

POLYLOGARITHMS AND MIXED MOTIVES

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§1. The category of mixed motives $\mathcal{D}_{finite}(k)$.

Let k be a field, $\square^1 = \mathbb{P}_k^1 - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \dots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square_{i,a}^{n-1} = \{x_i = a\}$ where $a = 0$ or ∞ . A face of dimension m is canonically isomorphic to $\square^m$.

Let Σ_n be the permutation group of the set $\{1, \dots, n\}$; it acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permutations. Let G_n be the semi-direct product of $(\mathbb{Z}/2\mathbb{Z})^n$ and Σ_n ; an element is of the form $(i_1, \dots, i_n; \tau)$ with $i_1, \dots, i_n \in \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ and $\tau \in \Sigma_n$. Σ_n acts on $\square^n$ by permutations of coordinates, and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by: $(i_1, \dots, i_n) \cdot (x_1, \dots, x_n) = (x_1^{i_1}, \dots, x_n^{i_n})$. So G_n acts on $\square^n$. There is a homomorphism $sgn : G_n \rightarrow \{\pm 1\}$, $(i_1, \dots, i_n; \tau) \mapsto i_1 \cdots i_n \cdot sgn \tau$.

Let X be an equi-dimensional variety (or a scheme) over a field k . Let $\mathcal{Z}^r(X, n)$ be the $\mathbb{Q}$ -vector space of $\mathbb{Q}$ -cycles z of codimension r on $X \times \square^n$ such that

- (i) each irreducible component of z meets each face $X \times \square^m$ properly, and
- (ii) z is alternating with respect to G_n , namely, for any $\sigma \in G_n$, $\sigma(z) = sgn(\sigma)z$.

The inclusions of codimension one faces $\delta_{i,a} : \square_{i,a}^{n-1} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^{i+a} \delta_{i,a}^* : \mathcal{Z}^r(X, n) \rightarrow \mathcal{Z}^r(X, n-1)$$

(namely, ∂ sends alternating cycles to alternating ones) and $(\mathcal{Z}^r(X, \cdot), \partial)$ is a homology complex. We call this the *cycle complex* (of codimension r) of X . By definition the (rational) higher Chow groups are the homology groups of this complex:

$$CH^r(X, n) = H_n \mathcal{Z}^r(X, \cdot).$$

Note $CH^r(X, 0) = CH^r(X)$, the rational Chow group of X .

The following notion will be of frequent use. A *C-complex* (of abelian groups) consists of

- (i) Complexes $A^m = (A^{m,\bullet}, d_{A^m})$ (each of which is not necessarily bounded) for $m \in \mathbb{Z}$, such that for all but finitely many m 's, $A^m = 0$; and
- (ii) For $m < n$, maps of graded groups

$$F^{m,n} : A^{m,\bullet} \rightarrow A^{n,\bullet-(n-m-1)}$$

subject to the condition

$$F^{m,n} \circ (-1)^m d_{A^m} + (-1)^n d_{A^n} \circ F^{m,n} + \sum_{m < \ell < n} F^{\ell,n} \circ F^{m,\ell} = 0$$

as a map $A^{m,\bullet} \rightarrow A^{n,\bullet-n+m+2}$.

One can associate with it a complex, the *total complex* $Tot(A) = (Tot(A)^\bullet, d)$, defined as:

$$Tot(A)^p = \bigoplus_{p \in \mathbb{Z}} A^{m, p-m}$$

(which is a finite sum), and the differential d is the direct sum, for m , of $(-1)^m d_{A^m} + F^{m,n} : A^{m,\bullet} \rightarrow \bigoplus_{n \geq m} A^{n,\bullet-(n-m-1)}$. The condition in (ii) is equivalent to d being a differential.

Let X and Y be smooth projective. For an element $f \in \mathcal{Z}^s(X \times Y, \ell)$, one has the partially defined map (of graded vector spaces)

$$f_* : \mathcal{Z}^r(X, n) \dashrightarrow \mathcal{Z}^{r+s}(Y, n + \ell), \quad f_*(z) = p_{Y*}[(f \times \square^n) \cdot (z \times Y \times \square^\ell)].$$

Here $(f \times \square^n) \cdot (z \times Y \times \square^\ell)$ denotes the alternation (with respect to the action of $G_{n+\ell}$) of the cycle theoretic intersection $(f \times \square^n) \odot (z \times Y \times \square^\ell)$, and $p_Y : X \times Y \times \square^{n+\ell} \rightarrow Y \times \square^{n+\ell}$ is the projection.

