

The Nilpotent Subvariety of the Vector Space

Associated to A Symmetric Pair

— Survey —

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§1. Introduction.

1.1. This note is a survey of my paper [S].

1.2. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{g}_0$ be its real form. Let θ be a Cartan involution of $\mathfrak{g}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{v}_0$ be the corresponding Cartan decomposition. Extend θ to $\mathfrak{g}$ as a complex linear involution and let $\mathfrak{k}, \mathfrak{v}$ be the complexifications of $\mathfrak{k}_0, \mathfrak{v}_0$, respectively. In this note, $(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair and $\mathfrak{v}$ is the vector space associated to it. Put $G = \text{Int } \mathfrak{g}$ and $K_\theta = \{g \in G; \theta g = g\}$. Let K be the identity component of K_θ . Let $\underline{N}(\mathfrak{v})$ be the nilpotent subvariety of $\mathfrak{v}$, that is, an element X of $\mathfrak{v}$ is contained in $\underline{N}(\mathfrak{v})$ if and only if $\text{ad}_{\mathfrak{g}}(X)$ is nilpotent (cf. [1]). Let $\mathbb{C}[\mathfrak{v}]$ be the polynomial ring over $\mathfrak{v}$ and let $\mathbb{C}[\mathfrak{v}]^K$ be the subring of $\mathbb{C}[\mathfrak{v}]$ consisting of K -invariant elements. Then there exist homogeneous polynomials $P_1, \dots, P_\ell$ of $\mathbb{C}[\mathfrak{v}]^K$ such that $\mathbb{C}[\mathfrak{v}]^K = \mathbb{C}[P_1, \dots, P_\ell]$ (cf. [1]). It follows that $\underline{N}(\mathfrak{v}) = \{X \in \mathfrak{v}; P_1(X) = \dots = P_\ell(X) = 0\}$ and that $\text{codim}_{\mathfrak{v}} \underline{N}(\mathfrak{v}) = \ell$, in other words, $\underline{N}(\mathfrak{v})$ is a complete intersection. The number ℓ is

nothing but the restricted rank of the corresponding Riemannian symmetric pair $(\underline{g}_0, \underline{k}_0)$.

1.3. The subjects of [S] are concerned with the following problems:

- (1) Determine the irreducible components of $\underline{N}(V)$.
- (2) Construct an analogue of Springer's resolution for $\underline{N}(V)$.
- (3) Examine the generic singularities of $\underline{N}(V)$.

We determine the number of the irreducible components of $\underline{N}(V)$ completely in [S] (cf. §2). The cotangent bundle over the complete flag manifold of G is regarded as a desingularization of the nilpotent subvariety of the Lie algebra $\underline{g}$. This is called the Springer's resolution (cf. [4]). It seems to be interesting to construct an analogue to the Springer's resolution for $\underline{N}(V)$. This will be done in §2. As to (3), we recall Brieskorn's result (cf. [3]). His famous result states: A simple singularity of type A_ℓ , D_ℓ or E_ℓ appears in the nilpotent subvariety $\underline{N}_\underline{g}$ of the corresponding simple Lie algebra $\underline{g}$. More precisely, take a subregular nilpotent element of $\underline{g}$ and a transversal slice S_X to the G -orbit of X at X . Then $S_X \cap \underline{N}_\underline{g}$ is nothing but the simple singularity. It is interesting to examine the analogue of Brieskorn's result for $\underline{N}(V)$. This is the precise meaning of (3). Although I do not obtain a complete answer to this at present, I shall explain the result for this problem in §4.

§2. The irreducibility of $\underline{N}(V)$.

The variety $\underline{N}(V)$ is not irreducible in general. In contrast

to the fact that the nilpotent subvariety of a simple Lie algebra is irreducible, this is a remarkable difference.

Example 1. Consider the case where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\theta(X) = -{}^tX$. Then $\underline{N}(V) = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix}; x^2 + y^2 = 0 \right\}$. It is clear that $\underline{N}(V)$ has two irreducible components defined by the equations $x + \sqrt{-1}y = 0$ and $x - \sqrt{-1}y = 0$.

In general, we show the following ([S, Th.1]).

Theorem 1. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair, $\mathfrak{g}$ simple.

(1) If the corresponding Riemannian symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ is Hermitian symmetric and the restricted root system of $\mathfrak{g}_0$ is reduced, then $\underline{N}(V)$ has two irreducible components. Each irreducible components is also a complete intersection.

(2) If $(\mathfrak{g}, \mathfrak{k})$ is one of the following. If d is the number of irreducible components of $\underline{N}(V)$, then d is given in the right column.

