - x - \nu)| \\
& \leq c_{n,N} \sup_{|\gamma| \leq N} \|\partial_{\xi_1}^\gamma (\phi(\xi_1)\xi_1^\alpha)\|_{L_{\xi_1}^1} \langle K^{-1}k_1 + \rho - x - \nu \rangle^{-N} \\
& \leq c_{n,N,\phi} \langle \alpha \rangle^N (1+K)^{|\alpha|} \langle K^{-1}k_1 + \rho - \nu \rangle^{-N}
\end{aligned}$$

and hence

$$\begin{aligned}
|F_{k_1,\rho,\alpha}(x)| & \leq c_{n,N,\phi} \langle \alpha \rangle^N (1+K)^{|\alpha|} \sum_{\nu \in \mathbb{Z}^n} \langle K^{-1}k_1 + \rho - \nu \rangle^{-N} |f_1(x + \nu)| \\
& = c_{n,N,\phi} \langle \alpha \rangle^N (1+K)^{|\alpha|} \sum_{\sigma \in \mathbb{Z}^n} \langle K^{-1}k_1 + \sigma \rangle^{-N} |f_1(x + \rho - \sigma)|.
\end{aligned}$$

Set $\epsilon_1 = \min\{1, p_1, q_1\}$ and choose $N \in \mathbb{N}$ so that $\epsilon_1 N > n$. Then

$$\begin{aligned}
& \left\| \|F_{k_1,\rho,\alpha}(x)\|_{L_x^{p_1}(Q)} \right\|_{\ell_\rho^{q_1}(\mathbb{Z}^n)} \\
& \leq c \langle \alpha \rangle^N (1+K)^{|\alpha|} \left\| \left\| \sum_{\sigma \in \mathbb{Z}^n} \langle K^{-1}k_1 + \sigma \rangle^{-N} |f_1(x + \rho - \sigma)| \right\|_{L_x^{p_1}(Q)} \right\|_{\ell_\rho^{q_1}(\mathbb{Z}^n)} \\
& \leq c \langle \alpha \rangle^N (1+K)^{|\alpha|} \\
& \quad \times \left(\sum_{\sigma \in \mathbb{Z}^n} \langle K^{-1}k_1 + \sigma \rangle^{-\epsilon_1 N} \left\| \|f_1(x + \rho - \sigma)\|_{L_x^{p_1}(Q)} \right\|_{\ell_\rho^{q_1}(\mathbb{Z}^n)}^{\epsilon_1} \right)^{1/\epsilon_1} \\
& = c \langle \alpha \rangle^N (1+K)^{|\alpha|} \left(\sum_{\sigma \in \mathbb{Z}^n} \langle K^{-1}k_1 + \sigma \rangle^{-\epsilon_1 N} \right)^{1/\epsilon_1} \|f_1\|_{(L^{p_1}, \ell^{q_1})} \\
& \leq c \langle \alpha \rangle^N (1+K)^{|\alpha|} \|f_1\|_{(L^{p_1}, \ell^{q_1})(\mathbb{R}^n)},
\end{aligned}$$

which implies (2.5). Now the latter inequality of Theorem 1 is proved. $\square$

Next, we prove the former inequality of Theorem 1.

Proof of the former inequality of Theorem 1. Here we shall prove the inequality

$$(2.7) \quad \|T_a^{\text{period}}\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq c \|T_{a,\Phi}\|_{(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)}.$$

From the assumption that Φ satisfies the condition (B), there exist a point $\xi^0 \in \mathbb{R}^{2n}$ and a sufficiently small $\epsilon > 0$ such that

$$(2.8) \quad \Phi(\xi^0) \neq 0,$$

$$(2.9) \quad |\xi - \xi^0| < 2\epsilon, \mu \in \mathbb{Z}^{2n}, \mu \neq 0 \Rightarrow \Phi(\xi - \mu) = 0.$$

We write $\xi^0 = (\xi_1^0, \xi_2^0)$. We then take functions $\theta_1, \theta_2 \in C_0^\infty(\mathbb{R}^n)$ such that

$$(2.10) \quad \text{supp } \theta_j \subset \{\xi \in \mathbb{R}^n \mid |\xi - \xi_j^0| < \epsilon\},$$

$$(2.11) \quad \left| \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} \Phi(\xi_1, \xi_2) \theta_1(\xi_1) \theta_2(\xi_2) d\xi_1 d\xi_2 \right| \geq 1 \text{ for all } x \in Q.$$

Hereafter we write

$$(2.12) \quad g(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} \Phi(\xi_1, \xi_2) \theta_1(\xi_1) \theta_2(\xi_2) d\xi_1 d\xi_2.$$

Take arbitrary $F_1, F_2 \in C^\infty(\mathbb{T}^n)$. We define $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ so that their Fourier transforms are given by

$$\widehat{f_j}(\xi) = \sum_{\nu \in \mathbb{Z}^n} \widehat{F_j}(\nu) \theta_j(\xi - \nu), \quad \xi \in \mathbb{R}^n, \quad j = 1, 2,$$

or equivalently by

$$f_j(x) = \sum_{\nu \in \mathbb{Z}^n} \widehat{F_j}(\nu) e^{2\pi i \nu \cdot x} (\mathcal{F}^{-1} \theta_j)(x) = F_j(x) (\mathcal{F}^{-1} \theta_j)(x), \quad j = 1, 2$$

