

LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

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§1. Introduction

Let L be a Lie group with finitely many components, K a maximal compact subgroup of L , and S a connected closed normal subgroup of L . Then KS is closed, and we have a fiber bundle

$$K \backslash KS \rightarrow K \backslash L \rightarrow KS \backslash L.$$

L acts on $K \backslash L$ by right multiplication. L acts also on $KS \backslash L$ by right multiplication; let N denote the kernel of this action, i.e., $N = \{ g \in L; KSxg = KSx \text{ for all } x \in L \}$. The action of N on $K \backslash L$ leaves all fibers invariant; in other words, we have a family of right N -spaces parametrized over $KS \backslash L$.

Lemma 1. The right N -spaces $K \backslash KSx$ ($x \in L$) are equivalent.

Proof. Since K is compact, $K \backslash L$ has an L -invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K \backslash KSx$ and $K \backslash KSy$ ($x, y \in L$). It suffices to construct an N -equivariant diffeomorphism from $K \backslash KSx$ onto $K \backslash KSy$ when they are sufficiently close to each other, because $KS \backslash L$ is connected.

Fix a point p of $K \backslash K S_x$ and let d be the distance between p and $K \backslash K S_y$. $K \backslash L$ is complete and $K \backslash K S_y$ is closed; therefore, d is positive and can be achieved as the length of a geodesic γ connecting p and a point q of $K \backslash K S_y$. S is contained in N and acts transitively on each fiber. The action of an element s of S sends γ to a geodesic $\gamma \cdot s$ of the same length d connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $K \backslash K S_x$ to $K \backslash K S_y$ is independent of the choice of the point, and γ is one of the shortest geodesic connecting $K \backslash K S_x$ and $K \backslash K S_y$. Therefore γ is perpendicular to $K \backslash K S_x$ at p . Let $(T_p(K \backslash K S_x))^{\perp}$ denote the orthogonal complement of the tangent space $T_p(K \backslash K S_x)$ of $K \backslash K S_x$ at p in the tangent space of $K \backslash L$ at p . As the exponential map Exp is a diffeomorphism near the origin, any fiber $K \backslash K S_z$ that meets $\text{Exp}(V)$ meets $\text{Exp}(V)$ exactly once, where V is a sufficiently small neighborhood in $(T_p(K \backslash K S_x))^{\perp}$ of the origin. This implies that γ is the unique geodesic of length d connecting p and $K \backslash K S_y$, as long as $K \backslash K S_y$ is sufficiently close to $K \backslash K S_x$. Let us suppose that this is the case. Then the correspondence $p \cdot s \mapsto q \cdot s$ ($s \in S$) defines a diffeomorphism $K \backslash K S_x \rightarrow K \backslash K S_y$, which is obviously N -equivariant because it sends a point in $K \backslash K S_x$ to the unique point closest to it in $K \backslash K S_y$ and N acts on $K \backslash L$ by isometries. $\square$

Remark. The N -equivariant diffeomorphism above defines a local trivialization of the fiber bundle $K \backslash L \rightarrow K S \backslash L$ so that the action of N on $K \backslash L$ is locally a product of the action of N on a fiber and the action of a trivial group on the base.

If G is a lattice of L , the action of L on $K \backslash L$ restricts

to an action of G on $K \backslash L$. $H = G \cap N$ is a normal subgroup of G which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K \backslash KS/H \rightarrow K \backslash L/H \rightarrow KS \backslash L.$$

The quotient group $\Gamma = G/H$ acts on $K \backslash L/H$ and $KS \backslash L$ such that $(K \backslash L/H)/\Gamma = K \backslash L/G$ and $(KS \backslash L)/\Gamma = KS \backslash L/G$; the fiber bundle map induces a map:

$$q: K \backslash L/G \rightarrow (KS \backslash L)/\Gamma.$$

Note that $KS \backslash L$ can be naturally identified with $(S \backslash KS) \backslash (S \backslash L)$, which has an $(S \backslash L)$ -invariant (and hence L -invariant) Riemannian metric. Thus Γ can be thought of as a subgroup of the group $I(KS \backslash L)$ of all the isometries of $KS \backslash L$ with respect to this Riemannian metric.