Similarly there is the partially defined map

$$\mathcal{Z}^r(X \times Y, \cdot) \otimes \mathcal{Z}^s(Y \times Z, \cdot) \dashrightarrow \mathcal{Z}^{r+s-\dim Y}(X \times Z, \cdot)$$

which sends $u \otimes v \in \mathcal{Z}^r(X \times Y, n) \otimes \mathcal{Z}^s(Y \times Z, m)$ to

$$v \circ u := \text{the alternation of } p_{XZ*}[(X \times v \times \square^n) \odot (u \times Z \times \square^m)].$$

For each smooth projective variety X there is a collection of *distinguished subcomplexes* of $\mathcal{Z}^r(X, \cdot)$ satisfying:

- (i) For a distinguished subcomplex $\mathcal{Z}^r(X, \cdot)'$, the inclusion into $\mathcal{Z}^r(X, \cdot)$ is a quasi-isomorphism;
- (ii) For any cycle $f \in \mathcal{Z}^s(X \times Y, \ell)$ and a distinguished subcomplex $\mathcal{Z}^{r+s-\dim X}(Y, \cdot)'$ there is a distinguished subcomplex $\mathcal{Z}^r(X, \cdot)'$ on which f_* is defined and induces a map $f_* : \mathcal{Z}^r(X, \cdot)' \rightarrow \mathcal{Z}^{r+s-\dim X}(Y, \cdot + \ell)'$;
- (iii) The intersection of a finite collection of distinguished subcomplexes is again distinguished.

For details on the category $\mathcal{D}(k)$ of mixed motives, we refer the reader to [Ha, II]. We will only need the subcategory $\mathcal{D}_{finite}(k)$ of mixed motives of finite type; the definitions are briefly recalled.

A *finite symbol* is a formal sum

$$\bigoplus_{\alpha \in I} (X_\alpha, r_\alpha)$$

where X_α is a smooth projective variety, I a finite index set and $r_\alpha \in \mathbb{Z}$. We write 0 for the corresponding symbol when I is an empty set.

Define dual, tensor product, and inner Hom of (a) finite symbol(s) as:

$$(\oplus (X_\alpha, r_\alpha))^\vee = \oplus (X_\alpha, \dim X_\alpha - r_\alpha).$$

$$(\oplus (X_\alpha, r_\alpha)) \otimes (\oplus (X'_{\alpha'}, r_{\alpha'})) = \oplus (X_\alpha \times X'_{\alpha'}, r_\alpha + r_{\alpha'});$$

and

$$\underline{Hom}(\oplus (X_\alpha, r_\alpha), \oplus (X'_{\alpha'}, r_{\alpha'})) = (\oplus (X_\alpha, r_\alpha))^\vee \otimes (\oplus (X'_{\alpha'}, r_{\alpha'})).$$

Define the *cycle complex* of a finite formal symbol by

$$\mathcal{Z}^0(\oplus(X_\alpha, r_\alpha), \cdot) = \bigoplus \mathcal{Z}^{r_\alpha}(X_\alpha, \cdot) .$$

One also uses cohomological notation $\mathcal{Z}^0(\oplus(X_\alpha, r_\alpha))^{-n} = \mathcal{Z}^0(\oplus(X_\alpha, r_\alpha), n)$.

Note there is a partially defined map

$$\begin{aligned} & \mathcal{Z}^0(\underline{\text{Hom}}((X_1, r_1), (X_2, r_2)), \cdot) \otimes \mathcal{Z}^0(\underline{\text{Hom}}((X_2, r_2), (X_3, r_3)), \cdot) \\ & \longrightarrow \mathcal{Z}^0(\underline{\text{Hom}}((X_1, r_1), (X_3, r_3)), \cdot) \end{aligned}$$

given by the composition of correspondences

$$u \otimes v \mapsto v \circ u = p_{13*}[(u \times X_3) \cdot (X_1 \times v)] .$$

By definition, an object of $\mathcal{D}_{\text{finite}}(k)$ is a set of data $K = (K^m) = (K^m, f^{m,n})$ where

(i) For each integer m , $K^m = \bigoplus_{\alpha \in I(m)} (X_\alpha, r_\alpha)$, a finite symbol.