(recall that $\widehat{F_j}$ denotes the Fourier coefficient of F_j). Then, since F_j is a periodic function and $\mathcal{F}^{-1} \theta_j$ is a function in $\mathcal{S}$, we have

$$(2.13) \quad \|f_j\|_{(L^{p_j}, \ell^{q_j})(\mathbb{R}^n)} \leq c \|F_j\|_{L^{p_j}(\mathbb{T}^n)}, \quad j = 1, 2.$$

On the other hand, from (2.9) and (2.10), we have

$$\begin{aligned} & \sigma_{a, \Phi}(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \\ &= \left(\sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) \Phi(\xi_1 - \mu_1, \xi_2 - \mu_2) \right) \\ & \quad \times \left(\sum_{\nu_1 \in \mathbb{Z}^n} \widehat{F_1}(\nu_1) \theta_1(\xi_1 - \nu_1) \right) \left(\sum_{\nu_2 \in \mathbb{Z}^n} \widehat{F_2}(\nu_2) \theta_2(\xi_2 - \nu_2) \right) \\ &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) \widehat{F_1}(\mu_1) \widehat{F_2}(\mu_2) \Phi(\xi_1 - \mu_1, \xi_2 - \mu_2) \theta_1(\xi_1 - \mu_1) \theta_2(\xi_2 - \mu_2) \end{aligned}$$

and thus

$$\begin{aligned} T_{a, \Phi}(f_1, f_2)(x) &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) \widehat{F_1}(\mu_1) \widehat{F_2}(\mu_2) \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} \\ & \quad \times \Phi(\xi_1 - \mu_1, \xi_2 - \mu_2) \theta_1(\xi_1 - \mu_1) \theta_2(\xi_2 - \mu_2) d\xi_1 d\xi_2 \\ (2.14) \quad &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) \widehat{F_1}(\mu_1) \widehat{F_2}(\mu_2) e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x) \\ &= T_a^{\text{period}}(F_1, F_2)(x) g(x). \end{aligned}$$

From this and (2.11)-(2.12), we have

$$(2.15) \quad |T_{a,\Phi}(f_1, f_2)(x)| \geq |T_a^{\text{period}}(F_1, F_2)(x)|, \quad x \in Q.$$

Now from (2.15) and (2.13), we obtain

$$\begin{aligned} \|T_a^{\text{period}}(F_1, F_2)\|_{L^p(Q)} &\leq \|T_{a,\Phi}(f_1, f_2)\|_{L^p(Q)} \leq \|T_{a,\Phi}(f_1, f_2)\|_{(L^p, \ell^q)(\mathbb{R}^n)} \\ &\leq \|T_{a,\Phi}\|_{(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)} \|f_1\|_{(L^{p_1}, \ell^{q_1})(\mathbb{R}^n)} \|f_2\|_{(L^{p_2}, \ell^{q_2})(\mathbb{R}^n)} \\ &\leq c \|T_{a,\Phi}\|_{(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)} \|F_1\|_{L^{p_1}(\mathbb{T}^n)} \|F_2\|_{L^{p_2}(\mathbb{T}^n)}, \end{aligned}$$

which implies (2.7). Now the former inequality of Theorem 1 is proved and proof of Theorem 1 is complete. $\square$

§ 3. The case of Wiener amalgam spaces

In this section, we shall give our second main theorem, which concerns the operator norm of $T_{a,\Phi}$ in Wiener amalgam spaces.

We begin with the definition of Wiener amalgam spaces. Let $\kappa \in C_0^\infty(\mathbb{R}^n)$ be a function satisfying

$$\left| \sum_{k \in \mathbb{Z}^n} \kappa(\xi - k) \right| \geq 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Then for $p, q \in (0, \infty]$, the *Wiener amalgam space* $W^{p,q} = W^{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{W^{p,q}} = \left\| \left\| \kappa(D - k)f(x) \right\|_{\ell_k^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)} < \infty.$$

It is known that the definition of Wiener amalgam space does not depend on the choice of the function κ up to the equivalence of quasi-norm. It is also known that the embedding $W^{p_1, q_1} \hookrightarrow W^{p_2, q_2}$ holds if $0 < p_1 \leq p_2 \leq \infty$ and $0 < q_1 \leq q_2 \leq \infty$. For these facts, see Feichtinger [10, 11], and Triebel [26].