Suppose that Γ is discrete in $I(KS \backslash L)$. Then the isotropy subgroup Γ_v of Γ at $v \in KS \backslash L$ is finite for each v , and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS \backslash L)/\Gamma$ is $((K \backslash KSx)/H)/\Gamma_v$, where $v = KSx$ ($x \in L$). Thus a "fiber" of q is homeomorphic to a quotient of the "general fiber" $K \backslash KS/H$ by an action of a finite group; i.e., q is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup S .

Theorem 2. Let L be a non-compact Lie group with finitely many components, K a maximal compact subgroup of L , G a lattice of L . Then there is an orbifold Seifert fibration

$$K \backslash L / G \rightarrow O^m,$$

where O^m is a Riemannian orbifold of dimension $m > 0$ and of non-positive sectional curvature. If L is amenable, O^m can be chosen to be flat.

The proof will occupy the following two sections. Some special cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group L with finitely many components is amenable if and only if L/R is compact, where R denotes the radical (= the unique maximal connected normal solvable subgroup) of L . See Milnor [6]. In this section we handle the case when L is not amenable. We use R as S , following [4]; i.e., we are going to show that

$$K \backslash L / G \rightarrow KR \backslash L / G$$

is a Seifert fibration with the desired property. As in the previous section, identify $KR \backslash L$ with $(R \backslash KR) \backslash (R \backslash L) = \mathbb{R}^m$ ($m > 0$). $R \backslash L$ is a non-compact semi-simple Lie group, and $R \backslash KR$ is a maximal compact subgroup of $R \backslash L$. Using the Cartan decomposition and the Killing form, one can introduce an

$(R \setminus L)$ -invariant (and hence L -invariant) Riemannian metric g on $\mathbb{R}^m$ with non-positive sectional curvature. In fact, any $(R \setminus L)$ -invariant Riemannian metric on $\mathbb{R}^m$ has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism $\Phi: L \rightarrow I(\mathbb{R}^m, g)$. Let Γ denote the image $\Phi(G)$ of G . To prove the theorem, it suffices to show that Γ is discrete in $I(\mathbb{R}^m, g)$. Let τ denote the natural projection $L \rightarrow R \setminus L$. If the image $\tau(G)$ of G in $R \setminus L$ is discrete, then Γ is obviously discrete. Unfortunately $\tau(G)$ may not be discrete in general. We remedy this situation as follows.

Let L_0 denote the identity component of L . $G \cap L_0$ is a subgroup of G with finite index. Therefore it suffices to show that $\Phi(G \cap L_0)$ is discrete in $I(\mathbb{R}^m, g)$. As $(R \setminus (K \cap L_0)R) \setminus (R \setminus L_0)$ can be naturally identified with $\mathbb{R}^m$, we may assume from the beginning that L is connected.

Now there is a semi-simple Lie subgroup S of L such that $L = SR$ and such that $S \cap R$ is discrete (Levi decomposition). Let $\sigma: S \rightarrow \text{Aut}(R)$ denote the action of S on R . A sufficient condition for $\tau(G)$ to be discrete in $R \setminus L$ is that the identity component $(\ker \sigma)_0$ of the kernel of σ has no compact factors (Raghunathan[9], p.150). Let C denote the unique maximal compact normal subgroups of $(\ker \sigma)_0$. It is a characteristic subgroup of $(\ker \sigma)_0$, and hence it is normal in $\ker \sigma$ and in S . On the other hand, C commutes with elements of R . Therefore C is normal in L . Let $\pi: L \rightarrow L/C$ denote the natural projection, and let $L' = \pi(L)$, $S' = \pi(S)$, $R' = \pi(R)$, $G' = \pi(G)$, $K' = \pi(K)$. Then S is semi-simple, R' is the radical of L' , G' is a lattice of L' ,

and K' is a maximal compact subgroup of L' . Let $\sigma': S' \rightarrow \text{Aut} R'$ denote the action of S' on R' . Then it is easily observed that $\ker \sigma' = (\ker \sigma)/C$, since $C \cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image G'' of G' in $R' \backslash L'$ is discrete. Thus the action of G on $\mathbb{R}^m$ factors through a properly discontinuous action of G'' on $K'R' \backslash L = KR \backslash L$. Therefore, Γ is discrete in $I(\mathbb{R}^m, g)$. This completes the proof of theorem 2 when L is not amenable.

