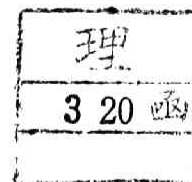


學位申請論文

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On the homotopy types of the groups of equivariant diffeomorphisms

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§0. Introduction

The purpose of this paper is to study the homotopy type of the group of the equivariant diffeomorphisms of a closed connected smooth G -manifold M , when G is a compact Lie group and the orbit space M/G is homeomorphic to a unit interval $[0,1]$.

Let $\text{Diff}_G^\infty(M)_0$ denote the group of equivariant C^∞ diffeomorphisms of the G -manifold M which are G -isotopic to the identity, endowed with C^∞ topology. If M/G is homeomorphic to $[0,1]$, then M has two or three orbit types G/H , G/K_0 and G/K_1 . We can choose the isotropy subgroups H , K_0 , K_1 satisfying $H \subset K_0 \cap K_1$. Moreover the G -manifold structure of M is determined by an element η of a factor group $N(H)/H$, where $N(H)$ is the normalizer of H in G (see §1). Let $\Omega(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(\eta K_1 \eta^{-1}))/H)_0$ denote the connected component of the identity of the space of paths $a: [0,1] \rightarrow N(H)/H$ satisfying $a(0) \in (N(H) \cap N(K_0))/H$ and $a(1) \in (N(H) \cap N(\eta K_1 \eta^{-1}))/H$.

Theorem. $\text{Diff}_G^\infty(M)_0$ has the same homotopy type as the path space $\Omega(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(\eta K_1 \eta^{-1}))/H)_0$.

The paper is organized as follows. In §1, we study the G -manifold structure of M and give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M . This differentiable structure is important to study the structure of $\text{Diff}_G^\infty(M)_0$. In §2, we define a group homomorphism $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]_0$ and prove that P is a continuous homomorphism between topological groups. In §3, we prove that there exists a global

continuous section of P and $\text{Ker } P$ is a deformation retract of $\text{Diff}_G^\infty(M)_0$.

In §4, we study the group structure of $\text{Ker } P$. In §5 and §6, we
prove our Theorem.

§1. G-manifold structure of M and the functional structure of M/G.

In this paper we assume that all manifolds and all actions are differentiable of class C^∞ .

In this section we study the G-manifold structure of M. First we see that it is sufficient for us to consider $n=1$ (see Lemma 1.1). Next we give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M (see Lemma 1.2).

Let M be a closed connected smooth G-manifold such that M/G is homeomorphic to $[0,1]$. We denote this homeomorphism by f . Let $\pi: M \rightarrow M/G$ be the natural projection. Put $M_0 = (f \circ \pi)^{-1}([0, 1/2])$ and $M_1 = (f \circ \pi)^{-1}([1/2, 1])$. Let x_i be a point of M with $f(\pi(x_i)) = i$ for $i = 0, 1$. Then M_i is a G-invariant closed tubular neighborhood of the orbit $G(x_i)$ (c.f. G. Bredon [3, Chapter VI, §6]). Moreover M is equivariantly diffeomorphic to a union of the G-manifolds M_0 and M_1 such that their boundaries are identified under a G-diffeomorphism $\eta': \partial M_0 \rightarrow \partial M_1$. Let V_i be a normal vector space of $G(x_i)$ at x_i and K_i be the isotropy subgroup of x_i for $i = 0, 1$. Then V_i is a representation space of K_i . From the differentiable slice theorem, M_i is equivariantly diffeomorphic to a smooth G-bundle $G \times_{K_i} D(V_i)$ which is associated to the principal K_i bundle $\pi_i: G \rightarrow G/K_i$, where $D(V_i)$ is a unit disc in V_i .

Let H be a principal isotropy subgroup of the G-manifold M. We can assume that H is a subgroup of $K_0 \cap K_1$. Let $e_i \in S(V_i)$ be a point such that the isotropy subgroup of e_i is H for $i = 0, 1$, where $S(V_i)$ is a unit sphere in V_i . There exists a G-diffeomorphism $h_i: G/H \rightarrow G \times_{K_i} S(V_i)$ given by $h_i(gH) = [g, e_i]$, $i = 0, 1$. Then we have a G-diffeomorphism

$$\eta'' : G/H \xrightarrow{h_0} G \times_{K_0} S(V_0) = \partial M_0 \xrightarrow{\eta'} \partial M_1 = G \times_{K_1} S(V_1) \xrightarrow{h_1^{-1}} G/H.$$

Since any G-map $G/H \rightarrow G/H$ is given by a right translation of an element

of $N(H)/H$, η'' defines an element $\eta \in N(H)/H$.

Put $x'_1 = \eta \cdot x_1$. Then the isotropy subgroup K'_1 of x'_1 is $\eta K_1 \eta^{-1}$. Let V'_1 be a normal vector space of the orbit $G(x'_1) = G(x_1)$ at x'_1 . Put $e'_1 = (d\eta)_{x_1}(e_1) \in S(V'_1)$. There exists a G -diffeomorphism $u: G \times_{K_1} D(V_1) \rightarrow G \times_{K'_1} D(V'_1)$ given by $u([g, v]) = [g\eta^{-1}, \eta \cdot v]$. Then $(u \circ \eta')([g, e_0]) = u([g\eta, e_1]) = [g, e'_1]$ for $[g, v] \in G \times_{K_0} S(V_0)$. Therefore M is equivariantly diffeomorphic to a union of the G -bundles $G \times_{K_0} D(V_0)$ and $G \times_{K'_1} D(V'_1)$ such that their boundaries are identified under a G -diffeomorphism $u \circ \eta'$. Now we have:

Lemma 1.1. Let M be a closed connected smooth G -manifold such that M/G is homeomorphic to $[0, 1]$. Then M has two or three orbit types G/H , G/K_0 and G/K_1 with $H \subset K_0 \cap K_1$, and there exist representation spaces V_i , $i = 0, 1$, of K_i such that M is equivariantly diffeomorphic to a union of G -bundles $G \times_{K_0} D(V_0)$ and $G \times_{K_1} D(V_1)$ with their boundaries identified under a G -diffeomorphism $\eta: G \times_{K_0} S(V_0) \rightarrow G \times_{K_1} S(V_1)$. Moreover we may assume that $\eta([g, e_0]) = [g, e_1]$, where e_i is a point of $S(V_i)$ such that the isotropy subgroup of e_i is H for $i = 0, 1$.

Hereafter we shall assume that M is a G -manifold as in Lemma 1.1. Let $\xi: [0, 1] \rightarrow \mathbb{R}$ be a smooth function such that

$$\begin{aligned}\xi(r) &= r^2 \quad \text{for } 0 \leq r \leq 1/2, \\ \xi'(r) &> 0 \quad \text{for } 0 < r \leq 1 \text{ and} \\ \xi(r) &= r - 1/2 \quad \text{for } 7/8 < r \leq 1.\end{aligned}$$

Let $\theta: M = G \times_{K_0} D(V_0) \cup_{\eta} G \times_{K_1} D(V_1) \rightarrow [0, 1]$ be a map given by

$$\begin{aligned}\theta([g, v]) &= \xi(\|v\|) & \text{for } [g, v] \in G \times_{K_0} D(V_0), \\ \theta([g, v]) &= 1 - \xi(\|v\|) & \text{for } [g, v] \in G \times_{K_1} D(V_1).\end{aligned}$$

Since θ is a G -map, there exists a map $\phi: M/G \rightarrow [0, 1]$ such that $\phi \circ \pi = \theta$. It is easy to see that ϕ is a homeomorphism. We give a differentiable structure of M/G by ϕ .

Lemma 1.2. In the above situation, we have

(1) θ is a smooth map,

(2) there exists a G -diffeomorphism $\alpha: \theta^{-1}((0,1)) \rightarrow G/H \times (0,1)$ such that $\theta \circ \alpha^{-1}$ is the projection on the second factor, and

(3) $f: M/G \rightarrow R$ is smooth if and only if $f \circ \pi: M \rightarrow R$ is smooth.

