

On two sufficient conditions for univalence of real coefficient functions

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Abstract

It is well known that if the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and satisfies one of the following conditions

$$1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in } |z| < 1$$

or

$$1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in } |z| < 1,$$

then $f(z)$ is univalent in $|z| < 1$. In this paper, we improve the above conditions for the function $f(z)$ whose coefficients are all real.

1. Introduction

Let $\mathcal{A}$ be the set of analytic functions defined in the unit disk $E = \{z \mid |z| < 1\}$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and let

$$\mathcal{S} = \{f(z) \mid f(z) \in \mathcal{A} \text{ and } f(z) \text{ is univalent in } E\}.$$

The late professor Ozaki [1] proved the following theorem.

Theorem A. *Let $f(z) \in \mathcal{A}$ and if $f(z)$ satisfies one of the following conditions*

$$(i) \quad 1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in } |z| < 1$$

or

$$(ii) \quad 1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in } |z| < 1,$$

then we have $f(z) \in \mathcal{S}$.

2. Theorems

First our theorem is contained in

Theorem 1. Let $f(z) \in \mathcal{A}$, all the coefficients a_n , $2 \leq n \in \mathbb{N} = \{1, 2, 3, \dots\}$ are real and suppose that

$$(1) \quad 1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > -1 \quad (z \in \mathbb{E}).$$

Then we have $f(z) \in \mathcal{S}$.

Proof. Suppose that if there exists a positive real number r , $0 < r < 1$ for which $f(z)$ is univalent in $|z| < r$ but $f(z)$ is not univalent in $|z| \leq r$, then from the hypothesis, there exists two points $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $\theta_1 < \theta_2$ and $\theta_2 - \theta_1 < \pi$ for which $f(z_1) = f(z_2)$.

From the hypothesis (1), we have $f'(z) \neq 0$ in $\mathbb{E}$, because if $f'(z)$ has a zero in $\mathbb{E}$, then it is impossible that $f(z)$ satisfies the condition (1).

Let us put

$$C = \{z \mid z = re^{i\theta}, \theta_1 \leq \theta \leq \theta_2\}$$

and

$$C_{f(z)} = \{f(z) \mid z \in C\}.$$

Then we have

$$\begin{aligned} \int_{C_{f(z)}} d \arg df(z) &= -\pi \\ &= \int_C d \arg f'(z) dz \\ &= \int_{\theta_1}^{\theta_2} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta \\ &> \int_{\theta_1}^{\theta_2} (-1) d\theta = \theta_1 - \theta_2 > -\pi. \end{aligned}$$

This is a contradiction and therefore it completes the proof.

Theorem 2. Let $f(z) \in \mathcal{A}$, all the coefficients a_n , $2 \leq n \in \mathbb{N} = \{1, 2, 3, \dots\}$ are real and suppose that

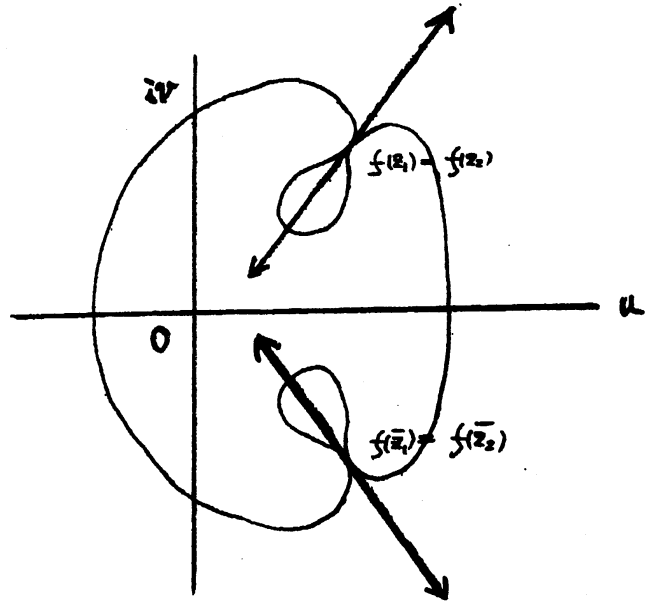
$$(2) \quad 1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < 2 \quad (z \in \mathbb{E}).$$

Then we have $f(z) \in \mathcal{S}$.

Proof. Applying the same method as the proof of Theorem 1, if there exists a positive real number r , $0 < r < 1$ for which $f(z)$ is univalent in $|z| < r$ but $f(z)$ is not univalent in $|z| \leq r$, then there are four points such as the proof of Theorem 1, $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $z_3 = re^{i(2\pi-\theta_2)}$ and $z_4 = re^{i(2\pi-\theta_1)}$, $0 < \theta_1 < \theta_2 < \pi$ for which we have $f(z_1) = f(z_2)$ and $f(z_3) = f(z_4)$. From the hypothesis, the tangent line at the point $f(z_1)$ and $f(z_2)$ is the common tangent and it is the same for the points $f(z_3)$ and $f(z_4)$.

Therefore, we have

$$\int_{\theta_1}^{\theta_2} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi$$



and

$$\int_{2\pi-\theta_2}^{2\pi-\theta_1} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi,$$

where $z = re^{i\theta}$.

From the hypothesis (2) and the same reason as the proof of Theorem 1, we have $f'(z) \neq 0$ in E .

From the hypothesis (2), we have

$$\begin{aligned} & \int_{|z|=r} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta = 2\pi \\ &= \int_{\theta_1}^{\theta_2} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{\theta_2}^{2\pi-\theta_2} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta \\ &+ \int_{2\pi-\theta_2}^{2\pi-\theta_1} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{2\pi-\theta_1}^{2\pi+\theta_1} \left(1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right) d\theta \\ &< -\pi + \int_{\theta_2}^{2\pi-\theta_2} 2d\theta - \pi + \int_{2\pi-\theta_1}^{2\pi+\theta_1} 2d\theta \\ &= \{4\pi - 2(\theta_2 - \theta_1)\} - 2\pi \\ &< 4\pi - 2\pi = 2\pi. \end{aligned}$$

This is a contradiction and so, it completes the proof.

Remark. A function $f(z) \in \mathcal{A}$ is typically real in E if $(\operatorname{Im} f(z))(\operatorname{Im} z) > 0$ for $E/\mathbb{R} = \{z \mid z \in E \cap z \notin \mathbb{R}\}$. In Theorem 1, if $f(z)$ is typically real and satisfies (1), then the conclusion continues to hold true.

References

- [1] S. Ozaki, On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku 4(1941), 45 - 86.

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