Remark. Let $q: K \backslash L/G \rightarrow KR \backslash L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRxG)$ over the point $KRxG \in KR \backslash L/G$ ($x \in L$) is homeomorphic to

$$(x^{-1}Kx) \backslash (x^{-1}KRx) / (x^{-1}KRx \cap G).$$

It is easily observed that $x^{-1}KRx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let L be a Lie group with finitely many components, K a maximal compact subgroup of L , R the radical of L , and G a lattice of L . Then $KR \cap G$ is a uniform lattice of KR .

§3. Amenable case

Now let us assume that L is non-compact and amenable. Let K be a maximal compact subgroup and R the radical of L as

before. Since L is amenable, $L = KR$.

We define a sequence $N^{(j)}$ ($j \geq -1$) of closed characteristic subgroups of L as follows:

- (1) $N^{(-1)}$ is the radical R ,
- (2) $N^{(0)}$ is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of L ,
- (3) $N^{(j)}$ is the commutator subgroup $[N^{(j-1)}, N^{(j-1)}]$ of $N^{(j-1)}$, for $j > 0$.

It may not be so obvious that $N^{(j)}$'s are closed when $j > 0$; in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer k such that $N^{(k)} = \{1\}$. Consider the following sequence:

$$L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \dots \supset KN^{(k)} = K.$$

There exists an integer i (≥ 0) such that

$$L = KN^{(-1)} = KN^{(0)} = \dots = KN^{(i-1)} \neq KN^{(i)},$$

because L is non-compact. Let us write $M = N^{(i-1)}$ and $N = N^{(i)}$. We introduce a flat L -invariant Riemannian metric on $KN \backslash L$.

Let us study the action of L on $KN \backslash L$ defined by right multiplication. An element ky of $KM = L$ ($k \in K$, $y \in M$) acts on an element KNx ($x \in M$) of $KN \backslash L$ as follows:

$$\begin{aligned} KNx \cdot (ky) &= KNxky \\ &= KN(k^{-1}xk)y. \end{aligned}$$

Note that we have $[M, M] \subset N$; we identify the coset space $KN \backslash L$ with the simply-connected abelian Lie group $(K \cap M)N \backslash M = \mathbb{R}^m$ ($m > 0$). Now the induced action of L on $\mathbb{R}^m$ is:

$$(K \cap M)N x \cdot (ky) = (K \cap M)N(k^{-1}xk)y.$$

The following are easily observed: (1) this action, when restricted to K , defines a homomorphism $\alpha: K \rightarrow \text{Aut}(\mathbb{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbb{R}^m$, and (2) if $k \in K \cap M$, then $(K \cap M)N x \cdot k = (K \cap M)N k^{-1}xk = (K \cap M)N(k^{-1}xkx^{-1})x = (K \cap M)N x$ for $x \in M$, and so $K \cap M$ acts trivially on $\mathbb{R}^m$. Let $\beta: M \rightarrow (K \cap M)N \backslash M$ denote the natural projection. We now define a map $\Phi: L = KM \rightarrow \alpha(K) \rtimes ((K \cap M)N \backslash M) \subset O(m) \rtimes \mathbb{R}^m = I(\mathbb{R}^m)$ by sending ky ($k \in K, y \in M$) to $(\alpha(k), \beta(y)) \in O(m) \rtimes \mathbb{R}^m$. This is a well-defined homomorphism. Here $\rtimes$'s denote the obvious semi-direct products. Let Γ denote the image of G by Φ in $I(\mathbb{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that Γ is a discrete subgroup of $I(\mathbb{R}^m)$. To do this we use the following lemma:

Lemma 4. If N is a connected nilpotent Lie group and H is a discrete cocompact subgroup of N , then the commutator subgroup $[N, N]$ is closed in N and $H \cap [N, N]$ is cocompact in $[N, N]$.