Proof. (1) Let $\alpha_i: G \times_{K_i} (D(V_i) - 0) \rightarrow G/H \times (0,1]$ be a map given by $\alpha_i([g, re_i]) = (gH, r)$ for $g \in G$ and $r \in (0,1]$ ($i=0,1$). Then it is easy to see that α_i is a G -diffeomorphism. Since $\alpha_1 \circ \eta = \alpha_0$ on $G \times_{K_0} S(V_0)$, the composition $\beta: \theta^{-1}((0,1)) = G \times_{K_0} (D(V_0) - 0) \bigcup_{\eta} G \times_{K_1} (D(V_1) - 0) \xrightarrow{\alpha_0 \cup \alpha_1} G/H \times (0,1] \bigcup_{1_{G/H} \times 1} G/H \times (0,1] = G/H \times (0,2]$ is a G -diffeomorphism. Note that

$$(\theta \circ \beta^{-1})(gH, r) = \begin{cases} \xi(r) & \text{for } 0 < r \leq 1, \\ 1 - \xi(2-r) & \text{for } 1 \leq r < 2. \end{cases}$$

Thus $\theta \circ \beta^{-1}$ is a smooth map, and θ is a smooth map on $\theta^{-1}((0,1))$. From the definition, θ is a smooth map on $\theta^{-1}(r)$ for $r \neq 1/2$. Therefore θ is a smooth map.

(2) Let $\bar{\theta}: (0,2) \rightarrow (0,1)$ be a smooth map given by

$$\bar{\theta}(r) = \begin{cases} \xi(r) & \text{for } 0 < r \leq 1, \\ 1 - \xi(2-r) & \text{for } 1 \leq r < 2. \end{cases}$$

Since $\bar{\theta}'(r) > 0$ for $0 < r < 2$, $\bar{\theta}$ is a diffeomorphism. Let $\alpha: \theta^{-1}((0,1)) \rightarrow G/H \times (0,1)$ be a G -diffeomorphism given by $\alpha = (1, \bar{\theta}) \circ \beta$. Then $(\theta \circ \alpha^{-1})(gH, r) = (\theta \circ \beta^{-1})(gH, \bar{\theta}^{-1}(r)) = r$, and $\theta \circ \alpha^{-1}$ is the projection on the second factor.

(3) Let $f: M/G \rightarrow R$ be a function such that $f \circ \pi: M \rightarrow R$ is smooth. We shall prove that $f \circ \phi^{-1}: [0,1] \rightarrow R$ is smooth. Since

$$(f \circ \pi \circ \alpha^{-1})(gH, r) = (f \circ \phi^{-1} \circ \theta \circ \alpha^{-1})(gH, r) = (f \circ \phi^{-1})(r) \text{ for } 0 < r < 1,$$

$f \circ \phi^{-1}$ is smooth on $(0,1)$. Let $i_0: D_{1/2}(V_0) = \{v \in D(V_0); \|v\| \leq 1/2\} \rightarrow G \times_{K_0} D(V_0)$ be an inclusion given by $i_0(v) = [1, v]$. Note that $(\theta \circ i_0)(v) = \|v\|^2$ for $v \in D_{1/2}(V_0)$. By Corollary 5.3 of G. Bredon [3, Chapter VI, §5], $f \circ \phi^{-1}$ is smooth if and only if $(f \circ \phi^{-1}) \circ (\theta \circ i_0)$ is

smooth. Since $(f \circ \phi^{-1}) \circ (\theta \circ i_0) = f \circ \pi \circ i_0$, which is smooth, then $f \circ \phi^{-1}$ is smooth on $[0, 1/4]$. Similarly we can prove that $f \circ \phi^{-1}$ is smooth on $[3/4, 1]$. Since $(f \circ \phi^{-1})(r) = (f \circ \phi^{-1} \circ \theta \circ \alpha^{-1})(1H, r) = (f \circ \pi \circ \alpha^{-1})(1H, r)$ for $0 < r < 1$, $f \circ \phi^{-1}$ is smooth on $(0, 1)$. This completes the proof of Lemma 1.2.

Remark 1.3. Lemma 1.2 is essentially proved by G. Bredon [3, Chapter VI, §5], and (3) implies that the functional structure of M/G is induced from that of M .

§2. On the group homomorphism P .

In this section we shall define a group homomorphism $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]_0$, and we shall prove P is continuous.

We shall identify the orbit space M/G with $[0,1]$ by the homeomorphism ϕ in §1, therefore the projection $\pi: M \rightarrow M/G$ is identified with the smooth map $\theta: M \rightarrow [0,1]$. Let $h: M \rightarrow M$ be a G -diffeomorphism of M which is G -isotopic to the identity 1_M , and let $f: [0,1] \rightarrow [0,1]$ be the orbit map of h . Since $f \circ \pi = \pi \circ h$ is a smooth map, f is a smooth map by Lemma 1.2 (3). Similarly the inverse map f^{-1} of f is smooth, and f is a diffeomorphism. Then there exists an abstract group homomorphism $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]$ which is given by $P(h) = f$, where $\text{Diff}^\infty[0,1]$ is the group of C^∞ diffeomorphism of $[0,1]$, endowed with C^∞ topology.

Proposition 2.1. $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]$ is a continuous homomorphism of topological groups.

Let $C^\infty(M_1, M_2)$ denote the set of all smooth maps from a compact smooth manifold M_1 to a smooth manifold M_2 , endowed with C^∞ topology. Before the proof of Proposition 2.1, we begin with some lemmas.

Lemma 2.2. Let M_i be a compact smooth manifold and N_i be a smooth manifold for $i=1,2$. Then we have

- (1) Let $\phi: N_1 \rightarrow N_2$ be a smooth map, and let $\phi_*: C^\infty(M_1, N_1) \rightarrow C^\infty(M_1, N_2)$ be a map which is given by $\phi_*(f) = \phi \circ f$. Then ϕ_* is continuous.
- (2) Let $\phi: M_1 \rightarrow M_2$ be a smooth map, and let $\phi^*: C^\infty(M_2, N_1) \rightarrow C^\infty(M_1, N_1)$ be a map which is given by $\phi^*(f) = f \circ \phi$. Then ϕ^* is continuous.
- (3) Let $\phi: M_1 \rightarrow N_2$ be a smooth map and let $\phi_\# : C^\infty(M_1, N_1) \rightarrow C^\infty(M_1, N_1 \times N_2)$ be a map which is given by $\phi_\#(f) = (f, \phi)$. Then $\phi_\#$ is continuous.

(4) Let $\phi: M_2 \rightarrow N_2$ be a smooth map and let $\phi_1: C^\infty(M_1, N_1) \rightarrow C^\infty(M_1 \times M_2, N_1 \times N_2)$ be a map given by $\phi_1(f) = f \times \phi$. Then ϕ_1 is continuous.

(5) Let $\kappa: C^\infty(M_1, N_1) \times C^\infty(M_1, N_2) \rightarrow C^\infty(M_1, N_1 \times N_2)$ be a map given by $\kappa(f, g)(x) = (f(x), g(x))$ for $x \in M_1$. Then κ is continuous.

(6) Let L be a smooth manifold. Let $\text{comp}: C^\infty(M_1, N_1) \times C^\infty(N_1, L) \rightarrow C^\infty(M_1, L)$ be a map given by $\text{comp}(f, g) = g \circ f$. Then comp is continuous.

Proof. (1) and (2) are proved by R. Abraham [2, Theorem 11.2 and 11.3]. It is easy to see (3), (4) and (5). From J. Cerf [4, Chapter I, §4, Proposition 5], (6) follows.