We write $X(\mathbb{Z}^n)$ to denote the set of all functions $b : \mathbb{Z}^n \rightarrow \mathbb{C}$ such that $b(\mu) = 0$ except for a finite number of $\mu \in \mathbb{Z}^n$. For $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$, we define the bilinear map $S_a : X(\mathbb{Z}^n) \times X(\mathbb{Z}^n) \rightarrow X(\mathbb{Z}^n)$ by

$$\begin{aligned} S_a(b_1, b_2)(\mu) &= \sum_{\mu_1 + \mu_2 = \mu} a(\mu_1, \mu_2) b_1(\mu_1) b_2(\mu_2), \\ \mu &\in \mathbb{Z}^n, \quad b_1, b_2 \in X(\mathbb{Z}^n). \end{aligned}$$

For $q_1, q_2, q \in (0, \infty]$, we define

$$\|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q} = \sup \left\{ \frac{\|S_a(b_1, b_2)\|_{\ell^q(\mathbb{Z}^n)}}{\|b_1\|_{\ell^{q_1}(\mathbb{Z}^n)} \|b_2\|_{\ell^{q_2}(\mathbb{Z}^n)}} \mid b_1, b_2 \in X(\mathbb{Z}^n) \setminus \{0\} \right\}.$$

The following is the second main theorem of this article.

Theorem 3. *Let $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the condition (B) and let $p_1, p_2, p, q_1, q_2, q \in (0, \infty]$ satisfy $1/p_1 + 1/p_2 \geq 1/p$. Then there exists a constant $c \in (0, \infty)$ depending only on $n, p_1, p_2, p, q_1, q_2, q$, and Φ , such that*

$$c^{-1} \|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q} \leq \|T_{a, \Phi}\|_{W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}} \leq c \|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q}$$

for all $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$.

Before we give the proof of this theorem, we give some remarks.

Remark 4. (1) The latter inequality

$$\|T_{a, \Phi}\|_{W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}} \leq c \|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q}$$

in the conclusion of Theorem 3 holds for all $\Phi \in C_0^\infty(\mathbb{R}^n)$, without the condition (B). This will be seen from the proof to be given below.

(2) The assumption $1/p_1 + 1/p_2 \geq 1/p$ in Theorem 3 gives no essential restriction. T_σ with a nontrivial $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ has a finite operator norm in $W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}$ only if $1/p_1 + 1/p_2 \geq 1/p$. For a proof of this fact, see Lemma 7 in Appendix.

In the proof of the latter inequality of Theorem 3, we use the following lemma.

Lemma 5. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and let $g_\mu \in \mathcal{S}'(\mathbb{R}^n)$, $\mu \in \mathbb{Z}^n$. Suppose the Fourier transform of each g_μ has a compact support and suppose there exists a number $K \in (0, \infty)$ such that $\text{diam}(\text{supp } \varphi) \leq K$ and $\text{diam}(\text{supp } \widehat{g_\mu}) \leq K$ for all $\mu \in \mathbb{Z}^n$. Then for each $p, q \in (0, \infty]$ there exists a constant c depending only on n, p, q, K , and φ such that*

$$\left\| \left\| \varphi(D - \mu)g_\mu(x) \right\|_{\ell_\mu^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \leq c \left\| \left\| g_\mu(x) \right\|_{\ell_\mu^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)}.$$

Proof. We use the following two well known facts. Firstly, if the Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ has a compact support and if R is a positive real number satisfying $\text{diam}(\text{supp } \widehat{f}) \leq R$ then for each r, s satisfying $0 < r \leq s \leq \infty$, there exists a constant c depending only on r, s , and n such that

$$(3.1) \quad R^{n/s} \|f\|_{L^s(\mathbb{R}^n)} \leq c R^{n/r} \|f\|_{L^r(\mathbb{R}^n)}.$$

For a proof of this inequality, see for example [27, Proposition 1.3.2]. Secondly,

$$(3.2) \quad \left\| \|f(x, y)\|_{L_x^p} \right\|_{L_y^q} \leq \left\| \|f(x, y)\|_{L_y^q} \right\|_{L_x^p} \quad \text{if } 0 < p \leq q \leq \infty,$$

which holds for all L^p and L^q quasi-norms defined on any σ -finite measure spaces. The inequality (3.2) can be easily proved by the use of Minkowski's inequality for integrals.

Now let φ and g_μ be as in Lemma 5. We write

$$\varphi(D - \mu)g_\mu(x) = \int e^{2\pi i \mu \cdot y} (\mathcal{F}^{-1}\varphi)(y) g_\mu(x - y) dy.$$

From our assumption, the Fourier transform of the function $y \mapsto (\mathcal{F}^{-1}\varphi)(y)g_\mu(x - y)$ has a compact support of diameter not exceeding $2K$. Thus by (3.1) we have

$$\begin{aligned} |\varphi(D - \mu)g_\mu(x)| &= \left| \int_{\mathbb{R}^n} e^{2\pi i \mu \cdot y} (\mathcal{F}^{-1}\varphi)(y) g_\mu(x - y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\varphi)(y)g_\mu(x - y)| dy \leq c_{n,\epsilon,K} \|(\mathcal{F}^{-1}\varphi)(y)g_\mu(x - y)\|_{L_y^\epsilon} \end{aligned}$$

for any ϵ satisfying $0 < \epsilon \leq 1$. Taking ϵ so that $\epsilon \leq \min\{1, p, q\}$, we use (3.2) to obtain

$$\begin{aligned} &\left\| \|\varphi(D - \mu)g_\mu(x)\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &\leq c_{n,\epsilon,K} \left\| \left\| (\mathcal{F}^{-1}\varphi)(y)g_\mu(x - y) \right\|_{L_y^\epsilon} \right\|_{\ell_\mu^q} \Big\|_{L_x^p} \\ &\leq c_{n,\epsilon,K} \left\| \left\| (\mathcal{F}^{-1}\varphi)(y)g_\mu(x - y) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \Big\|_{L_y^\epsilon} = c_{n,\epsilon,K} \|\mathcal{F}^{-1}\varphi\|_{L^\epsilon} \left\| \|g_\mu(x)\|_{\ell_\mu^q} \right\|_{L_x^p}. \end{aligned}$$