(ii) For (m, n) with $m < n$, given $f^{m,n} = (f_{\alpha\beta}^{m,n}) \in \mathcal{Z}^0(\underline{\text{Hom}}(K^m, K^n))^{-n+m+1}$, which are subject to the conditions:

For $f^{m_k, m_{k+1}} \in \mathcal{Z}^0(\underline{\text{Hom}}(K^{m_k}, K^{m_{k+1}}))^{-m_{k+1}+m_k+1}$ ($k = 1, 2, \dots, r$) one has

$$f^{m_r, m_{r+1}} \circ f^{m_r, m_{r-1}} \circ \dots \circ f^{m_1, m_2} \text{ is defined and } \in \mathcal{Z}^0(\underline{\text{Hom}}(K^{m_1}, K^{m_{r+1}}))^{-m_{r+1}+m_1+r} .$$

For $m < n$, one has

$$(-1)^n \partial f^{m,n} + \sum_{m < \ell < n} f^{\ell,n} \circ f^{m,\ell} = 0 , .$$

On the left side the compositions of the correspondences are *required* to be defined.

There is the functor of *cycle complexes* $\mathcal{Z}^0$ from $\mathcal{D}_{\text{finite}}(k)$ to the derived category of $\mathbb{Q}$ -vector spaces. To define $\mathcal{Z}^0(K, \cdot)$, for each m and $\alpha \in I(m)$, take a distinguished subcomplex $\mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)'$ so that each $f_{\alpha\beta}^{m,n}$ induces the map $f_{\alpha\beta}^{m,n} * : \mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)' \rightarrow \mathcal{Z}^0((X_\beta, r_\beta), \cdot)'$. We then let

$$\mathcal{Z}^0(K^m, \cdot)' := \bigoplus_{\alpha} \mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)'$$

and have $f^{m,n} * : \mathcal{Z}^0(K^m, \cdot)' \rightarrow \mathcal{Z}^0(K^n, \cdot + (n - m - 1))'$ is defined. We define $\mathcal{Z}^0(K, \cdot)$ to be the *total complex* $\text{Tot}(\mathcal{Z}^0(K^m, \cdot)', f_*^{m,n})$, namely the complex $(\mathbb{K}, d)$ with

$$\mathbb{K}^i = \bigoplus_{j \geq i} \mathcal{Z}^0(K^j, j - i)' ,$$

and

$$d^i = \sum_j ((-1)^j \partial_j + \sum_{j < \ell} f_*^{j,\ell}) .$$

Let (K, f) and (L, g) be objects in $\mathcal{D}_{\text{finite}}(k)$. The *function cycle complex* $\text{Hom}(K, L)^\bullet$ is defined as follows. Let $\mathcal{Z}^0(\underline{\text{Hom}}(K^m, L^{m'}), \cdot)'$ be distinguished subcomplexes such that

$$\text{For } u \in \mathcal{Z}^0(\underline{\text{Hom}}(K^m, L^{m'}), \cdot), \text{ both } u \circ f^{n,m} \text{ and } g^{m',n'} \circ u \text{ are defined.}$$

(This is possible since there are only finitely many non-zero $f^{n,m}$'s and $g^{m',n'}$'s.) The cohomological complex to be defined has the group of N -cochains

$$\mathrm{Hom}(K, L)^N = \bigoplus_{-m+m'-p=N} \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{m'}), p)'$$

The differential of this complex, which we denote by D , is the sum of the three kinds of maps:

$$(-1)^{p+m'+n+1} (\circ f^{n,m}) : \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{m'}), p)' \rightarrow \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^n, L^{m'}), p+n'-m'-1)' ,$$

$$(-1)^{m'+n'} (g^{m',n'} \circ) : \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{m'}), p)' \rightarrow \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{n'}), p+n'-m'-1)' ,$$

and

$$(-1)^{m'} \partial : \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{m'}), \cdot)' \rightarrow \mathcal{Z}^0(\underline{\mathrm{Hom}}(K^m, L^{m'}), \cdot - 1)' .$$

Given three objects K, L and M , the partially defined composition map

$$\begin{aligned} \mathrm{Hom}(K, L)_\bullet \otimes \mathrm{Hom}(L, M)^\bullet &\longrightarrow \mathrm{Hom}(K, M)^\bullet \\ u \otimes v &\mapsto v \circ u; \quad (v \circ u)^{m,n} = \sum_{\ell \in \mathbb{Z}} v^{\ell,n} \circ u^{m,\ell} \end{aligned}$$

satisfies the Leibniz formula

$$D(v \circ u) = Dv \circ u + (-1)^{\deg v} v \circ Du ,$$

where $\deg v$ is the total degree of v in the cohomological complex. There is a quasi-isomorphic subcomplex of $\mathrm{Hom}(K, L)_\bullet \otimes \mathrm{Hom}(L, M)^\bullet$ on which the composition is defined. See [Ha, II, §1].