$(\mathfrak{g}, \mathfrak{k})$	d
$(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}))$	2
$(\mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{so}(n+1, \mathbb{C}) + \mathfrak{so}(n, \mathbb{C}))$	2
$(\mathfrak{so}(4n, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}) + \mathfrak{so}(2n, \mathbb{C}))$	4
$(\mathfrak{so}(4n+2, \mathbb{C}), \mathfrak{so}(2n+1, \mathbb{C}) + \mathfrak{so}(2n+1, \mathbb{C}))$	2
$(\mathfrak{so}(4n+k, \mathbb{C}), \mathfrak{so}(2n+k, \mathbb{C}) + \mathfrak{so}(2n, \mathbb{C}))$ $(k, n \geq 2)$	2
$(\mathfrak{e}_7^{\mathbb{C}}, \mathfrak{sl}(8, \mathbb{C}))$	2

(3) If $(\mathfrak{g}, \mathfrak{k})$ is not the one treated in (1) and (2), then $\underline{N}(V)$ is irreducible.

Problem. In the case of (2), each irreducible component of $\underline{N}(V)$ is not a complete intersection. Determine the defining ideal of each irreducible component.

I have no idea to this problem at present.

Put

$$\underline{N}(V)_r = \{X \in \underline{N}(V); dP_1, \dots, dP_\ell \text{ are linearly independent at } X\}.$$

$$\underline{N}(V)_{pr} = \{X \in \underline{N}(V); K \cdot X \text{ is open in } \underline{N}(V)\}.$$

An element of $\underline{N}(V)_{pr}$ is called principal nilpotent (cf. [1]). It is clear that $\underline{N}(V)_{pr}$ is contained in $\underline{N}(V)_r$. But in general they do not coincide. We now give an example of such a pair that $\underline{N}(V)_{pr} \subsetneq \underline{N}(V)_r$.

Example 2. $(\underline{sl}(n+1, \mathbb{C}), \underline{gl}(n, \mathbb{C}))$ ($n > 1$).

Let $\underline{g} = \underline{sl}(n+1, \mathbb{C})$ and let θ be an involution of $\underline{g}$ defined by $\theta(X) = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} X \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$. Then $\underline{k} = \{X \in \underline{g}; \theta(X) = X\}$ is isomorphic to the Lie algebra $\underline{gl}(n, \mathbb{C})$. We identify $\mathbb{C}^n \times \mathbb{C}^n$ with $V = \{X \in \underline{g}; \theta(X) = -X\}$ by the map $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto$

$$\begin{pmatrix} 0 & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ y_1 \cdots y_n & 0 \end{pmatrix}. \text{ Under the identification, we find that}$$

$$\underline{N}(V) = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n ; x_1 y_1 + \cdots + x_n y_n = 0 \}.$$

By direct calculation, we also find that $\underline{N}(V)$ has four K -orbits O_i ($i = 1, \dots, 4$) defined by

$$O_1 = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n ; x_1 y_1 + \dots + x_n y_n = 0, \\ x \neq 0, y \neq 0 \}$$

$$O_2 = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n ; x = 0, y \neq 0 \}$$

$$O_3 = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n ; x \neq 0, y = 0 \}$$

$$O_4 = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n ; x = y = 0 \}$$

and that $O_1 = \underline{N}(V)_{pr}$, $O_1 \cup O_2 \cup O_3 = \underline{N}(V)_r$. Hence $\underline{N}(V)_{pr} \subsetneq \underline{N}(V)_r$ in this case.

In general, we have the following.

Theorem 2 ([S, Th.4]). Let (g, k) be a symmetric pair and let Σ be the restricted root system of the corresponding Riemannian symmetric pair (g_0, k_0) .

(1) If Σ is reduced, then $\underline{N}(V)_{pr} = \underline{N}(V)_r$.

(2) If Σ is not reduced, then $\underline{N}(V)_{pr} \subsetneq \underline{N}(V)_r$. Moreover, in this case, if X is in $\underline{N}(V)_r - \underline{N}(V)_{pr}$, then $\text{Ad}(G)X \cap V = \underline{N}(V)_r - \underline{N}(V)_{pr}$.

§3. A resolution of the nilpotent subvariety.

Take $X_0 \in \underline{N}(V)_{pr}$ and fix it. It follows from [1] that there exist $H_0 \in k$ and $Y_0 \in V$ such that

$$[H_0, X_0] = 2X_0, \quad [H_0, Y_0] = -2Y_0, \quad [X_0, Y_0] = H_0.$$

We define

$$\mathfrak{g}(j) = \{ A \in \mathfrak{g}; [H_0, A] = jA \}$$

$$\tilde{\mathfrak{l}} = \bigoplus_{j \geq 0} \mathfrak{g}(j), \quad \tilde{\mathfrak{n}} = \bigoplus_{j > 0} \mathfrak{g}(j)$$

$$\underline{\mathfrak{l}} = \tilde{\mathfrak{l}} \cap \underline{\mathfrak{k}}, \quad \underline{\mathfrak{n}} = \tilde{\mathfrak{n}} \cap \underline{\mathfrak{v}}.$$

We note here that $\tilde{\mathfrak{l}}$ is a parabolic subalgebra of $\mathfrak{g}$, that $\tilde{\mathfrak{n}}$ is its nilpotent radical and that $[\tilde{\mathfrak{l}}, \tilde{\mathfrak{n}}] \subset \tilde{\mathfrak{n}}$. Let $\tilde{L}$ be the parabolic subgroup of G with lie algebra $\tilde{\mathfrak{l}}$ and put $L_\theta = \tilde{L} \cap K_\theta$.