Lemma 5 is proved. $\square$

Now we shall prove Theorem 3. The proof is a modification of the argument given in [22]. We shall divide the proof into two parts, proof of the latter inequality and proof of the former inequality. In the proofs, we assume $a \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n)$. For nonnegative quantities A and B , we write $A \lesssim B$ if there exists a constant c with the same properties as the constant c of Theorem 3. Also we write $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

Proof of the latter inequality of Theorem 3. Here we shall prove the inequality

$$(3.3) \quad \|T_{a,\Phi}\|_{W^{p_1,q_1} \times W^{p_2,q_2} \rightarrow W^{p,q}} \leq c \|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q}.$$

Here we don't need the condition (B). By virtue of the embedding $W^{\tilde{p},q} \hookrightarrow W^{p,q}$, $\tilde{p} \leq p$, it is sufficient to show it in the case $1/p_1 + 1/p_2 = 1/p$.

Take K and ϕ in the same way as in Proof of the latter inequality of Theorem 1. In the present case, we take ϕ so that it satisfies the additional condition

$$\sum_{m \in \mathbb{Z}^n} \phi(\xi - m) \geq 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Then we have

$$\|f\|_{W^{r,s}} \approx \left\| \|\phi(D - \mu)f(x)\|_{\ell_\mu^s} \right\|_{L_x^r}$$

for each $r, s \in (0, \infty]$.

We use the same representations as in Proof of the latter inequality of Theorem 1:

$$\begin{aligned}\Phi(\xi_1, \xi_2) &= \sum_{k_1, k_2 \in \mathbb{Z}^n} b(k_1, k_2) \phi_{k_1} \otimes \phi_{k_2}(\xi_1, \xi_2), \\ \sigma_{a, \Phi}(\xi_1, \xi_2) &= \sum_{k_1, k_2 \in \mathbb{Z}^n} b(k_1, k_2) \sigma_{a, \phi_{k_1} \otimes \phi_{k_2}}(\xi_1, \xi_2)\end{aligned}$$

(see (2.2) and (2.3)). Recall that $\{b(k_1, k_2)\}$ is a rapidly decreasing sequence. Hence in order to prove (3.3) it is sufficient to prove the estimate

$$(3.4) \quad \|T_{a, \phi_{k_1} \otimes \phi_{k_2}}\|_{W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}} \leq c \|S_a\|_{\ell^{q_1} \times \ell^{q_2} \rightarrow \ell^q}$$

(with c independent of k_1, k_2).

Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}(3.5) \quad T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)(x) &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} a(\mu_1, \mu_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} \\ &\quad \times \phi_{k_1}(\xi_1 - \mu_1) \phi_{k_2}(\xi_2 - \mu_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) d\xi_1 d\xi_2 \\ &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) g_{\mu_1, k_1}^1(x) g_{\mu_2, k_2}^2(x),\end{aligned}$$

where

$$g_{\mu_j, k_j}^j(x) = \phi_{k_j}(D - \mu_j) f_j(x) = e^{-2\pi i K^{-1} k_j \cdot \mu_j} \phi(D - \mu_j) f_j(x + K^{-1} k_j), \quad j = 1, 2.$$

Notice that g_{μ_j, k_j}^j , $j = 1, 2$, satisfy

$$(3.6) \quad \text{supp } \mathcal{F}(g_{\mu_j, k_j}^j) \subset \{\zeta \mid |\zeta - \mu_j| \lesssim 1\},$$

$$(3.7) \quad \left\| \|g_{\mu_j, k_j}^j(x)\|_{\ell_{\mu_j}^{q_j}} \right\|_{L_x^{p_j}} = \left\| \|\phi(D - \mu_j) f_j(x + K^{-1} k_j)\|_{\ell_{\mu_j}^{q_j}} \right\|_{L_x^{p_j}} \approx \|f_j\|_{W^{p_j, q_j}};$$

notice that the quantities in (3.7) do not depend on k_1, k_2 .