Lemma 2.3. Let X be a topological space. Let M be a compact smooth manifold and N be a smooth manifold. Choose an open covering $\{U_i\}$ of M such that each closure $\bar{U}_i$ of U_i is a regular submanifold of M which is contained in a coordinate neighborhood of M . Then a map $\Psi: X \rightarrow C^\infty(M, N)$ is continuous if and only if each composition $\Psi_i: X \xrightarrow{\Psi} C^\infty(M, N) \xrightarrow{j_i^*} C^\infty(\bar{U}_i, N)$ is continuous for each i , where $j_i: \bar{U}_i \hookrightarrow M$ is an inclusion.

Proof. From Lemma 2.2 (2), if Ψ is continuous, then Ψ_i is continuous for each i . We can choose $\{U_i\}$ as a coordinate covering of M . Let $\{V_\lambda\}$ be a coordinate covering of N . Let $f \in C^\infty(M, N)$ and $K \subset U_i$ be a compact subset such that $f(K) \subset V_\lambda$ for some λ . $N^r(f, U_i, V_\lambda, K, \epsilon)$ ($r = 0, 1, 2, \dots, 0 < \epsilon \leq \infty$) denote the set of C^r maps $g: M \rightarrow N$ such that $g(K) \subset V_\lambda$ and $\|D^k f(x) - D^k g(x)\| < \epsilon$ for any $x \in K, k = 0, 1, 2, \dots, r$. Then the C^∞ topology on $C^\infty(M, N)$ is generated by these sets $N^r(f, U_i, V_\lambda, K, \epsilon)$ (see M. Hirsch [6, Chapter 2, §1]).

Let $x \in X$ and let $f = \Psi(x)$. For any open neighborhood W of f , there exist above sets $N_k = N^{r_k}(f, U_{i_k}, V_{\lambda_k}, K_k, \epsilon_k)$, $k = 1, 2, \dots, n$, such that $\bigcap_{k=1}^n N_k \subset W$. Note that $j_{i_k}^*: C^\infty(M, N) \rightarrow C^\infty(\bar{U}_{i_k}, N)$ is an open map and $(j_{i_k}^*)^{-1}(j_{i_k}^*(N_k)) = N_k$. Since Ψ_{i_k} is continuous, $\Psi^{-1}(N_k) = \Psi_{i_k}^{-1}(j_{i_k}^*(N_k))$ is an open neighborhood of x in X , for each k . Then $\bigcap_{k=1}^n \Psi^{-1}(N_k)$

$\psi^{-1}(N_k)$ is an open neighborhood of x in X . Since $\psi(\bigcap_{k=1}^n \psi^{-1}(N_k)) \subset \bigcap_{k=1}^n N_k \subset W$, ψ is continuous at x . This completes the proof of Lemma 2.3.

Remark. Lemma 2.2 and Lemma 2.3 are hold in the cases of manifolds with corneres.

Let $C_e^\infty([-1/2, 1/2], \mathbb{R})$ denote the set of all smooth functions $f: [-1/2, 1/2] \rightarrow \mathbb{R}$ satisfying $f(-x) = f(x)$, endowed with C^∞ topology. Let $T: C_e^\infty([-1/2, 1/2], \mathbb{R}) \rightarrow C^\infty([0, 1/4], \mathbb{R})$ denote a map defined by $T(f)(x) = f(\sqrt{x})$. Then we have

Lemma 2.4. The above map T is well defined and continuous.

Proof. Put $f = T(\hat{f})$ for each $\hat{f} \in C_e^\infty([-1/2, 1/2], \mathbb{R})$. Since $\hat{f}$ is a C^∞ even function, we have the Taylor expansion

$$\begin{aligned} \hat{f}(x) = & \hat{f}(0) + (\hat{f}''(0)/2)x^2 + \dots + (\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{2n-2} \\ & + \left(\int_0^1 ((1-t)^{2n-1}/(2n-1)!) \hat{f}^{(2n)}(tx) dt \right) x^{2n} \end{aligned}$$

for $-1/2 \leq x \leq 1/2$, $n=1, 2, \dots$. Thus we have

$$\begin{aligned} f(x) = & \hat{f}(0) + (\hat{f}''(0)/2)x + \dots + (\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{n-1} \\ & + \left(\int_0^1 ((1-t)^{2n-1}/(2n-1)!) \hat{f}^{(2n)}(t\sqrt{x}) dt \right) x^n \end{aligned}$$

for $0 \leq x \leq 1/4$. By the composite mapping formula, we can compute the n -th derivative

$$\begin{aligned} & D^n (f^{(2n)}(t\sqrt{x})x^n) \\ = & \sum_{p=0}^n \sum_{q=0}^p \sum_{\substack{i_1+\dots+i_q=p \\ i_1>0, \dots, i_q>0}} B(p, i_1, \dots, i_q) \hat{f}^{(2n+q)}(t\sqrt{x}) x^{q/2} t^q, \end{aligned}$$

where $B(p, i_1, \dots, i_q)$ is a real number. Put $f_i = T(\hat{f}_i)$ for $\hat{f}_i \in C_e^\infty([-1/2, 1/2], \mathbb{R})$ ($i=1, 2$). Then there exists a positive number A_n such that

$$\begin{aligned} & \sup_{0 \leq x \leq 1/4} |D^n f_1(x) - D^n f_2(x)| \\ \leq & A_n \cdot \max_{0 \leq q \leq 3n} \left(\sup_{-1/2 \leq x \leq 1/2} |D^q \hat{f}_1(x) - D^q \hat{f}_2(x)| \right) \end{aligned}$$

for each $n = 1, 2, \dots$. Note that

$$\sup_{0 \leq x \leq 1/4} |f_1(x) - f_2(x)| = \sup_{-1/2 \leq x \leq 1/2} |\hat{f}_1(x) - \hat{f}_2(x)|.$$

Therefore T is a continuous map, and this completes the proof of Lemma 2.4.

Proof of Proposition 2.1. Let J denote a closed interval $[0, 1/4]$, $[1/5, 4/5]$ or $[3/4, 1]$. By Lemma 2.3, it is sufficient to show that the composition $P_J: \text{Diff}_G^\infty(M)_0 \xrightarrow{P} \text{Diff}^\infty[0, 1] \xrightarrow{j^*} C^\infty(J, [0, 1])$ is continuous, where $j: J \hookrightarrow [0, 1]$ is an inclusion map.

We shall first consider the case $J = [0, 1/4]$. Let $\iota: [-1/2, 1/2] \rightarrow [0, 1/4]$ be a map given by $\iota(x) = x^2$. Let $\hat{\iota}: [-1/2, 1/2] \rightarrow G \times_{K_0} D(V_0) \hookrightarrow M$ be a map given by $\hat{\iota}(r) = [1, re_0]$, where e_0 is a point of $S(V_0)$ as in §1. Then $\pi \circ \hat{\iota} = \iota$. Let $\hat{P}_J$ denote the composition $\text{Diff}_G^\infty(M)_0 \xrightarrow{\hat{\iota}^*} C^\infty([-1/2, 1/2], M) \xrightarrow{\pi_*} C^\infty([-1/2, 1/2], [0, 1])$. Then $\hat{P}_J(h) = \pi \circ h \circ \hat{\iota} = P(h) \circ \iota = \iota^* P(h)$ for $h \in \text{Diff}_G^\infty(M)_0$, and the image of $\hat{P}_J$ is contained in $C_e^\infty([-1/2, 1/2], \mathbb{R})$. Note that $P_J = T \circ \hat{P}_J$. Combining Lemma 2.2 and Lemma 2.4, P_J is continuous.

Next consider the case $J = [1/5, 4/5]$. By Lemma 1.2, there is a G -diffeomorphism $\alpha: \pi^{-1}([1/5, 4/5]) \rightarrow G/H \times [1/5, 4/5]$. Let $i: \pi^{-1}([1/5, 4/5]) \hookrightarrow M$ be the inclusion map and let $k: [1/5, 4/5] \rightarrow G/H \times [1/5, 4/5]$ be a map given by $k(r) = (1H, r)$. Then P_J is the composition

$$\text{Diff}_G^\infty(M)_0 \xrightarrow{(i \circ \alpha^{-1} \circ k)^*} C^\infty([1/5, 4/5], M) \xrightarrow{\pi_*} C^\infty([1/5, 4/5], [0, 1])$$

which is continuous by Lemma 2.2.

