

ON AUSLANDER-REITEN QUIVERS OF FINITE GROUPS

大阪市立大学理学部 河田成人 (Shigeto KAWATA)

1. Introduction

Let G be a finite group and k a field of characteristic $p > 0$. Let $\Gamma_s(kG)$ be the stable Auslander-Reiten quiver of the group algebra kG . By Webb's Theorem, the tree class of a connected component Δ of $\Gamma_s(kG)$ is restricted. We summarize results from [W, O1, Bt1, E-S] on the graph structure of connected components of $\Gamma_s(kG)$.

Theorem 1.1([W], [O1], [Bt1], [E-S]). Let Δ be a connected component of $\Gamma_s(kG)$. Then the tree class of Δ is A_n , $\tilde{A}_{1,2}$, $\tilde{B}_3$, A_∞ , B_∞ , C_∞ , D_∞ or A_∞^∞ . If k is algebraically closed, then the tree class is not B_∞ or C_∞ . Moreover if the tree class or the reduced graph of Δ is Euclidean, then the modules in Δ lie in a block whose defect group is a Klein four group $C_2 \times C_2$.

Moreover if Δ contains the trivial kG -module k , then the graph structure of Δ has been investigated [W, L, O1, E2].

Theorem 1.2([W], [L], [O1], [E2]). Let Δ_0 be the connected component containing the trivial kG -module k and T the tree class of Δ_0 . Let P be a Sylow p -subgroup of G . Then;

- (1) If P is cyclic, then $T = A_n$ for some n .
- (2) If $P = C_2 \times C_2$ and $N_G(P) = C_G(P)$, then $T = \tilde{A}_{1,2}$.
- (3) If $P = C_2 \times C_2$ and $N_G(P) \neq C_G(P)$ but k does not contain a primitive cube root of unity, then $T = \tilde{B}_3$.
- (4) If P is a dihedral 2-group and neither (2) nor (3) holds, then $T = A_\infty$. Moreover if P is dihedral of order at least 8, then $\Delta_0 \cong ZA_\infty$.
- (5) If P is a semidihedral 2-group, then $T = D_\infty$ and $\Delta_0 \cong ZD_\infty$.
- (6) If P is a generalized quaternion 2-group, then $T = A_\infty$ and Δ_0 is a 2-tube.
- (7) $T = A_\infty$ and $\Delta_0 \cong ZA_\infty$ otherwise.

Here we study a connected component of $\Gamma_s(kG)$ containing an indecomposable kG -module whose k -dimension is not divided by p . Suppose that M is an indecomposable kG -module and $p \nmid \dim_k M$. In Section 2, we will show that M lies in a connected component isomorphic to ZA_∞ if k is an algebraically closed field of odd characteristic and a Sylow p -subgroup of G is not cyclic. In Sections 3 and 4 we consider the situation where $p = 2$ and a Sylow 2-subgroup of G is dihedral of order at least 8 or semidihedral. In Section 5 we make some remarks on tensoring the component containing the trivial kG -module k with M .

The notation is almost standard. For an indecomposable non-projective kG -module W , we write $A(W)$ to denote the Auslander-Reiten sequence (AR-sequence) $0 \rightarrow \Omega^2 W \rightarrow m(W) \rightarrow W \rightarrow 0$

terminating at W , where Ω is the Heller operator. The symbol $\otimes$ denotes tensor product over the coefficient field k . For an exact sequence of kG -modules $S : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and a kG -module W , we write $S \otimes W$ to denote the tensor sequence $0 \rightarrow A \otimes W \rightarrow B \otimes W \rightarrow C \otimes W \rightarrow 0$. For tensoring the AR-sequence with an indecomposable kG -module, see [A-C, B-C]. If an exact sequence of kG -modules S is of the form $0 \rightarrow \Omega^2 W \oplus U' \rightarrow m(W) \oplus U \oplus U' \rightarrow W \oplus U \rightarrow 0$, where W is an indecomposable non-projective kG -module, and U and U' are projective or 0, we say that S is the AR-sequence $A(W)$ *modulo projectives*. Concerning some basic facts and terminologies used here, we refer to [Bn], [F] and [G].