Proof: This is well-known if N is simply-connected, so consider the universal cover $p: U \rightarrow N$ of N ; it can be identified with the natural projection $U \rightarrow U/\Pi$, where Π is the kernel of

p . To see that $[N, N]$ is closed in N , it suffices to show that $N/[N, N]$ is Hausdorff. As $p^{-1}([N, N]) = \Pi[U, U]$, we have homeomorphisms:

$$\begin{aligned} N/[N, N] &\cong U/\Pi[U, U] \\ &\cong (U/[U, U]) / (\Pi[U, U]/[U, U]). \end{aligned}$$

Here $U/[U, U]$ is a Lie group, because U is simply-connected and hence its commutator subgroup $[U, U]$ is closed. Note that the preimage $p^{-1}(H)$ of H is discrete and cocompact in U . Since U is simply-connected, $p^{-1}(H) \cap [U, U]$ is cocompact in $[U, U]$. Therefore, the image $p^{-1}(H)[U, U]/[U, U]$ of $p^{-1}(H)$ by projection: $U \rightarrow U/[U, U]$ is discrete. As $\Pi \subset p^{-1}(H)$, $\Pi[U, U]/[U, U]$ is also discrete and hence closed in $U/[U, U]$. Therefore $(U/[U, U]) / (\Pi[U, U]/[U, U])$ is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

$$\begin{aligned} [N, N]/H \cap [N, N] &\cong \Pi[U, U] / p^{-1}(H) \cap \Pi[U, U] \\ &\cong [U, U] / p^{-1}(H) \cap [U, U], \end{aligned}$$

the second statement is obvious. $\square$

Now we prove

Lemma 5. $N^{(j)}$, s are closed subgroups of L , and Γ is a crystallographic subgroup of $I(\mathbb{R}^m)$.

Proof: If $G \cap R$ is cocompact in $R = N^{(-1)}$, then $G \cap N^{(0)}$ is a discrete cocompact subgroup of $N^{(0)}$ and we can apply lemma 4 to

prove that $N^{(j)}$'s are closed for $j \geq 1$. Unfortunately, $G \cap R$ may not be cocompact in R , in general. To remedy this situation we introduce a quotient Lie group L' of L as in the previous section. We may assume that L is connected. We have Levi decomposition $L = SR$, where S is a connected semi-simple (and hence compact) subgroup, R is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma: S \rightarrow \text{Aut}(R)$ denote the action of S on R . The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of L , because it commutes with elements of R . In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi: L \rightarrow L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then G' is a lattice of L' , K' is a maximal compact subgroup of L' , S' is a semi-simple subgroup of L' , R' is the radical of L' , and the action $\sigma': S' \rightarrow \text{Aut}(R')$ of S' on R' is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N'^{(j)}$ ($j \geq -1$) of characteristic subgroups of L' by:

- (1) $N'^{(-1)} = R'$
- (2) $N'^{(0)} = \text{the nil-radical of } L'$
- (3) $N'^{(j)} = [N'^{(j-1)}, N'^{(j-1)}]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N'^{(0)}$ are cocompact in R' and $N'^{(0)}$

respectively. By successively using lemma 4, we know that all $N'^{(j)}$'s are closed. Note that $\pi|_R: R \rightarrow R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi|_R)^{-1}(N'^{(j)})$ for each j . Therefore $N^{(j)}$'s are closed in L .