We can prove that P_J is continuous in the case $J = [3/4, 1]$ similarly as in the case $J = [0, 1/4]$, and this completes the proof of Proposition 2.1.

§3. A continuous global section of P.

In §2 we have proved that $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]$ is continuous. Thus the image of P is contained in the connected component $\text{Diff}^\infty[0,1]_0$ of the identity. In this section we shall construct a continuous global section of $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]_0$.

Let f be an element of $\text{Diff}^\infty[0,1]_0$. We shall define a map $\Psi(f): M \rightarrow M$ as follows: $\Psi(f)$ is defined on $\pi^{-1}((0,1))$ by the composition $\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{(1,f)} G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1))$, and $\Psi(f) = 1$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$.

Proposition 3.1. $\Psi(f)$ is a G-diffeomorphism of M.

In order to prove Proposition 3.1, we need the following lemma and notations.

Lemma 3.2. Let $\Psi_1: \text{Diff}^\infty[0,1]_0 \rightarrow \text{Diff}^\infty(D^n)$ be a map defined by

$$\Psi_1(f)(v) = \begin{cases} (\sqrt{f(\|v\|^2)} / \|v\|)v & \text{for } v \neq 0, \\ 0 & \text{for } v = 0, \end{cases}$$

where D^n denotes an n-dimensional unit disc. Then Ψ_1 is a well defined and continuous map.

Notations 3.3. For $i=0,1$, we shall use the following notations:

$\pi_i: G \rightarrow G/K_i$ the natural projection,

U_i an open disc neighborhood of $1K_i$ in G/K_i ,

$\sigma_i: U_i \rightarrow G$ a smooth local cross section of π_i ,

$\sigma_{i,a}: aU_i \rightarrow G$ ($a \in G$) a smooth local cross section of π_i defined by $\sigma_{i,a}(x) = a \cdot \sigma_i(a^{-1}x)$.

Put $M_i = G \times_{K_i} D(V_i)$ and $M_i(r) = G \times_{K_i} D_r(V_i)$, where $D_r(V_i)$ is a closed r -disc in V_i ($0 < r \leq 1$).

$p_i: M_i \rightarrow G/K_i$, $p_{i,r}: M_i(r) \rightarrow G/K_i$ the bundle projections,

$\phi_{i,a}: p_i^{-1}(aU_i) \rightarrow U_i \times D(V_i)$ ($a \in G$) a chart of p_i over aU_i defined

by $\phi_{i,a}([g,v]) = (a^{-1}\pi_i(g), ((\sigma_{i,a} \circ \pi_i)(g))^{-1}g \cdot v)$,

$\pi_2: G \rightarrow G/H$ the natural projection,

U_2 an open disc neighborhood of $1H$ in G/H ,

$\sigma_2: U_2 \rightarrow G$ a smooth local cross section of π_2 .

Proof of Proposition 3.1. Put $h = \Psi(f)$. We shall first prove that h is smooth on $\pi^{-1}(0)$. Since $f(0)=0$, there exists a real number ϵ such that $0 < \epsilon \leq 1/2$ and $f(\epsilon^2) \leq 1/4$. Then $h(\pi^{-1}([0, \epsilon^2])) \subset \pi^{-1}([0, 1/4])$, and $h(M_0(\epsilon)) \subset M_0(1/2)$. For $[g, re_0] \in G \times_{K_0} D_\epsilon(V_0 - 0)$ ($0 < r \leq \epsilon$), $h([g, re_0]) = (\alpha^{-1} \circ (1, f) \circ \alpha)([g, re_0]) = (\alpha^{-1} \circ (1, f))(gH, r^2) = \alpha^{-1}(gH, f(r^2)) = [g, \sqrt{f(r^2)}e_0]$. Then, for $[g, v] \in G \times_{K_0} D_\epsilon(V_0 - 0)$, $h([g, v]) = [g, \sqrt{f(\|v\|^2)}\|v\|v] = [g, \Psi_1(f)(v)]$. Since $h([g, 0]) = [g, 0]$, $h([g, v]) = [g, \Psi_1(f)(v)]$ for any $[g, v] \in M_0(\epsilon)$. Then the composition

$$\begin{aligned} \tilde{h}: U_0 \times D_\epsilon(V_0) &\xrightarrow{(\phi_{0,a})^{-1}} p_{0,\epsilon}^{-1}(aU_0) \\ &\xrightarrow{h} p_{0,1/2}^{-1}(aU_0) \\ &\xrightarrow{\phi_{0,a}} U_0 \times D_{1/2}(V_0) \end{aligned}$$

is given by $\tilde{h}(x, v) = (x, \Psi_1(f)(v))$ for $a \in G$. Since $\Psi_1(f)$ is a smooth map by Lemma 3.2, h is smooth on $\pi^{-1}(0)$. Similarly we can prove that h is smooth on $\pi^{-1}(1)$. Since h is smooth on $\pi^{-1}((0, 1))$ by the definition, h is a smooth map. Since $h^{-1} = \Psi(f^{-1})$, h^{-1} is also a smooth map. Thus h is a G -diffeomorphism of M , and this completes the proof of Proposition 3.1.

In order to prove Lemma 3.2, we need the following assertion.

Assertion 3.4. Let $\phi: \text{Diff}^\infty[0, 1]_0 \rightarrow C^\infty([0, 1], \mathbb{R})$ be a map given by

$$\phi(f)(x) = \begin{cases} \sqrt{f(x)/x} & \text{for } x \neq 0, \\ \sqrt{f'(0)} & \text{for } x = 0. \end{cases}$$

Then ϕ is a well defined continuous map.

Proof. For $f \in \text{Diff}^\infty[0, 1]_0$, we have the Taylor expansion

$$f(x) = f'(0)x + x^2 \int_0^1 (1-t)f''(tx) dt \quad \text{for } 0 \leq x \leq 1.$$

Put $F(x) = f'(0) + x \int_0^1 (1-t)f''(tx) dt$ for $0 \leq x \leq 1$. Then $\phi(f) = \sqrt{F}$.

Note that $F(x) > 0$ for $0 \leq x \leq 1$. It is easy to see that ϕ is continuous.

Proof of Lemma 3.2. Let $N: D^n \rightarrow [0,1]$ be a map given by $N(v) = \|v\|^2$. Let $i: D^n \hookrightarrow \mathbb{R}^n$ be the inclusion and let $\mu: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the scalar multiplication. Since $\psi_1(f) \overset{(v)}{=} \phi(f)(\|v\|^2)v$, $\psi_1(f)$ is a smooth map by Assertion 3.4. Since $\psi_1(f^{-1}) = \psi_1(f)^{-1}$, $\psi_1(f)^{-1}$ is also a smooth map. Thus $\psi_1(f)$ is a diffeomorphism of D^n . Note that ψ_1 is the composition $\text{Diff}^\infty[0,1]_0 \xrightarrow{\phi} C^\infty([0,1], \mathbb{R}) \xrightarrow{N^*} C^\infty(D^n, \mathbb{R}) \xrightarrow{i\#} C^\infty(D^n, \mathbb{R} \times \mathbb{R}^n) \xrightarrow{\mu_*} C^\infty(D^n, \mathbb{R}^n)$. Combining Assertion 3.4 and Lemma 2.2, ψ is continuous. This completes the proof of Lemma 3.2.

Proposition 3.5. $\psi: \text{Diff}^\infty[0,1]_0 \rightarrow \text{Diff}_G^\infty(M)$ is continuous.