2. ZA_∞ -Component

Throughout this section, we assume that

(#2) k is algebraically closed and a Sylow p -subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

First of all, we show

Theorem 2.1. Suppose that Θ is a connected component of $\Gamma_s(kG)$ containing an indecomposable kG -module whose k -dimension is not divided by p . Then

- (1) Θ is isomorphic to ZA_∞ or ZD_∞ .
- (2) If p is odd, then Θ is isomorphic to ZA_∞ .

(3) All modules in Θ have the same vertex P .

Remark. The above (3) follows from [U, Theorem 4.3].

Let M be an indecomposable kG -module with a Sylow p -subgroup P of G as vertex, and let S be a P -source of M . Then $p \nmid \dim_k M$ if and only if $p \nmid \dim_k S$ from [B-C, Proposition 2.4].

Proposition 2.2. Let M be an indecomposable kG -module such that $p \nmid \dim_k M$, and let S be a P -source of M . Let Θ be the connected component of $\Gamma_s(kG)$ containing M , and let Ξ be the connected component of $\Gamma_s(kP)$ containing S . Then

- (1) Θ is isomorphic to ZA_∞ if and only if Ξ is isomorphic to ZA_∞ .
- (2) M lies at the end of ZA_∞ -component if and only if S lies at the end of ZA_∞ -component.
- (3) Suppose that Θ is isomorphic to ZA_∞ and M lies at the end of Θ . Let $M = M_2 \cdots M_n \cdots$ is a maximal tree of Θ with an irreducible map $M_{n+1} \rightarrow M_n$ ($n \geq 1$). Then there is a P -source S_n of M_n ($n \geq 2$) such that $S = S_2 \cdots S_n \cdots$ is a maximal tree of Ξ with an irreducible map $S_{n+1} \rightarrow S_n$ ($n \geq 1$).

Now we give examples of indecomposable kG -modules lying at the ends of ZA_∞ -components.

Proposition 2.3. Let M be an indecomposable kG -module whose k -dimension is not divided by p . Let Q be a proper subgroup of P . Suppose that M satisfies the following conditions (with respect to Q);

(1) The trivial kQ -module k is a direct summand of $(M \otimes M) \downarrow_Q$ with multiplicity one;

(2) If Q is generalized quaternion, then $\Omega^2 k \nmid (M \otimes M) \downarrow_Q$.

Then M lies at the end of ZA_∞ -component.

Remark. The above condition (1) is equivalent to the following condition: (1') We have an indecomposable direct sum decomposition $N \oplus (\oplus_i W_i)$ of $M \downarrow_Q$, where $p \nmid \dim_k N$ and $p \mid \dim_k W_i$ for all i .

From Proposition 2.3, we have following

Example 2.4. (1) Suppose that p is odd. Let M be an indecomposable kG -module with vertex P and S a P -source of M . Suppose that $\dim_k S = 2$. Then M lies at the end of ZA_∞ -component.

(2) Suppose that $p \neq 3$. Let M be an indecomposable kG -module with vertex P and S a P -source of M . Suppose that $\dim_k S = 3$. Then M lies at the end of ZA_∞ -component.

Proof. There exists an element x of P such that x does not act on S trivially. Let $Q = \langle x \rangle$. Then S satisfies the conditions (with respect to Q) in Proposition 2.3.

Remark. In [E3], Erdmann proved that if k is algebraically closed and a p -group P is not cyclic, dihedral, semidihedral or generalized quaternion, then there are infinitely many kP -modules of dimension 2 or 3 lying at the ends of ZA_∞ -components ([E3], Propositions 4.2 and 4.4.). Using this result, she consequently showed that for a block B over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components of the

form ZA_∞ if a defect group of B is not cyclic, dihedral, semidihedral or generalized quaternion.

3. Dihedral 2-group

In this section we consider the following situation:

(#3) k is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup P of G is dihedral of order at least 8.

Let Δ_0 be the connected component containing the trivial kG -module k . Then Δ_0 is isomorphic to ZA_∞ by Theorem 1.2. It is known that all modules in Δ_0 are endotrivial kG -modules (see, e.g., [Bt2]). Hence the following holds.