Every element p of L_θ induces an automorphism of $K_\theta \times \underline{\mathfrak{n}}$ in the following way: $(k, X) \rightarrow (kp, \text{Ad}(p^{-1})X)$. We denote by $\tilde{\underline{N}}(V)$ the quotient of $K_\theta \times \underline{\mathfrak{n}}$ by the action of L_θ and put $k^*X = (k, X)L_\theta$ for any $(k, X) \in K_\theta \times \underline{\mathfrak{n}}$. Let π be the canonical mapping of $\tilde{\underline{N}}(V)$ to $\underline{N}(V)$. By the construction, connected components of $\tilde{\underline{N}}(V)$ correspond to irreducible components of $\underline{N}(V)$. Hence in general $\tilde{\underline{N}}(V)$ is not connected (cf. Th.1).

The following theorem shows that $\tilde{\underline{N}}(V)$ is an analogue of the Springer's resolution of the nilpotent subvariety of a simple Lie algebra (cf. [4]).

Theorem 3 ([S, Th.5]). The mapping $\pi : \tilde{\underline{N}}(V) \rightarrow \underline{N}(V)$ has the following properties.

- (a) $\tilde{\underline{N}}(V)$ is smooth.
- (b) π is proper and surjective.
- (c) π induces an isomorphism $\pi^{-1}(\underline{N}(V)_{pr}) \rightarrow \underline{N}(V)_{pr}$.

We give here examples which illustrate the resolution of the nilpotent variety $\underline{N}(V)$.

Example 3. $(\underline{so}(n+1, \mathbb{C}), \underline{so}(n, \mathbb{C}))$ ($n \geq 2$).

In this case, V is identified with $\mathbb{C}^n$ and the nilpotent subvariety $\mathbb{C}(V)$ with the set

$$S = \{ x \in \mathbb{C}^n; x_1^2 + \dots + x_n^2 = 0 \}.$$

Then the resolution $\tilde{\underline{N}}(V)$ is identified with

$$\begin{aligned} \tilde{S} = \{ (x, \xi) \in \mathbb{C}^n \times \mathbb{P}^{n-1}; x_1^2 + \dots + x_n^2 = 0, \\ \xi_1^2 + \dots + \xi_n^2 = 0, x // \xi \}. \end{aligned}$$

Define $S_1 = \{x \in S; x \neq 0\}$ and $S_2 = \{0\}$. Then the K -orbits of S are S_1 and S_2 . If x is contained in S_1 , then $\pi^{-1}(x)$ is clearly a single point. On the other hand, if $x = 0$, then $\pi^{-1}(x) = \mathbb{P}^{n-1}$.

Example 2 (continued).

In this case, V is identified with $\mathbb{C}^{2n}$ and $\underline{N}(V)$ is with the set

$$S = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n; x_1 y_1 + \dots + x_n y_n = 0 \}.$$

The resolution $\tilde{\underline{N}}(V)$ is identified with

$$\begin{aligned} \tilde{S} = \{ ((x, y), (\xi, \eta)) \in S \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}; \xi_1 \eta_1 + \dots + \xi_n \eta_n = 0, \\ x // \xi, y // \eta \}. \end{aligned}$$

We may regard O_i ($i = 1, 2, 3, 4$) as subsets of S . Then S_i ($i = 1, \dots, 4$) are the K -orbits of S . In particular, S_1 is the totality of the principal nilpotent elements and $S' = S_1 \cup S_2 \cup S_3$ is non-singular and identified with $\underline{N}(V)_r$. If (x, y) is in S_1 , then $\pi^{-1}((x, y))$ consists of a single point. On the other hand, for any $x \in \mathbb{C}^n$ ($x \neq 0$), we have $\pi^{-1}((x, 0)) = \mathbb{P}^{n-1}$. Moreover $\pi^{-1}(0, 0) = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Remark. Put $L = L_\theta \cap K$. Then L is a parabolic subgroup of K . But L is not the Borel subgroup of K in general.

Problem. Does there exist an analogue of Grothendieck's simultaneous resolution for V ?

\$4. The generic singularities of $\underline{N}(V)$.