Proof. Let $B_i \subset U_i$ be a closed disc neighborhood of $1K_i$ in G/K_i for $i=0,1$. Let $B_2 \subset U_2$ be a closed disc neighborhood of $1H$ in G/H . We can choose $\{ \text{int}(p_{0,\epsilon}^{-1}(aB_0)), \text{int}(p_{1,\epsilon}^{-1}(aB_1)), \text{int}(\alpha^{-1}(aB_2 \times [\epsilon/2, 1-\epsilon/2])) ; a \in G \}$ as an open covering of M for $0 < \epsilon < 1/2$. Put $W = \{ f \in \text{Diff}^\infty[0,1]_0 ; f([0, \epsilon^2]) \subset [0, 1/4], f([1-\epsilon^2, 1]) \subset (3/4, 1] \}$. Then W is an open neighborhood of the identity in $\text{Diff}^\infty[0,1]_0$. Since ψ is a homomorphism

as an abstract group, it is enough to show that ψ is continuous

on W . Let C denote one of the sets $p_{0,\epsilon}^{-1}(aB_0)$, $p_{1,\epsilon}^{-1}(aB_1)$ or $\alpha^{-1}(aB_2 \times [\epsilon/2, 1-\epsilon/2])$ for a G . If we can prove that the composition

$$\psi_C: W \xrightarrow{\psi} \text{Diff}_G^\infty(M)_0 \xrightarrow{i^*} C^\infty(C, M)$$

is continuous for each C , then ψ is continuous on W by Lemma 2.3,

where $i: C \hookrightarrow M$ is an inclusion map.

First consider in the case $C = p_{0,\epsilon}^{-1}(aB_0)$. $\psi(f)(C)$ is contained in $p_{0,1/2}^{-1}(aU_0)$ for each $f \in W$. Note that $\psi(f)([g,v]) = [g, \psi_1(f)(v)]$ for $[g,v] \in C$ and $(\phi_{0,a} \circ \psi(f) \circ \phi_{0,a}^{-1})(x,v) = (x, \psi_1(f)(v))$ for $(x,v) \in B_0 \times D_\epsilon(V_0)$. Thus ψ_C is given by the composition

$$\begin{aligned}
W & \xrightarrow{\psi_1} C^\infty(D_\varepsilon(V_0), D(V_0)) \\
& \xrightarrow{j!} C^\infty(B_0 \times D_\varepsilon(V_0), U_0 \times D(V_0)) \\
& \xrightarrow{\phi_{0,a}^*} C^\infty(C, U_0 \times D(V_0)) \\
& \xrightarrow{(k \circ \phi_{0,a}^{-1})^*} C^\infty(C, M),
\end{aligned}$$

where $j: B_0 \hookrightarrow U_0$ and $k: p_0^{-1}(aU_0) \hookrightarrow M$ are inclusions. Combining Lemma 3.2 and Lemma 2.2, ψ_C is continuous.

Now consider the case $C = \alpha^{-1}(B_0 \times [\varepsilon/2, 1-\varepsilon/2])$. ψ_C is given by the composition

$$\begin{aligned}
W & \xrightarrow{\iota^*} C^\infty([\varepsilon/2, 1-\varepsilon/2], (0,1)) \\
& \xrightarrow{j!} C^\infty(B_0 \times [\varepsilon/2, 1-\varepsilon/2], G/H \times (0,1)) \\
& \xrightarrow{\alpha^*} C^\infty(C, G/H \times (0,1)) \\
& \xrightarrow{(k \circ \alpha^{-1})^*} C^\infty(C, M),
\end{aligned}$$

where $\iota: [\varepsilon/2, 1-\varepsilon/2] \hookrightarrow [0,1]$. $j: B_0 \hookrightarrow G/H$ and $k: \pi^{-1}((0,1)) \hookrightarrow M$ are inclusion maps. By Lemma 2.2, ψ_C is continuous.

We can prove that ψ_C is continuous in the case $C = p_{1,\varepsilon}^{-1}(aB_1)$ similarly as in the case $C = p_{0,\varepsilon}^{-1}(aB_0)$, and this completes the proof of Proposition 3.5.

By Proposition 3.5, $P: \text{Diff}_G^\infty(M)_0 \rightarrow \text{Diff}^\infty[0,1]_0$ is a globally trivial fibration. Then we have

Corollary 3.6. $\text{Diff}_G^\infty(M)_0$ is homeomorphic to $\text{Diff}^\infty[0,1]_0 \times \text{Ker } P$.

§4. On the group $\text{Ker } P$.

In this section we shall define a group homomorphism $L: \text{Ker } P \rightarrow Q$, where Q is a subgroup of $C^\infty([0,1], N(H)/H)$, and we shall prove that L is a group monomorphism between topological groups (see Lemma 4.5 and Proposition 4.6).

Let h be an element of $\text{Ker } P$. Let $\hat{h}$ be the composition

$$G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1)) \xrightarrow{h} \pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1).$$

Then $\hat{h}$ is a level preserving G -diffeomorphism. Let $a: (0,1) \rightarrow N(H)/H$ be a smooth map satisfying $h(gH, r) = (ga(r), r)$ for $(gH, r) \in G/H \times (0,1)$.

Proposition 4.1. With the above notations, there exists a smooth map $\bar{a}: [0,1] \rightarrow N(H)/H$ such that

- (1) $\bar{a} = a$ on $(0,1)$,
- (2) $\bar{a}(i) \in (N(H) \cap N(K_i))/H$ for $i = 0,1$.

To prove Proposition 3.1, we need the following lemma.

Lemma 4.2. Let G be a compact Lie group. Let K and N be closed subgroups of G . Let $\pi: G \rightarrow G/K$ be the natural projection. Then there exists a smooth local section σ of π , which is defined on an open neighborhood U of $1K$, such that $\sigma(1K) = 1$ and $\sigma(x) \in N$ for $x \in \pi(N) \cap U$.

Proof. Let $\pi_1: N \rightarrow N/(N \cap K)$ be a natural projection. Let $i: N \hookrightarrow G$ be the inclusion and let $I: N/(N \cap K) \rightarrow G/K$ be a map satisfying $\pi \circ i = I \circ \pi_1$. Since $I(N/(N \cap K)) = \pi(N)$ is an orbit of the natural action $N \times G/K \rightarrow G/K$, I is an imbedding. Let U be a disc neighborhood around $\pi(1)$ in G/K and let U_1 be a disc neighborhood around $\pi_1(1)$ in $N/(N \cap K)$. Since I is an imbedding, we can assume $I(U_1) = U \cap I(N/(N \cap K)) = U \cap \pi(N)$. Let $\sigma_1: U_1 \rightarrow N$ be a smooth local section of π_1 satisfying $\sigma_1(\pi_1(1)) = 1$. Then $\sigma_1 \circ I^{-1}$ is a smooth section

defined on $I(U_1)$. We can extend $\sigma_1 \circ I^{-1}$ to a smooth local section defined on U . Then $\sigma(\pi(1)) = 1$ and $\sigma(U \cap \pi(N)) \subset N$. This completes the proof of Lemma 4.2.

Lemma 4.3. Let G be a compact connected Lie group. Let V be a representation of G such that G acts transitively and effectively on a unit sphere $S(V)$ of V . Let H be the isotropy subgroup of a point of $S(V)$. Then we have the following list:

G	$SO(n) \ (n \geq 3)$	$SU(n) \ (n \geq 2)$	$U(n) \ (n \geq 1)$	$Sp(n) \ (n \geq 1)$	$Sp(n) \times_{\mathbb{Z}_2} S^3 \ (n \geq 1)$
H	$SO(n-1)$	$SU(n-1)$	$U(n-1)$	$Sp(n-1)$	H_1
$N(H)/H$	$\mathbb{Z}_2$	$U(1)$	$U(1)$	$Sp(1)$	$\mathbb{Z}_2$

$Sp(n) \times_{\mathbb{Z}_2} S^1 \ (n \geq 1)$	G_2	$Spin(7)$	$Spin(9)$
H_2	$SU(3)$	G_2	$Spin(7)$
S^1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

where $H_1 = \{[(q, A), q^{-1}] \in Sp(n) \times_{\mathbb{Z}_2} S^3; (q, A) \in Sp(1) \times Sp(n-1) \subset Sp(n)\}$ and $H_2 = \{[(z, A), z^{-1}] \in Sp(n) \times_{\mathbb{Z}_2} S^1; (z, A) \in S^1 \times Sp(n-1) \subset Sp(n)\}$.