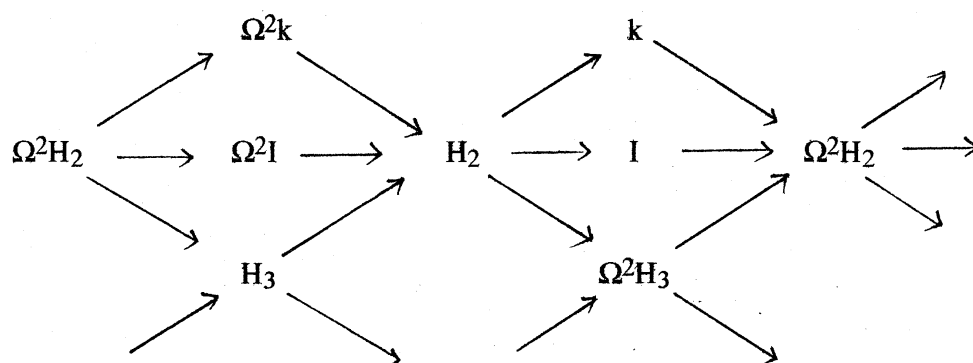
Proposition 3.1. Assume (#3). Let M be an odd dimensional indecomposable kG -module. Let Θ be the connected component of $\Gamma_s(kG)$ containing M and Δ_0 the connected component containing k . Then Θ is isomorphic to ZA_∞ and tensoring with M induces a graph isomorphism from Δ_0 onto Θ . Moreover all modules in Θ have the same vertex P .

4. Semidihedral 2-group

Throughout this section, we assume that

(#4) k is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup P of G is semidihedral.

Let Λ_0 be the connected component of $\Gamma_s(kP)$ containing the trivial kP -module k . Then Λ_0 is isomorphic to ZD_∞ (see [E2, p.76, II. 10.7 Remark]). Thus a part of Λ_0 is as follows for some indecomposable kG -modules H_2, H_3 and I .



Let $P = \langle x, y ; x^2 = y^{2^n-1} = 1, y^x = y^{-1+2^{n-2}} \rangle$ and $\mathfrak{K} = \{\langle x \rangle\}$. Then an $\mathfrak{K}$ -projective cover resolution of k is $0 \rightarrow \Omega_{\mathfrak{K}} k \rightarrow (k \downarrow_{\langle x \rangle})^{\uparrow P} \rightarrow k \rightarrow 0$, where $(k \downarrow_{\langle x \rangle})^{\uparrow P} \rightarrow k$ is a canonical epimorphism and $\Omega_{\mathfrak{K}} k$ is its kernel. Concerning some basic facts on relative projective cover, we refer to [Kn, T, O2].

In [O2], Okuyama showed the following

Theorem 4.1[O2]. With the same assumption and notations as above,

- (1) $I \cong \Omega(\Omega_{\mathfrak{K}} k)$ and I is an endotrivial kP -module.
- (2) I is self-dual and odd dimensional.
- (3) If I' is self-dual, odd dimensional and indecomposable, then

$I' \cong k$ or I .

Applying Theorem 4.1, we have

Lemma 4.2. Let S be an odd dimensional indecomposable kP -module. Then $S \not\cong S \otimes I$.

If S is an odd dimensional indecomposable kP -module, then the projective-free part S' of $S \otimes I$ is odd dimensional indecomposable and $S \not\cong S'$ by Theorem 4.1 and Lemma 4.2. Moreover it follows that the projective-free part of $S \otimes H_2$ is indecomposable. Therefore the following holds.

Proposition 4.3. Let S be an odd dimensional indecomposable kP -module and Ξ the connected component of $\Gamma_s(kP)$ containing S . Then

- (1) Ξ is isomorphic to ZD_∞ .
- (2) All indecomposable kP -modules in Ξ have the same vertex P .

Remark. The above (2) follows from [E1, Theorem A].

Let $k \text{---} H_2 \text{---} H_3 \text{---} \cdots \text{---} H_n \text{---} \cdots$ be a maximal tree of Λ_0 .

$$\begin{array}{c} | \\ I \end{array}$$

If S is an odd dimensional indecomposable kG -module, then the projective-free part S_n of $H_n \otimes S$ is indecomposable and the tensor sequence $A(H_n) \otimes S$ is the AR-sequence $A(S_n)$ modulo projectives. Hence the following holds.