By definition

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{finite}}(k)}(K, L) = H^0 \mathcal{Z}^0(\underline{\mathrm{Hom}}(K, L))^\bullet .$$

The composition of morphisms is induced from the composition of the function complexes. A morphism $u : K \rightarrow L$ is represented by $u^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m}$ (non-zero only for $m \leq n$) subject to the condition

$$(-1)^n \partial u^{m,n} - \sum (-1)^{m+\ell} u^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ u^{m,\ell} = 0 .$$

It defines the zero morphism if there exist $U^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m-1}$ (non-zero only for $m \leq n-1$) such that

$$u^{m,n} = (-1)^n \partial U^{m,n} + \sum (-1)^{m+\ell} U^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ U^{m,\ell} .$$

We have the following. Let $\mathbb{Q}(r) = (pt, r)[2r]$, the Tate objects.

(1.1) **Theorem.** *The category $\mathcal{D}_{finite}(k)$ has a structure of triangulated category. Moreover*

(1) *$\mathcal{D}_{finite}(k)$ has dual, tensor product, inner Hom, the unit object $\mathbb{Q}$, and the Tate objects $\mathbb{Q}(r)$.*

(2) *There is a contravariant functor $h : (\text{Smooth Proj.}/k) \rightarrow \mathcal{D}_{finite}(k)$.*

(3) *If X is smooth and projective, one has*

$$\text{Hom}_{\mathcal{D}_{finite}(k)}(\mathbb{Q}, h(X)(r)[2r - m]) = K_m(X)_{\mathbb{Q}}^{(r)}.$$

Here the right hand side is an Adams-graded piece of the K -group of X .

(4) *There is the cycle complex functor $Z^0 : \mathcal{D}_{finite}(k) \rightarrow D(\mathbb{Q})$.*

For the rest we will simply write $\mathcal{D}(k)$ for $\mathcal{D}_{finite}(k)$.

§2. Etale realization.

For a smooth projective variety X over a field k and $\ell \neq \text{ch } k$, we have the ℓ -adic cohomology $H^*(X, \mathbb{Q}_{\ell}(r))$. We use complexes calculating the etale cohomology, which behave well with respect to composition of correspondences. Let X, Y be smooth projective, and D any variety. One can define a complex of $\mathbb{Q}_{\ell}$ -vector spaces $\text{Hom}(X, Y)_D(r)$ satisfying the following properties (cf. [Ha, II, §5] for details in case of Betti cohomology).

(1) $H^i \text{Hom}(X, Y)_D(r) = H^{i+2r}(X \times Y \times D, \mathbb{Q}_{\ell}(r))$. If the first variety of the pair is $\text{pt} = \text{Spec } k$, $\text{Hom}(\text{pt}, X)_{\text{pt}}(r) = \Gamma(X, C^{\bullet}(\mathbb{Q}_{\ell}(r)))[2r] := \varprojlim \Gamma(X, C^{\bullet}(\mathbb{Z}/\ell^v(r))) \otimes \mathbb{Q}_{\ell}[2r]$. Here $C^{\bullet}$ denotes the Godement resolution.

(2) There is a map of complexes

$$\text{Hom}(X, Y)_D(r) \otimes \text{Hom}(Y, Z)_D(s) \rightarrow \text{Hom}(X, Z)_D(r + s - \dim Y)$$

which gives rise to the composition of correspondences

$$H^*(X \times Y \times D, \mathbb{Q}_{\ell}(r)) \otimes H^*(Y \times Z \times D, \mathbb{Q}_{\ell}(r)) \rightarrow H^*(X \times Z, \mathbb{Q}_{\ell}(r + s - \dim Y)).$$

The map is associative.

(3) To a map $\alpha : D' \rightarrow D$, there corresponds to a map of complexes $\alpha^* : \text{Hom}(X, Y)_D(r) \rightarrow \text{Hom}(X, Y)_{D'}(r)$.