Put

$$\underline{N}(V)_s = \{X \in \underline{N}(V); dP_1, \dots, dP_\ell \text{ are}$$

linearly dependent at $X\}$.

It is pathological that $\underline{N}(V)_s$ is neither irreducible nor equidimensional in general. Let $O_1, \dots, O_r$ be K -orbits of $\underline{N}(V)$ such that $O_i \cap O_j = \emptyset$ ($i \neq j$) and $\underline{N}(V)_s = \bigcup_{i=1}^r \overline{O}_i$. In particular, each $\overline{O}_i$ is an irreducible component of $\underline{N}(V)_s$. Put $\underline{N}(V)'_s = \bigcup_{i=1}^r O_i$. Take $X \in \underline{N}(V)'_s$. Let S_X be a transversal slice

to the K -orbit of X at X . A standard one is constructed as follows. Let $H \in \underline{k}$ and $Y \in V$ be such that $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$ (cf. [1]). Then $S_X = X + [Y, \underline{k}]$ is a transversal slice. It seems to be interesting to determine the intersection $S_X \cap \underline{N}(V)$. In fact, as stated in the Introduction, a simple singularity is appeared in this manner if we consider a symmetric pair $(\underline{g} \oplus \underline{g}, \underline{g})$.

Let $X \in \underline{N}(V)'_S$. It is not clear whether the intersection $S_X \cap \underline{N}(V)$ is a hypersurface singularity or not. But I conjecture that this is true.

From now on, assume that $(\underline{g}, \underline{k})$ is of the classical type, that is, $\underline{g}$ is simple of the classical type. I give the concrete expression of $S_X \cap \underline{N}(V)$ for such an $X \in \underline{N}(V)'_S$ that $S_X \cap \underline{N}(V)$ is a hypersurface singularity (cf. [S, §§3, 4]).

$(\underline{sl}(n, \mathbb{C}), \underline{so}(n, \mathbb{C}))$	$x^n + y^2 = 0$
$(\underline{sl}(2n, \mathbb{C}), \underline{sp}(n, \mathbb{C}))$	$x^n + u_1 v_1 + u_2 v_2 = 0$
$(\underline{sl}(2n+k, \mathbb{C}), \underline{sl}(n+k, \mathbb{C}) + \underline{sl}(n, \mathbb{C}) + \mathbb{C})$	$x^n + yz = 0$ $x_1 y_1 + \cdots + x_{k+1} y_{k+1} = 0$
$(\underline{so}(n+1, \mathbb{C}), \underline{so}(n, \mathbb{C}))$	$x_1^2 + \cdots + x_n^2 = 0$
$(\underline{so}(n+3, \mathbb{C}), \underline{so}(n+1, \mathbb{C}) + \underline{so}(2, \mathbb{C}))$	$xy = 0$ $x_1^2 + \cdots + x_n^2 = 0$
$(\underline{so}(2n, \mathbb{C}), \underline{so}(n, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$	$x^{n-1} + xy^2 = 0$

$(\underline{so}(2n+1, \mathbb{C}), \underline{so}(n+1, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$	$\frac{x^{2n} + y^2 = 0}{xy = 0}$
$(\underline{so}(2n+k, \mathbb{C}), \underline{so}(n+k, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$ ($k > 1$)	$\frac{x^n + y^2 = 0}{x_1^2 + \cdots + x_{k+1}^2 = 0}$
$(\underline{so}(4n, \mathbb{C}), \underline{gl}(2n, \mathbb{C}))$	$\frac{x^n + u_1 v_1 + u_2 v_2 = 0}{xy = 0}$
$(\underline{so}(4n+2, \mathbb{C}), \underline{gl}(2n+1, \mathbb{C}))$	$\frac{x^n + u_1 v_1 + u_2 v_2 = 0}{u_1 v_1 + u_2 v_2 + u_3 v_3 = 0}$
$(\underline{sp}(n, \mathbb{C}), \underline{gl}(n, \mathbb{C}))$	$\frac{x^{2n} + y^2 = 0}{x^{2n} + xy^2 = 0}$
$(\underline{sp}(2n, \mathbb{C}), \underline{sp}(n, \mathbb{C}) + \underline{sp}(n, \mathbb{C}))$	$u_1 v_1 + u_2 v_2 = 0$
$(\underline{sp}(2n+k, \mathbb{C}), \underline{sp}(n+k, \mathbb{C}) + \underline{sp}(n, \mathbb{C}))$	$\frac{x^n + u_1 v_1 + u_2 v_2 = 0}{\sum_{i=1}^{2k+2} u_i v_i = 0}$

Remark. In [2], Mr. Y. Shimizu and I treaded the case where $(\underline{g}, \underline{k})$ is of normal type, that is, $\underline{g}_0$ is a normal real form of $\underline{g}$.

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