Proof. It is known that G and H are the above Lie groups (c.f. W. C. Hsiang and W. Y. Hsiang [7, §1]). We can determine the Lie group $N(H)/H$ by an immediate calculation except for $G = G_2, Spin(7), Spin(9)$. For the cases $G = G_2, Spin(7), Spin(9)$, we can determine $N(H)/H$ by using I. Yokota's definitions of these Lie groups in [9, Chapter 5].

Lemma 4.4. (1) Let $F: [-1, 1] \rightarrow \mathbb{R}$ be a smooth function such that $F(0) = 0$. Put $f(x) = F(x)/x$ for $x \neq 0$ and $f(x) = F'(0)$ for $x = 0$. Then $f: [-1, 1] \rightarrow \mathbb{R}$ is a well defined smooth function.

(2) Put $C_0^\infty([-1, 1], \mathbb{R}) = \{F \in C^\infty([-1, 1], \mathbb{R}); F(0) = 0\}$, ~~endowed with~~

endowed with C^∞ topology. Let $\phi: C_0^\infty([-1,1], R) \rightarrow C^\infty([-1,1], R)$

be a map given by $\phi(F)(x) = f(x)$. Then ϕ is continuous.

Proof. For $F \in C_0^\infty([-1,1], R)$, we have $\phi(F)(x) = f(x) = F'(0) + x \int_0^1 (1-t) F''(tx) dt$. Then the n -th derivative $f^{(n)}(x) = x \int_0^1 (1-t) \cdot t^n F^{(n+2)}(tx) dt + n \int_0^1 (1-t) t^{n-1} F^{(n+1)}(tx) dt$. Thus there exists a positive number A such that $\|\phi(F)\|_n \leq A \|F\|_{n+2}$, and Lemma 4.4 follows.

Proof of Proposition 4.1. Let ϵ ($0 < \epsilon \leq 1/2$) be a real number. Let W_i and U_i be open neighborhoods of $1K_i$, satisfying $\bar{W}_i \subset U_i$ for $i = 0, 1$. Put $O = \{h \in \text{Ker } P; h(p_{i,\epsilon}^{-1}(\bar{W}_i)) \subset p_{i,\epsilon}^{-1}(U_i) \text{ for } i = 0, 1\}$. Then O is an open neighborhood of the identity in $\text{Ker } P$. By Corollary 3.6, $\text{Ker } P$ is connected, and O generates the topological group $\text{Ker } P$. Thus we can assume $h \in O$.

Let $\tilde{h}$ be the composition

$$W_0 \times_{D_\epsilon}(V_0) \xrightarrow{(\phi_{0,1})^{-1}} p_{0,\epsilon}^{-1}(W_0) \xrightarrow{h} p_{0,\epsilon}^{-1}(U_0) \xrightarrow{\phi_{0,1}} U_0 \times_{D_\epsilon}(V_0).$$

Let $\rho_1: U_0 \times_{D_\epsilon}(V_0) \rightarrow U_0$ and $\rho_2: U_0 \times_{D_\epsilon}(V_0) \rightarrow D_\epsilon(V_0)$ be projections on the first factor and the second factor, respectively. Let $i: [-\epsilon, \epsilon] \rightarrow W_0 \times_{D_\epsilon}(V_0)$ be an imbedding given by $i(r) = (1K_0, re_0)$. Then the compositions $\tilde{h}_1 = \rho_1 \circ \tilde{h} \circ i: [-\epsilon, \epsilon] \rightarrow U_0$ and $\tilde{h}_2 = \rho_2 \circ \tilde{h} \circ i: [-\epsilon, \epsilon] \rightarrow D_\epsilon(V_0)$ are smooth maps. Let $\bar{\pi}_0: G/H \rightarrow G/K_0$ be the natural projection. Note that

$$\begin{aligned} (\alpha \circ h \circ \phi_{0,1}^{-1})(1K_0, re_0) &= (\alpha \circ h)([1, re_0]) \\ &= (\hat{h} \circ \alpha)([1, re_0]) \\ &= \hat{h}(1H, r^2) \\ &= (a(r^2), r^2) \quad \text{for } |r| \leq \epsilon, r \neq 0. \end{aligned}$$

Then

$$\begin{aligned} \tilde{h}(1K_0, re_0) &= (\phi_{0,1} \circ \alpha^{-1})(a(r^2), r^2) \\ &= (\bar{\pi}_0(a(r^2)), (\sigma_0 \circ \bar{\pi}_0)(a(r^2))^{-1} \cdot a(r^2) \cdot re_0), \end{aligned}$$

and

$$\begin{aligned}\tilde{h}_1(r) &= \bar{\pi}_0(a(r^2)), \\ \tilde{h}_2(r) &= (\sigma_0 \circ \bar{\pi}_0)(a(r^2))^{-1} \cdot a(r^2) \cdot re_0,\end{aligned}$$

for $|r| \leq \varepsilon$, $r \neq 0$.

Here we can assume that $\sigma_0(1K_0) = 1$ and $\sigma_0(\pi_0(N(H)) \cap U_0) \subset N(H)$ by Lemma 4.2. Let $b: [-\varepsilon, \varepsilon] \rightarrow G$ be a smooth map given by $b(r) = \sigma_0(\tilde{h}_1(r))$. Then $b(r) = \sigma_0(\bar{\pi}_0(a(r^2))) \in \sigma_0(\pi_0(N(H)) \cap U_0)$, and $b(r) \in N(H)$ for $r \neq 0$. Since b is a smooth map, $b(r) \in N(H)$ for $r = 0$. For $[1, 0] \in \pi^{-1}(0)$, we have $h([1, 0]) = (h \circ \phi_{0,1}^{-1})(1K_0, 0) = \underbrace{h \circ \phi_{0,1}^{-1}}_{(\phi_{0,1}^{-1})}(i(0)) = \phi_{0,1}^{-1}(\tilde{h}_1(0), 0) = [b(0), 0]$. Note that p_0 is a G -diffeomorphism on the zero section of p_0 and $h(\pi^{-1}(0)) = \pi^{-1}(0)$. Then the composition $p_0 \circ h \circ p_0^{-1}: G/K_0 \rightarrow G/K_0$ is a G -diffeomorphism, and $(p_0 \circ h \circ p_0^{-1})(1K_0) = (p_0 \circ h)([1, 0]) = p_0([b(0), 0]) = b(0)K_0$. Thus $b(0) \in N(K_0)$, and $b(0) \in N(H) \cap N(K_0)$.

Put $J = [-\varepsilon, 0) \cup (0, \varepsilon]$. Let $c: J \rightarrow N(H)/H$ be a smooth map given by $c(r) = b(r)^{-1} \cdot a(r^2)$. Since $\bar{\pi}_0(c(r)) = \bar{\pi}_0(\sigma_0(\bar{\pi}_0(a(r^2)))^{-1} \cdot a(r^2)) = 1K_0$, then $c(r) \in K_0/H$. Thus $c(r) \in N(H, K_0)/H$ for $r \in J$. Since $\text{Ker } P$ is connected, the maps a, b and c are homotopic to the constant maps. Note that the identity component $(N(H, K_0)/H)^0$ of $N(H, K_0)/H$ is contained in $(N(H, K_0) \cap K_0^0) \cdot H/H$, and there exists an isomorphism $(N(H, K_0) \cap K_0^0) \cdot H/H \simeq (N(H, K_0) \cap K_0^0)/(H \cap K_0^0)$ as a Lie group, where K_0^0 is the identity component of K_0 . Then there exists a smooth map $\hat{c}: J \xrightarrow{c} (N(H, K_0)/H)^0 \hookrightarrow (N(H, K_0) \cap K_0^0) \cdot H/H \simeq (N(H, K_0) \cap K_0^0)/(H \cap K_0^0) \hookrightarrow N(H \cap K_0^0, K_0^0)/(H \cap K_0^0)$. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$, and so is c .