Next, we show that Γ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. Note that we have

$$L' = K'N'^{(-1)} = K'N'^{(0)} = \dots = K'N'^{(i-1)} \neq K'N'^{(i)}$$

for the same i and that $K'N' \setminus K'M' = KN \setminus KM$, where $M' = N'^{(i-1)}$, $N' = N'^{(i)}$. $G' \cap N'^{(j)}$ is cocompact in $N'^{(j)}$ for all j . In particular $G' \cap M'$ is cocompact in M' . So the image of G' in $M' \setminus L'$ is discrete; furthermore, it is finite, because $M' \setminus L'$ is compact. Looking at the diagram:

$$\begin{array}{ccccc}
 & G & \xrightarrow{\quad\quad\quad} & \text{finite} & \\
 & \cap & & \cap & \\
 \pi^{-1}(M') & \xrightarrow{\quad\quad\quad} & L & \xrightarrow{\quad\quad\quad} & \pi^{-1}(M') \setminus L \\
 \downarrow & & \downarrow & \pi & \downarrow \cong \\
 M' & \xrightarrow{\quad\quad\quad} & L' & \xrightarrow{\quad\quad\quad} & M' \setminus L' \\
 & & \cup & & \cup \\
 & & G' & \xrightarrow{\quad\quad\quad} & \text{finite}
 \end{array}$$

we know that $G \cap \pi^{-1}(M')$ has a finite index in G . So it suffices to show that the image $\Phi(G \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. As $\ker \sigma \subset \ker \alpha$, Φ sends elements in $\pi^{-1}(M') = (\ker \sigma)_0 M$ to elements in $\mathbb{R}^m \subset I(\mathbb{R}^m)$. Now consider the following commutative diagram:

$$\begin{array}{ccc}
 & \xrightarrow{\quad\quad\quad \Phi \quad\quad\quad} & O(m) \times \mathbb{R}^m \\
 \begin{array}{c} L \\ \cup \\ G \cap \pi^{-1}(M') \subset \pi^{-1}(M') = (\ker \sigma)_0 M \end{array} & & \begin{array}{c} \xrightarrow{\quad\quad\quad \Phi \quad\quad\quad} \\ \mathbb{R}^m = (KN) \setminus M \end{array}
 \end{array}$$

$$\begin{array}{ccccc}
 \downarrow & & \downarrow \pi & & \downarrow (\pi|_M)_* \\
 G' \cap M' & \subset & M' & \xrightarrow{\Phi'} & (K' \cap M') N' \setminus M'
 \end{array}$$

where Φ' is the natural map and $(\pi|_M)_*$ is the map induced by the restriction of π to M , $\pi|_M: M \rightarrow M'$. $K \cap M$ and $K' \cap M'$ are maximal compact subgroups of M and M' , respectively, and $\pi(K \cap M) = K' \cap M'$; therefore, $(\pi|_M)^{-1}(K' \cap M') = K \cap M$. Using this, it is easily verified that $(\pi|_M)^{-1}((K' \cap M') N') = (K \cap M) N$. Therefore $(\pi|_M)_*$ is an isomorphism. Since $(G' \cap M') \cap N' = G' \cap N'$ is cocompact in N' , $(G' \cap M') \cap (K' \cap M') N'$ is cocompact in $(K' \cap M') N'$; so $\Phi'(G' \cap M')$ is a discrete cocompact subgroup of $(K' \cap M') N' \setminus M'$. Therefore $\Phi(G \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $\mathbb{R}^m$ (and hence in $I(\mathbb{R}^m)$). This completes the proof of lemma 5. $\square$

Thus $K \backslash L / G \rightarrow K N \backslash L / G$ is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form $K \backslash K N x G / G$, and is homeomorphic to

$$(x^{-1} K x) \backslash (x^{-1} K N x) / (x^{-1} K N x \cap G).$$

If G is a lattice of L (which is automatically uniform), then $x^{-1} K N x \cap G$ is a uniform lattice of $x^{-1} K N x$.

§4. A rational computation of Wall's L -groups

Let L be an amenable Lie group with finitely many components, K a maximal compact subgroup of L , and G a uniform lattice of L . Such a discrete group G is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the L -groups of G in terms of certain generalized homology of $K \backslash L/G$.