Lemma 4.4. Let S be an odd dimensional indecomposable kP -module and Ξ the connected component of $\Gamma_s(kP)$ containing S . Then tensoring with S induces a graph isomorphism from Δ_0 onto Ξ .

Using [Ka1, Theorem and Ka2, Theorem], we obtain

Proposition 4.5. Let M be an odd dimensional indecomposable kG -module and Θ the connected component containing M . Let Δ_0 be the connected component containing the trivial kG -module k . Then

- (1) Θ is isomorphic to ZD_∞ and tensoring with M induces a graph isomorphism from Δ_0 onto Θ .
- (2) All indecomposable kG -modules in Θ have the same vertex P .

5. Remarks on tensoring with a certain module

Suppose that M is an indecomposable kG -module and $p \nmid \dim_k M$. Let Θ be the connected component of $\Gamma_s(kG)$ containing M and Δ_0 the connected component containing the trivial kG -module k . If a Sylow p -subgroup P of G is dihedral of order at least 8 or semidihedral, then tensoring with M induces a graph isomorphism from Δ_0 onto Θ as we have seen in Propositions 3.1 and 4.5.

In this section we consider on tensoring modules in Δ_0 with M under the same hypothesis as in Section 2. Throughout this section, we assume that

(#2) k is algebraically closed and a Sylow p -subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

Hence the connected component Δ_0 of $\Gamma_s(kG)$ containing the trivial kG -module k is of the form ZA_∞ by Theorem 1.2.

Proposition 5.1. Suppose that M is indecomposable kG -module and $p \nmid \dim_k M$. Let Θ be the connected component of $\Gamma_s(kG)$ containing M . Let S be a P -source of M and Ξ the connected component of $\Gamma_s(kP)$ containing S . Suppose that Θ is isomorphic to ZA_∞ and M lies at the end of Θ . Then the following are equivalent.

- (1) Tensoring with M induces a graph isomorphism from Δ_0 onto Θ .
- (2) Tensoring with S induces a graph isomorphism from the connected component of $\Gamma_s(kP)$ containing the trivial kP -module k onto Ξ .

Note that the hypothesis of Proposition 5.1 implies that $\Xi \cong ZA_\infty$ and S lies at the end of Ξ by Proposition 2.2.

Example 5.2. Let M be a trivial source module with vertex P . Let Θ be the connected component of $\Gamma_s(kG)$ containing M . Then Θ is isomorphic to ZA_∞ and M lies at the end of Θ . Moreover tensoring with M induces a graph isomorphism from Δ_0 onto Θ .

We consider an indecomposable kG -module M lying at the end of its connected component Θ isomorphic to ZA_∞ . In the following,

we give conditions which imply that tensoring with M induces a graph isomorphism from Δ_0 onto Θ .

Proposition 5.3. Let M be an indecomposable kG -module with $p \nmid \dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M . Suppose that M lies at the end of Θ and $M \otimes M^* \cong k \oplus (\oplus_i W_i)$, where each W_i is indecomposable and $p \mid \dim_k W_i$. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ .

Example 5.4. Suppose that M is an endotrivial kG -module. Let Θ be the connected component containing M . Then M satisfies the condition in Proposition 5.5. Hence tensoring with M induces a graph isomorphism from Δ_0 onto Θ .

Remark. Without the assumption (#2), if M is an endotrivial kG -module, then tensoring with M induces a graph isomorphism from the connected component containing the trivial kG -module onto the connected component containing M (Bt2, Theorem 2.3]). For related results on endotrivial modules, see also [Bt2].

Proposition 5.5. Let M be an indecomposable kG -module with $p \nmid \dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M . Let Q be a proper subgroup of P . Suppose that M satisfies the conditions (with respect to Q) in Proposition 2.3. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ .

Example 5.6. (1) Suppose that p is odd. Let M be an indecomposable kG -module with vertex P and S a P -source of M . Suppose that $\dim_k S = 2$. Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M .

(2) Suppose that $p = 2$. Let M be an indecomposable kG -module with vertex P and S a P -source of M . Suppose that $\dim_k S = 3$. Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M .

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