There is also the supported theory. Given a closed set $V \subset X \times Y \times D$, there is a subcomplex $\text{Hom}(X, Y)_{V,D}(r) \subset \text{Hom}(X, Y)_D(r)$ satisfying $H^i \text{Hom}(X, Y)_{V,D}(r) = H_V^{i+2r}(X \times Y \times D, \mathbb{Q}_{\ell}(r))$, and properties analogous to (2) and (3).

Now we take as D the cubical scheme $\square^{\bullet}$. For $r, s \in \mathbb{Z}$, we define a double complex $C((X, r), (Y, s))^{a,b}$ as follows.

$$C((X, r), (Y, s))^{\bullet,b} = \text{Hom}(X, Y)_{\square^{-b}}(s - r + \dim X)^{\text{Alt}}$$

where Alt denotes the alternating part with respect to the action of G_{-b} on $\square^{-b}$. Define

$$\partial : C^{\bullet,b}((X, r), (Y, s)) \rightarrow C^{\bullet,b+1}((X, r), (Y, s))$$

to be the alternating sum of the maps induced by the face maps $\square^{-b-1} \rightarrow \square^{-b}$. Denote the associated simple complex by $C((X, r), (Y, s))$ with differential $D = d + (-1)^b \partial$.

The above definitions can be extended to finite symbols. For K and L finite symbols we have the complex $C(K, L)$. Set $C(X, r) = C((\text{pt}, 0), (X, r)) = \text{Hom}(\text{pt}, X)(r)$, and $C(K) = C((\text{pt}, 0), K)$. There is composition map

$$C(K, L) \otimes C(L, M) \rightarrow C(K, M), \quad f \otimes g \mapsto g \circ f,$$

satisfying associativity. An element $F \in C(K, L)^n$ induces a map $F_* : C(K) \rightarrow C(L)[n]$.

Given $Z \in \mathcal{Z}^r(X, n)$ define a subcomplex

$$\begin{aligned} C_{|Z|}(X, r)^{\bullet\bullet} &\subset C(X, r)^{\bullet\bullet \geq -n} \\ &\subset C(X, r)^{\bullet\bullet} \end{aligned}$$

where $C_{|Z|}(X, r)^{a,b}$ is defined by the support condition with respect to $\cup_\delta |\delta^* Z|$, δ varying over the face maps $\delta : \square^{-b} \rightarrow \square^n$. Similarly given $f \in \text{Hom}(K, L)^{-n}$ there is a subcomplex $C_{|f|}(K, L)$. Here $|f|$ denotes the support of f (we sometimes write just f). An element $f \in \text{Hom}(K, L)^{-n}$ has cycle class $cl(f) \in H^0 C_{|f|}^{\bullet,b}(K, L)$.

(2.1) Proposition. *Given an object $(K^m, f^{m,n})$ of $\mathcal{D}(k)$, there exist, for $m < n$, elements*

$$F^{m,n} \in \bigoplus_{a+b=-(n-m-1), a \leq 0} C_{|f|}^{a,b}(K^m, K^n)$$

such that its $(0, -n + m + 1)$ -component ${}^0 F^{m,n} \in C_f^{\bullet, -(n-m-1)}(K^m, K^n)$ satisfies

$$[{}^0 F^{m,n}]_d = cl(f^{m,n}) \in H^0 C_f^{\bullet, -(n-m-1)}(K^m, K^n)$$

and one has the relation

$$\begin{aligned} (*) \quad & (-1)^n D(F^{m,n}) + \sum_{m < \ell < n} F^{\ell,n} \circ F^{m,\ell} = 0 \\ & \text{in } \bigoplus_{a+b=-(n-m-2), a \leq 0} C_f^{a,b}(K^m, K^n). \end{aligned}$$

We call such $(F^{m,n})$ a representative of $(cl(f^{m,n}))$.

Choose $(F^{m,n})$; then the maps

$$F^{m,n}_* : C(K^m) \rightarrow C(K^n)[-(n-m-1)]$$

satisfy

$$(-1)^n D \circ F^{m,n}_* + F^{m,n}_* \circ (-1)^m D + \sum F^{\ell,n}_* \circ F^{m,\ell}_* = 0$$

where D is the differential of $C(K^m)$ or $C(K^n)$, namely $(C(K^m), F^{m,n}_*)$ is a C -complex. So we have the total complex,

$$C(K) := \text{Tot}(\oplus C(K^m), D + \sum F^{m,n}_*).$$

It can be shown that $C(K)$ is well-defined independent of the choice of representatives $F^{m,n}$. By definition $H^*(K) = H^*(K, \mathbb{Q}_\ell) := H^* C(K)$.