$K \backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_y = x^{-1}Kx \cap G$ of G at $y=Kx$ ($x \in L$) is finite. The action of G on $\mathbb{R}^n$ is free if G is torsion-free; in general, $\mathbb{R}^n/G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let WG be a contractible free G -complex, and p denote the projection: $(\mathbb{R}^n \times WG)/G \rightarrow \mathbb{R}^n/G$, where G acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y] \in \mathbb{R}^n/G$ by p is homeomorphic to WG/G_y , and p is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^\infty(G)$ denote the limit of Ranicki's lower L -groups $L^{(-j)}(\mathbb{Z}G)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $L^\infty(-)$ from the category of spaces to the category of Ω -spectra such that the homotopy group of $L^\infty(X)$ is equal to $L_*^\infty(\pi_1 X)$. Applying $L^\infty(-)$ to each fiber of p , we obtain a sheaf of spectra, denoted $L^\infty(p)$. F. Quinn defines the homology group $H_*(\mathbb{R}^n/G; L^\infty(p))$ of $\mathbb{R}^n/G$ with coefficients $L^\infty(p)$. See [7], [10]. The following is a rational computation of $L_*^\infty(G)$ in terms of this homology.

Theorem 6. Let G be as above, then there is a natural isomorphism

$$H_*(\mathbb{R}^n/G; \mathbb{L}^{-\infty}(p)) \otimes \mathbb{Z}[1/2] \rightarrow L_*^{-\infty}(G) \otimes \mathbb{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations.

$$\begin{array}{ccc} (\mathbb{R}^n \times WG)/G & \xrightarrow{\text{id.}} & (\mathbb{R}^n \times WG)/G \\ \downarrow p & & \downarrow \\ \mathbb{R}^n/G & \xrightarrow{\quad} & \text{pt.} \end{array}$$

Note that $(\mathbb{R}^n \times WG)/G = BG$ is a classifying space for G and that $H_*(\text{pt.}; \mathbb{L}^{-\infty}(BG \rightarrow \text{pt.})) = L_*^{-\infty}(G)$ [10].

It is to be noted that theorem 6 says that the $\mathbb{L}^{-\infty}(p)$ coefficient homology of $\mathbb{R}^n/G$ is independent (modulo 2 torsion) of the action of G on $\mathbb{R}^n$. It is conceivable that the orbifold $\mathbb{R}^n/G$ has a certain strong rigidity.

Proof of theorem 6. The proof is by induction on the dimension n of $K \setminus L$. Let $q: \mathbb{R}^n/G \rightarrow \mathbb{R}^m/\Gamma$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$\begin{aligned} H_*(\mathbb{R}^n/G; \mathbb{L}^{-\infty}(p)) & \\ \cong H_*(\mathbb{R}^m/\Gamma; \bigcup_{w \in \mathbb{R}^m/\Gamma} H(q^{-1}(w); \mathbb{L}^{-\infty}(p|_{q^{-1}(w)})) & \\ \cong H_*(\mathbb{R}^m/\Gamma; \bigcup_w \mathbb{L}^{-\infty}((qp)^{-1}(w))) & \end{aligned}$$

$$= H_*(\mathbb{R}^m/\Gamma; \mathbb{L}^{-\infty}(qp))$$

by induction hypothesis, where H denote the homology theory spectrum [ibid.]. We can prove that $H_*(\mathbb{R}^m/\Gamma; \mathbb{L}^{-\infty}(qp)) \otimes \mathbb{Z}[1/2]$ is naturally isomorphic to $L_*^{-\infty}(G)$ using the proof of the main theorem of [ibid.] with only some obvious modifications, and this completes the proof of theorem 6. $\square$

Corollary 7. (Novikov Conjecture) Let G be as above, then the assembly map

$$H_*(BG; \mathbb{L}^{-\infty}(1)) \rightarrow L_*^{-\infty}(G)$$

is rationally split injective.

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