From (3.6), it follows that

$$(3.8) \quad \text{supp } \mathcal{F}(g_{\mu_1, k_1}^1 g_{\mu_2, k_2}^2) \subset \{\zeta \mid |\zeta - \mu_1 - \mu_2| \lesssim 1\}.$$

Let κ be the function used in the definition of the quasi-norm of Wiener amalgam spaces. Then, since κ has a compact support, we see that $\kappa(D - \mu)(g_{\mu_1, k_1}^1 g_{\mu_2, k_2}^2) \neq 0$ only if $|\mu_1 + \mu_2 - \mu| \lesssim 1$. This fact and (3.5) yield

$$(3.9) \quad \kappa(D - \mu)(T_{a, \phi_{k_1} \otimes \phi_{k_2}}(f_1, f_2)) = \kappa(D - \mu) h_{\mu, k_1, k_2}$$

with

$$h_{\mu, k_1, k_2} = \sum_{|\tau| \lesssim 1} \sum_{\mu_1 + \mu_2 = \mu + \tau} a(\mu_1, \mu_2) g_{\mu_1, k_1}^1 g_{\mu_2, k_2}^2.$$

By (3.8), the Fourier transform of h_{μ,k_1,k_2} has a compact support of diameter $\lesssim 1$. Hence (3.9) and Lemma 5 imply

$$\begin{aligned} \|T_{a,\phi_{k_1}\otimes\phi_{k_2}}(f_1, f_2)\|_{W^{p,q}} &= \left\| \left\| \kappa(D - \mu)(T_{a,\phi_{k_1}\otimes\phi_{k_2}}(f_1, f_2))(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &= \left\| \left\| \kappa(D - \mu)h_{\mu,k_1,k_2}(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \lesssim \left\| \left\| h_{\mu,k_1,k_2}(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p}. \end{aligned}$$

Using the definition of $\|S_a\|_{\ell^{q_1}\times\ell^{q_2}\rightarrow\ell^q}$ and Hölder's inequality with exponents $1/p_1 + 1/p_2 = 1/p$, we obtain

$$\begin{aligned} &\left\| \left\| h_{\mu,k_1,k_2}(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &= \left\| \left\| \sum_{|\tau|\lesssim 1} \sum_{\mu_1+\mu_2=\mu+\tau} a(\mu_1, \mu_2) g_{\mu_1,k_1}^1(x) g_{\mu_2,k_2}^2(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &\lesssim \left\| \left\| \sum_{\mu_1+\mu_2=\mu} a(\mu_1, \mu_2) g_{\mu_1,k_1}^1(x) g_{\mu_2,k_2}^2(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &\leq \left\| \|S_a\|_{\ell^{q_1}\times\ell^{q_2}\rightarrow\ell^q} \|g_{\mu_1,k_1}^1(x)\|_{\ell_{\mu_1}^{q_1}} \|g_{\mu_2,k_2}^2(x)\|_{\ell_{\mu_2}^{q_2}} \right\|_{L_x^p} \\ &\leq \|S_a\|_{\ell^{q_1}\times\ell^{q_2}\rightarrow\ell^q} \left\| \|g_{\mu_1,k_1}^1(x)\|_{\ell_{\mu_1}^{q_1}} \right\|_{L_x^{p_1}} \left\| \|g_{\mu_2,k_2}^2(x)\|_{\ell_{\mu_2}^{q_2}} \right\|_{L_x^{p_2}}. \end{aligned}$$

Now combining the above inequalities with (3.7), we obtain

$$\|T_{a,\phi_{k_1}\otimes\phi_{k_2}}(f_1, f_2)\|_{W^{p,q}} \lesssim \|S_a\|_{\ell^{q_1}\times\ell^{q_2}\rightarrow\ell^q} \|f_1\|_{W^{p_1,q_1}} \|f_2\|_{W^{p_2,q_2}},$$

which implies (3.4). Thus the latter inequality of Theorem 3 is proved. $\square$

Next, we shall prove the former inequality of Theorem 3.

Proof of the former inequality of Theorem 3. Here we shall prove the inequality

$$(3.10) \quad \|S_a\|_{\ell^{q_1}\times\ell^{q_2}\rightarrow\ell^q} \leq c \|T_{a,\Phi}\|_{W^{p_1,q_1}\times W^{p_2,q_2}\rightarrow W^{p,q}}.$$

Since Φ satisfies the condition (B), by the same reason as in Proof of the former inequality of Theorem 1, we can take $\xi^0 = (\xi_1^0, \xi_2^0) \in \mathbb{R}^{2n}$ and functions $\theta_1, \theta_2 \in C_0^\infty(\mathbb{R}^n)$ that satisfy (2.8), (2.9), (2.10), and (2.11).

We take a function $\kappa \in C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{\mu \in \mathbb{Z}^n} \kappa(\xi - \mu) = 1 \quad \text{for all } \xi \in \mathbb{R}^n$$

and

$$(3.11) \quad |\xi| < 2\epsilon \Rightarrow \kappa(\xi - \mu) = \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

where ϵ is the number in (2.9). Such a κ certainly exists if ϵ is chosen sufficiently small.