Note that K_0 acts transitively on the unit sphere $S(V_0)$ of V_0 . If $\dim S(V_0) = 0$, then $K_0/H = \mathbb{Z}_2$ and $N(H, K_0)/H = \mathbb{Z}_2$. In this case $\hat{c}$ is a trivial map, and it is clear that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. Now we assume $\dim S(V_0) > 0$. Since $S(V_0)$ is connected, K_0^0 acts transitively on $S(V_0)$ and $K_0^0/(K_0^0 \cap H)$ is diffeomorphic to $S(V_0)$.

Put $D = \bigcap_{g \in K_0^0} g(K_0^0 \cap H)g^{-1}$ which is the kernel of the action $K_0^0 \times S(V_0) \rightarrow S(V_0)$. Put $\bar{K}_0 = K_0^0/D$ and $\bar{H} = (H \cap K_0^0)/D$. Then $\bar{K}_0$ acts transitively and effectively on $S(V_0)$ and $\bar{K}_0/\bar{H}$ is diffeomorphic to $S(V_0)$. Put $\bar{N}_0 = N(\bar{H}, \bar{K}_0)/\bar{H}$ which is isomorphic to $N(H \cap K_0^0, K_0^0)/(H \cap K_0^0)$ as a Lie group. The pair $(\bar{K}_0, \bar{N}_0)$ is one of pairs $(G, N(H)/H)$ in the list of Lemma 4.3. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\epsilon, \epsilon]$. If $\bar{N}_0 = Z_2$, this is clear since $\hat{c}$ is a trivial map.

Consider the case $\bar{K}_0 = SU(n)$ ($n \geq 1$) and $\bar{N}_0 = U(1)$. In this case V_0 is an n -dimensional complex vector space and $\bar{N}_0 = U(1)$ acts on V_0 as a scalar multiplication. We can regard C^n as a $2n$ -dimensional real vector space R^{2n} and $\bar{N}_0$ as $SO(2)$. Then there exist smooth functions $c_i: J \rightarrow R$, $i = 1, 2$, such that

$$\hat{c}(r) = \begin{bmatrix} c_1(r) & -c_2(r) \\ c_2(r) & c_1(r) \end{bmatrix} \in SO(2) \quad \text{for } r \in J.$$

Note that $\tilde{h}_2: [-\epsilon, \epsilon] \rightarrow D_\epsilon(V_0)$ is a smooth map and $\tilde{h}_2(r) = c(r) \cdot re_0 = \hat{c}(r) \cdot re_0$ for $r \neq 0$. In this case $e_0 = (1, 0, \dots, 0) \in S^{2n-1}$ and $\tilde{h}_2(r) = (c_1(r)r, c_2(r)r, 0, \dots, 0)$ for $r \in J$. Put $c_i(0) = \lim_{r \rightarrow 0} c_i(r)$ for $i = 1, 2$. From Lemma 4.4, $c_i: [-\epsilon, \epsilon] \rightarrow R$, $i = 1, 2$, are smooth functions and $\hat{c}$ can be extended to a smooth maps on $[-\epsilon, \epsilon]$.

Now consider the case $\bar{K}_0 = Sp(n)$ ($n \geq 1$) and $\bar{N} = Sp(1)$. In this case V_0 is an n -dimensional quaternionic vector space H^n and $\bar{N}_0 = Sp(1)$ acts on V_0 as a scalar multiplication ~~on the right~~. We can regard H^n as R^{4n} and $Sp(1)$ as a subgroup of $SO(4)$ naturally. By the similar way as in the case $K_0 = SU(n)$, there exist smooth functions $c_i: J \rightarrow R$, $i = 1, 2, 3, 4$, such that $h_2(r) = (c_1(r)r, c_2(r)r, c_3(r)r, c_4(r)r, 0, \dots, 0)$ for $r \in J$, and we can extend $\hat{c}$ to a smooth map on $[-\epsilon, \epsilon]$.

The proof of the other cases are similar to those of the above cases. Thus we can extend c to a smooth map on $[-\epsilon, \epsilon]$. Since $c(r) \in N(H, K_0)/H$

for $r \neq 0$, we see $c(0) \in N(H, K_0)/H$. Put $\bar{a}(0) = b(0) \cdot c(0)$. Since $b(0) \in N(H) \cap N(K_0)$ and $c(0) \in N(H, K_0)/H$, we have $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Let $\hat{a}: [-1/2, 1/2] \rightarrow N(H)/H$ be a map given by $\hat{a}(r) = \bar{a}(r^2)$. Since $\hat{a}(r) = b(r) \cdot c(r)$ for $-\varepsilon \leq r \leq \varepsilon$, $\hat{a}$ is a smooth map. Since $\hat{a}$ is an even map and $\bar{a}(r) = \hat{a}(\sqrt{r})$ for $0 \leq r \leq 1/4$, $\bar{a}$ is a smooth map on $[0, 1/4]$ by Lemma 2.4. Thus we can extend the map a to a smooth map $\bar{a}$ on $[0, 1)$ satisfying $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Similarly we can extend a to a smooth map $\bar{a}$ on $[0, 1]$ satisfying $\bar{a}(1) \in (N(H) \cap N(K_1))/H$. This completes the proof of Proposition 4.1.

Let Q denote the set of smooth maps $f: [0, 1] \rightarrow N(H)/H$ satisfying $f(i) \in (N(H) \cap N(K_i))/H$ for $i = 0, 1$, endowed with C^∞ topology. Using Proposition 4.1, we define a map $L: \text{Ker } P \rightarrow Q$ by $L(h) = \bar{a}^{-1}$.

Lemma 4.5. $L: \text{Ker } P \rightarrow Q$ is a group monomorphism.

Proof. Let $h_i \in \text{Ker } P$ for $i = 1, 2$. For $0 < r < 1$ and $g \in G$, we have

$$\begin{aligned} (g \cdot L(h_2 \circ h_1)(r)^{-1}, r) &= (\alpha \circ h_2 \circ h_1 \circ \alpha^{-1})(gH, r) \\ &= ((\alpha \circ h_2 \circ \alpha^{-1}) \circ (\alpha \circ h_1 \circ \alpha^{-1}))(gH, r) \\ &= (\alpha \circ h_2 \circ \alpha^{-1})(g \cdot L(h_1)(r)^{-1}, r) \\ &= (g \cdot L(h_1)(r)^{-1} \cdot L(h_2)(r)^{-1}, r). \end{aligned}$$

Thus $L(h_2 \circ h_1) = L(h_2) \cdot L(h_1)$ on $(0, 1)$. Since $L(h_1)$, $L(h_2)$ and $L(h_1 \circ h_2)$ are smooth maps on $[0, 1]$, $L(h_2 \circ h_1) = L(h_2) \cdot L(h_1)$ on $[0, 1]$. Thus L is a group homomorphism. Suppose $L(h) = 1$ for $h \in \text{Ker } P$. Then $(h \circ \alpha^{-1})(gH, r) = \alpha^{-1}(gH, r)$ for $g \in G$ and $0 < r < 1$, and $h = 1$ on $\pi^{-1}((0, 1))$. Thus $h = 1$ on M , and L is a monomorphism.

Proposition 4.6. L is a continuous map.