(2.2) Theorem. *We have the functor of ℓ -adic etale cohomology*

$$H^*(-, \mathbb{Q}_\ell) : \mathcal{D}(k) \rightarrow \text{Vec}_{\mathbb{Q}_\ell}.$$

For K in $\mathcal{D}(k)$, we have $K \otimes_k \bar{k}$ in $\mathcal{D}(\bar{k})$; the cohomology

$$H^*(K \otimes_k \bar{k}, \mathbb{Q}_\ell)$$

is a $G(\bar{k}/k)$ -module. We have the functor to the category of Galois modules

$$\mathcal{D}(k) \rightarrow (G(\bar{k}/k) - \text{Vec}_{\mathbb{Q}_\ell}) .$$

§3. Polylogarithmic objects.

We define the category of mixed Tate motives to be the triangulated subcategory generated by $\mathbb{Q}(r)$, $r \in \mathbb{Z}$.

We give two types of algebraic cycles, each parametrized by $a \in k^*$. For $a \in k^*$, $f_a := \text{Alt}\{t = a\} \in \mathcal{Z}^1(\text{pt}, 1)$, the alternation of the cycle $\{t = a\}$. For $a \in k^* - \{1\}$ and $r \geq 2$, using cubical coordinates and parameters $t_1, \dots, t_{r-2}$,

$$V_a^r := [t_1, t_2, \dots, t_{r-1}, 1 - t_1, 1 - \frac{t_2}{t_1}, \dots, 1 - \frac{t_{r-1}}{t_{r-2}}]$$

and $C_a^r := \text{Alt} V_a^r \in \mathcal{Z}^r(\text{pt}, 2r - 1)$. This was considered by Totaro ($r = 2$) and Bloch ($r \geq 3$). Note

$$\partial C_a^r = \begin{cases} f_{1-a} \circ f_a & \text{if } r = 2 \\ C_a^{r-1} \circ f_a & \text{if } r \geq 3 \end{cases}$$

Define the object

$$L(a) := [(\text{pt}, 0) \xrightarrow{f_a} (\text{pt}, 1)]$$

where $(\text{pt}, 0)$, $(\text{pt}, 1)$ are placed in degrees 0 and 2, respectively. More precisely, $L(a) = (L^m, f^{m,n})$ consists of

$$L^{2m} = (\text{pt}, m) \quad \text{for } m = 0, 1, \quad \text{and } L^n = 0 \quad \text{otherwise;}$$

$$f^{0,2} = f_a, \quad f^{m,n} = 0 \quad \text{otherwise.}$$

For $p \geq 1$ and $a \in k^* - \{1\}$, define the object $K_p(a)$ (called the polylogarithmic object of weight p) by

$$K^{2m} = (\text{pt}, m) \quad \text{for } m = 0, 1, \dots, p, \quad K^n = 0 \quad \text{otherwise;}$$

$$f^{0,2} = f_{1-a}, \quad f^{2m, 2m+2} = f_a \quad \text{for } m = 1, \dots, p-1,$$

$$f^{0,2m} = C_a^m \quad \text{for } m = 2, \dots, p \quad \text{and } f^{m,n} = 0 \quad \text{otherwise.}$$

(3.1) Proposition. (1) $H^\nu(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$. There is an exact sequence

$$0 \rightarrow \mathbb{Q}_\ell(1) \rightarrow H^0(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell(0) \rightarrow 0$$

whose extension class is $[a] \in H^1(G(\bar{k}/k), \mathbb{Q}_\ell(1))$.

(2) $H^\nu(K_p(a) \otimes \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$; The cohomology $H^0(K_p(a) \otimes \bar{k}, \mathbb{Q}_\ell)$ has a filtration $W_\bullet$ (the weight filtration), $H^0 = W_0 \supset W_{-2} \supset \dots \supset W_{-2p} \supset W_{-2p-2} = 0$ such that $\text{Gr}_{-2q}^W = \mathbb{Q}_\ell(q)$ for $q = 0, \dots, p$ and the extension class of the exact sequence

$$0 \rightarrow \text{Gr}_{-2q-2}^W = \mathbb{Q}_\ell(q+1) \rightarrow W_{-2q}/W_{-2q-4} \rightarrow \text{Gr}_{-2q}^W = \mathbb{Q}_\ell(q) \rightarrow 0$$

is $[1-a]$ for $q = 0$ and $[a]$ for $q = 1, \dots, p-1$.

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