Now let $b_1, b_2 \in X(\mathbb{Z}^n)$. We define $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ through Fourier transform by

$$\widehat{f_j}(\xi) = \sum_{\nu \in \mathbb{Z}^n} b_j(\nu) \theta_j(\xi - \nu), \quad \xi \in \mathbb{R}^n.$$

From (2.10) and (3.11), we have

$$\kappa(\xi - \xi_j^0 - \mu) \widehat{f_j}(\xi) = \kappa(\xi - \xi_j^0 - \mu) \sum_{\nu \in \mathbb{Z}^n} b_j(\nu) \theta_j(\xi - \nu) = b_j(\mu) \theta_j(\xi - \mu)$$

and hence

$$\kappa(D - \xi_j^0 - \mu) f_j(x) = b_j(\mu) e^{2\pi i \mu \cdot x} \mathcal{F}^{-1} \theta_j(x).$$

Thus

$$\begin{aligned} (3.12) \quad \|f_j\|_{W^{p_j, q_j}} &\approx \left\| \left\| \kappa(D - \xi_j^0 - \mu) f_j(x) \right\|_{\ell_\mu^{q_j}} \right\|_{L_x^{p_j}} \\ &= \left\| \left\| b_j(\mu) e^{2\pi i \mu \cdot x} \mathcal{F}^{-1} \theta_j(x) \right\|_{\ell_\mu^{q_j}} \right\|_{L_x^{p_j}} = \|b_j\|_{\ell^{q_j}} \|\mathcal{F}^{-1} \theta_j(x)\|_{L_x^{p_j}} = c \|b_j\|_{\ell^{q_j}}. \end{aligned}$$

On the other hand, just in the same way as we obtained (2.14) in Proof of the former inequality of Theorem 1, we obtain

$$\begin{aligned} (3.13) \quad T_{a, \Phi}(f_1, f_2)(x) &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} \sum_{\nu_1, \nu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) b_1(\nu_1) b_2(\nu_2) \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} \\ &\quad \times \Phi(\xi_1 - \mu_1, \xi_2 - \mu_2) \theta_1(\xi_1 - \nu_1) \theta_2(\xi_2 - \nu_2) d\xi_1 d\xi_2 \\ &= \sum_{\mu_1, \mu_2 \in \mathbb{Z}^n} a(\mu_1, \mu_2) b_1(\mu_1) b_2(\mu_2) e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x) \end{aligned}$$

with $g(x)$ given by (2.12). Observe that

$$\begin{aligned} \text{supp } \mathcal{F}(e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x)) &\subset \{ \xi_1 + \xi_2 + \mu_1 + \mu_2 \mid \xi_1 \in \text{supp } \theta_1, \xi_2 \in \text{supp } \theta_2, \} \\ &\subset \{ \xi_1^0 + \xi_2^0 + \zeta + \mu_1 + \mu_2 \mid \zeta \in \mathbb{R}^n, |\zeta| < 2\epsilon \}. \end{aligned}$$

Hence our choice of κ (see (3.11)) implies that

$$\begin{aligned} \mu = \mu_1 + \mu_2 &\Rightarrow \kappa(\xi - \xi_1^0 - \xi_2^0 - \mu) = 1 \text{ on } \text{supp } \mathcal{F}(e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x)), \\ \mu \neq \mu_1 + \mu_2 &\Rightarrow \kappa(\xi - \xi_1^0 - \xi_2^0 - \mu) = 0 \text{ on } \text{supp } \mathcal{F}(e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x)), \end{aligned}$$

and hence

$$\kappa(D_x - \xi_1^0 - \xi_2^0 - \mu) (e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x)) = \begin{cases} e^{2\pi i x \cdot (\mu_1 + \mu_2)} g(x) & \text{if } \mu_1 + \mu_2 = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

This relation and (3.13) imply

$$\begin{aligned} & \kappa(D_x - \xi_1^0 - \xi_2^0 - \mu)T_{a,\Phi}(f_1, f_2)(x) \\ &= \sum_{\mu_1 + \mu_2 = \mu} a(\mu_1, \mu_2)b_1(\mu_1)b_2(\mu_2) e^{2\pi i x \cdot \mu} g(x). \end{aligned}$$

Recall that $|g(x)| \geq 1$ on Q (see (2.11)). Hence

$$\begin{aligned} & \|T_{a,\Phi}(f_1, f_2)\|_{W^{p,q}} \\ & \approx \left\| \left\| \kappa(D_x - \xi_1^0 - \xi_2^0 - \mu)T_{a,\Phi}(f_1, f_2)(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &= \left\| \left\| \sum_{\mu_1 + \mu_2 = \mu} a(\mu_1, \mu_2)b_1(\mu_1)b_2(\mu_2) e^{2\pi i x \cdot \mu} g(x) \right\|_{\ell_\mu^q} \right\|_{L_x^p} \\ &= \left\| \sum_{\mu_1 + \mu_2 = \mu} a(\mu_1, \mu_2)b_1(\mu_1)b_2(\mu_2) \right\|_{\ell_\mu^q} \|g\|_{L^p} \\ &\geq \left\| \sum_{\mu_1 + \mu_2 = \mu} a(\mu_1, \mu_2)b_1(\mu_1)b_2(\mu_2) \right\|_{\ell_\mu^q} = \|S_a(b_1, b_2)\|_{\ell^q}. \end{aligned}$$

Combining the above inequalities with (3.12), we obtain

$$\begin{aligned} & \|S_a(b_1, b_2)\|_{\ell^q} \lesssim \|T_{a,\Phi}(f_1, f_2)\|_{W^{p,q}} \\ & \leq \|T_{a,\Phi}\|_{W^{p_1,q_1} \times W^{p_2,q_2} \rightarrow W^{p,q}} \|f_1\|_{W^{p_1,q_1}} \|f_2\|_{W^{p_2,q_2}} \\ & \lesssim \|T_{a,\Phi}\|_{W^{p_1,q_1} \times W^{p_2,q_2} \rightarrow W^{p,q}} \|b_1\|_{\ell^{q_1}} \|b_2\|_{\ell^{q_2}}, \end{aligned}$$

which implies (3.10). Now the former inequality of Theorem 3 is proved and hence the proof of Theorem 3 is complete. $\square$

§ 4. Appendix

Here we give proofs of the facts mentioned in Remark 2 (3) and Remark 4 (2).

Lemma 6. *Let $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\sigma \neq 0$, $p_1, p_2, p, q_1, q_2, q \in (0, \infty]$, and suppose T_σ is bounded in $(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)$. Then $1/q \leq 1/q_1 + 1/q_2$.*

Proof. Take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \widehat{\varphi} \subset \{|\xi| \leq 1\}$ and $|\varphi(x)| \geq 1$ for $x \in Q$. Take a Lebesgue point (ξ_0, η_0) of σ such that $\sigma(\xi_0, \eta_0) \neq 0$ and define f_ϵ and g_ϵ for $0 < \epsilon < 1$ by

$$\begin{aligned} \widehat{f}_\epsilon(\xi) &= \epsilon^{-n} \widehat{\varphi}(\epsilon^{-1}(\xi - \xi_0)), \quad \widehat{g}_\epsilon(\eta) = \epsilon^{-n} \widehat{\varphi}(\epsilon^{-1}(\eta - \eta_0)), \\ f_\epsilon(x) &= e^{2\pi i \xi_0 \cdot x} \varphi(\epsilon x), \quad g_\epsilon(x) = e^{2\pi i \eta_0 \cdot x} \varphi(\epsilon x). \end{aligned}$$

Then $\mathbf{1}_Q(\epsilon x) \leq |f_\epsilon(x)| = |g_\epsilon(x)| \lesssim (1 + \epsilon|x|)^{-N}$ with any $N > 0$. From this we easily see that $\|f_\epsilon\|_{(L^{p_1}, \ell^{q_1})} \approx \epsilon^{-n/q_1}$ and $\|g_\epsilon\|_{(L^{p_2}, \ell^{q_2})} \approx \epsilon^{-n/q_2}$ for $0 < \epsilon < 1$.

On the other hand, $T_\sigma(f_\epsilon, g_\epsilon)(x)$ is written as

$$\begin{aligned} T_\sigma(f_\epsilon, g_\epsilon)(x) &= \iint e^{2\pi i x \cdot (\xi + \eta)} (\sigma(\xi, \eta) - \sigma(\xi_0, \eta_0)) \widehat{f}_\epsilon(\xi) \widehat{g}_\epsilon(\eta) d\xi d\eta \\ &\quad + \iint e^{2\pi i x \cdot (\xi + \eta)} \sigma(\xi_0, \eta_0) \widehat{f}_\epsilon(\xi) \widehat{g}_\epsilon(\eta) d\xi d\eta = A + B, \quad \text{say.} \end{aligned}$$

Since (ξ_0, η_0) is a Lebesgue point of σ , the term A tends to 0 uniformly in $x \in \mathbb{R}^n$ as $\epsilon \rightarrow 0$. For the term B , we have $B = \sigma(\xi_0, \eta_0) e^{2\pi i x \cdot (\xi_0 + \eta_0)} \varphi(\epsilon x)^2$, and hence our choice of φ implies $|B| \geq |\sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon x)$. Hence for all sufficiently small ϵ we have $|T_\sigma(f_\epsilon, g_\epsilon)(x)| \geq 2^{-1} |\sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon x)$ and thus

$$\begin{aligned} \|T_\sigma(f_\epsilon, g_\epsilon)\|_{(L^p, \ell^q)} &= \left\| \|T_\sigma(f_\epsilon, g_\epsilon)(z + \rho)\|_{L_z^p(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \\ &\gtrsim \left\| \|\sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon(z + \rho))\|_{L_z^p(Q)} \right\|_{\ell_\rho^q(\mathbb{Z}^n)} \approx |\sigma(\xi_0, \eta_0)| \epsilon^{-n/q}. \end{aligned}$$

If T_σ is bounded in $(L^{p_1}, \ell^{q_1}) \times (L^{p_2}, \ell^{q_2}) \rightarrow (L^p, \ell^q)$, then the inequalities obtained above imply $\epsilon^{-n/q} = O(\epsilon^{-n/q_1} \epsilon^{-n/q_2})$ as $\epsilon \rightarrow 0$, which holds only when $1/q \leq 1/q_1 + 1/q_2$. $\square$

Lemma 7. *Let $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\sigma \neq 0$, $p_1, p_2, p, q_1, q_2, q \in (0, \infty]$, and suppose T_σ is bounded in $W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}$. Then $1/p \leq 1/p_1 + 1/p_2$.*

Proof. Take $\varphi, (\xi_0, \eta_0), f_\epsilon$, and g_ϵ in the same way as in Proof of Lemma 6.

To estimate the quasi-norms of f_ϵ and g_ϵ in Wiener-amalgam spaces, take a function $\kappa \in C_0^\infty(\mathbb{R}^n)$ such that $\sum_{\mu \in \mathbb{Z}^n} \kappa(\xi - \mu) = 1$ for all $\xi \in \mathbb{R}^n$ and that

$$|\xi| < 1/10 \Rightarrow \kappa(\xi) = 1 \text{ and } \kappa(\xi - \mu) = 0 \text{ for } \mu \in \mathbb{Z}^n \setminus \{0\}.$$

Then, for $0 < \epsilon < 1/10$, we have $\kappa(D - \xi_0)f_\epsilon = f_\epsilon$ and $\kappa(D - \xi_0 - \mu)f_\epsilon = 0$ for $\mu \in \mathbb{Z}^n \setminus \{0\}$, and thus

$$\begin{aligned} \|f_\epsilon\|_{W^{p_1, q_1}} &\approx \left\| \|\kappa(D - \xi_0 - \mu)f_\epsilon(x)\|_{\ell_\mu^{q_1}(\mathbb{Z}^n)} \right\|_{L_x^{p_1}(\mathbb{R}^n)} \\ &= \|f_\epsilon(x)\|_{L_x^{p_1}(\mathbb{R}^n)} = \|\varphi(\epsilon x)\|_{L_x^{p_1}(\mathbb{R}^n)} \approx \epsilon^{-n/p_1}. \end{aligned}$$

Similarly we have $\|g_\epsilon\|_{W^{p_2, q_2}} \approx \epsilon^{-n/p_2}$ for $0 < \epsilon < 1/10$.

To estimate the $W^{p, q}$ -quasi-norm of $T_\sigma(f_\epsilon, g_\epsilon)$, we take a function $\tilde{\kappa} \in C_0^\infty(\mathbb{R}^n)$

such that $\tilde{\kappa}(\xi_0 + \eta_0) \neq 0$ and $\sum_{\mu \in \mathbb{Z}^n} \tilde{\kappa}(\xi - \mu) \geq 1$ for all $\xi \in \mathbb{R}^n$. Then

$$\begin{aligned} \tilde{\kappa}(D)T_\sigma(f_\epsilon, g_\epsilon)(x) &= \iint e^{2\pi i x \cdot (\xi + \eta)} \tilde{\kappa}(\xi + \eta) \sigma(\xi, \eta) \hat{f}_\epsilon(\xi) \hat{g}_\epsilon(\eta) d\xi d\eta \\ &= \iint e^{2\pi i x \cdot (\xi + \eta)} \left[\tilde{\kappa}(\xi + \eta) \sigma(\xi, \eta) - \tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0) \right] \hat{f}_\epsilon(\xi) \hat{g}_\epsilon(\eta) d\xi d\eta \\ &\quad + \iint e^{2\pi i x \cdot (\xi + \eta)} \tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0) \hat{f}_\epsilon(\xi) \hat{g}_\epsilon(\eta) d\xi d\eta = A + B, \quad \text{say.} \end{aligned}$$

Since (ξ_0, η_0) is a Lebesgue point of $\tilde{\kappa}(\xi + \eta) \sigma(\xi, \eta)$, the term A tends to 0 uniformly in $x \in \mathbb{R}^n$ as $\epsilon \rightarrow 0$. For the term B , we have $B = \tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0) e^{2\pi i x \cdot (\xi_0 + \eta_0)} \varphi(\epsilon x)^2$ and hence $|B| \geq |\tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon x)$. Hence for all sufficiently small ϵ we have $|\tilde{\kappa}(D)T_\sigma(f_\epsilon, g_\epsilon)(x)| \geq 2^{-1} |\tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon x)$ and thus

$$\begin{aligned} \|T_\sigma(f_\epsilon, g_\epsilon)\|_{W^{p,q}} &\approx \left\| \left\| \tilde{\kappa}(D - \mu)T_\sigma(f_\epsilon, g_\epsilon)(x) \right\|_{\ell_\mu^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \\ &\geq \left\| \tilde{\kappa}(D)T_\sigma(f_\epsilon, g_\epsilon)(x) \right\|_{L_x^p(\mathbb{R}^n)} \gtrsim \left\| |\tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0)| \mathbf{1}_Q(\epsilon x) \right\|_{L_x^p(\mathbb{R}^n)} \\ &\approx |\tilde{\kappa}(\xi_0 + \eta_0) \sigma(\xi_0, \eta_0)| \epsilon^{-n/p}. \end{aligned}$$

If T_σ is bounded in $W^{p_1, q_1} \times W^{p_2, q_2} \rightarrow W^{p, q}$, then the inequalities obtained above imply $\epsilon^{-n/p} = O(\epsilon^{-n/p_1} \epsilon^{-n/p_2})$ as $\epsilon \rightarrow 0$, which holds only if $1/p \leq 1/p_1 + 1/p_2$. $\square$

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