Proof. We shall use the notations in the proof of Proposition 4.1. Since L is a group homomorphism, it is sufficient to show ~~Proposition 4.6~~ that $L: O \rightarrow Q$ is continuous. Let I denote a closed

interval $[0, \varepsilon^2]$, $[\varepsilon^2/2, 1-\varepsilon^2/2]$ or $[1-\varepsilon^2, 1]$. By Lemma 2.3, it is sufficient to prove that $L_I: O \xrightarrow{L} Q \xrightarrow{j^*} C^\infty(I, N(H)/H)$ is continuous, where $j: I \hookrightarrow [0, 1]$ is an inclusion map.

First we shall consider the case $I = [0, \varepsilon^2]$. Let L_1 be the composition

$$O \xrightarrow{(k \circ \phi_{0,1}^{-1} \circ i)^*} C^\infty([- \varepsilon, \varepsilon], p_{0,\varepsilon}^{-1}(U_0)) \\ \xrightarrow{(\sigma_0 \circ \rho_1 \circ \phi_{0,1})^*} C^\infty([- \varepsilon, \varepsilon], G),$$

where $k: p_{0,\varepsilon}^{-1}(\bar{W}_0) \hookrightarrow M$ is an inclusion map. Then L_1 is continuous by Lemma 2.2. Note that $L_1(h) = b$.

Let $L_2: O \longrightarrow C^\infty([- \varepsilon, \varepsilon], (N(H, K_0)/H)^0)$ be a map given by $L_2(h) = c$. We shall prove that L_2 is continuous. This is trivial in the case $N(H, K_0)/H = \mathbb{Z}_2$. Consider the case $\bar{K}_0 = \text{SU}(n)$ ($n \geq 2$). In this case $V_0 = C^n = \mathbb{R}^{2n}$ and $\bar{N}_0 = \text{U}(1) = \text{SO}(2)$. Put $C_0^\infty([- \varepsilon, \varepsilon], V_0) = \{F \in C^\infty([- \varepsilon, \varepsilon], V_0); F(0) = 0\}$, endowed with C^∞ topology. Let $\Phi: C_0^\infty([- \varepsilon, \varepsilon], V_0) \rightarrow C^\infty([- \varepsilon, \varepsilon], \mathbb{R}^2)$ be a map defined by $\Phi(F) = (\Phi(F^1), \Phi(F^2))$, where $F = (F^1, \dots, F^{2n})$ and $\Phi(F^i)$ is a map defined in Lemma 4.4. Then Φ is continuous by Lemma 4.4. Let $m: \mathbb{R}^2 \rightarrow M_2(\mathbb{R})$ denote a smooth map defined by

$$m(x, y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where $M_2(\mathbb{R})$ denote the set of all 2×2 matrices over $\mathbb{R}$. Let L'_2 denote the composition

$$O \xrightarrow{(k \circ \phi_{0,1}^{-1} \circ i)^*} C^\infty([- \varepsilon, \varepsilon], p_{0,1}^{-1}(U_0)) \\ \xrightarrow{(\rho_2 \circ \phi_{0,1})^*} C^\infty([- \varepsilon, \varepsilon], D_\varepsilon(V_0)).$$

From Lemma 2.2, L'_2 is continuous. Note that $L'_2(h) = \tilde{h}_2$ and $L'_2(O)$ is contained in $C_0^\infty([- \varepsilon, \varepsilon], V_0)$. Let $\hat{L}_2$ denote the composition

$$O \xrightarrow{L'_2} C_0^\infty([- \varepsilon, \varepsilon], V_0) \\ \xrightarrow{\Phi} C^\infty([- \varepsilon, \varepsilon], \mathbb{R}^2) \\ \xrightarrow{m_*} C^\infty([- \varepsilon, \varepsilon], M_2(\mathbb{R})).$$

Then $\hat{L}_2(h) = \hat{c}$ and $\hat{L}_2$ is continuous. This implies that L_2 is

continuous by using Lemma 2.2. Similarly we can see that L_2 is continuous in the other cases.

Let $\mu: G \times G/H \rightarrow G/H$ be a map defined by the left translation and let $\iota: (N(H, K_0)/H)^0 \hookrightarrow G/H$ be an inclusion map. Then the composition

$$\begin{aligned} \hat{L}: 0 &\xrightarrow{(L_1, \iota_* \circ L_2)} C^\infty([- \varepsilon, \varepsilon], G) \times C^\infty([- \varepsilon, \varepsilon], G/H) \\ &\xrightarrow{\kappa} C^\infty([- \varepsilon, \varepsilon], G \times G/H) \\ &\xrightarrow{\mu_*} C^\infty([- \varepsilon, \varepsilon], G/H) \end{aligned}$$

is continuous by Lemma 2.2, where κ is defined by $\kappa(f_1, f_2)(r) = (f_1(r), f_2(r))$. Note that $\hat{L}(h) = b \cdot c = \hat{a}$ and $\hat{L}(0)$ is contained in $C_e^\infty([- \varepsilon, \varepsilon], N(H)/H)$. Here $C_e^\infty([- \varepsilon, \varepsilon], N(H)/H)$ denote the set of all smooth even maps $f: [- \varepsilon, \varepsilon] \rightarrow N(H)/H$, endowed with C^∞ topology.

Let $T: C_e^\infty([- \varepsilon, \varepsilon], N(H)/H) \rightarrow C^\infty([0, \varepsilon^2], N(H)/H)$ be a map defined by $T(f)(r) = f(\sqrt{r})$. By the same argument as in the proof in Lemma 2.4, we can prove that T is continuous. Thus $L_I = T \circ L$ is continuous.

Now consider the case $I = [\varepsilon^2/2, 1 - \varepsilon^2/2]$. L_I is given by the composition

$$\begin{aligned} 0 &\xrightarrow{k^*} C^\infty(\pi^{-1}(I), \pi^{-1}(I)) \\ &\xrightarrow{(\alpha^{-1} \circ \iota)^*} C^\infty(I, \pi^{-1}(I)) \\ &\xrightarrow{(q_2 \circ \alpha)_*} C^\infty(I, G/H), \end{aligned}$$

where $k: \pi^{-1}(I) \hookrightarrow M$ is an inclusion, $\iota: I \rightarrow G/H \times I$ is a map given by $\iota(r) = (lH, r)$ and $q_2: G/H \times I \rightarrow G/H$ is the projection on the first factor. Thus L_I is continuous. We can see that L_I is continuous in the case $I = [1 - \varepsilon^2, 1]$ similarly as in the case $I = [0, \varepsilon^2]$, and this completes the proof of Proposition 4.6.

§5. Subgroups of the topological groups Q and $\text{Ker } P$.

In this section we shall consider subgroups Q_1 and S of the topological groups Q and $\text{Ker } P$, respectively, such that $L(S) = Q_1$, and we shall prove that the inclusions $Q_1 \hookrightarrow Q_0$ and $S \hookrightarrow \text{Ker } P$ are homotopy equivalence^s, where Q_0 is the identity component of Q .

Put $Q_1 = \{a \in Q_0; a(r) = a(0) \text{ for } 0 \leq r \leq 1/4 \text{ and } a(r) = a(1) \text{ for } 3/4 \leq r \leq 1\}$. Then Q_1 is a topological subgroup of Q_0 . Let $i: Q_1 \hookrightarrow Q_0$ be an inclusion.

Lemma 5.1. $i: Q_1 \hookrightarrow Q_0$ is a homotopy equivalence.

Proof. Let $\sigma: [0,1] \rightarrow [0,1]$ be a smooth map such that

$$\sigma(r) = 0 \quad \text{for } 0 \leq r \leq 1/4,$$

$$\sigma(r) = 1 \quad \text{for } 3/4 \leq r \leq 1.$$

Let $\mu_t: [0,1] \rightarrow [0,1]$ ($0 \leq t \leq 1$) be a smooth homotopy given by $\mu_t(r) = t\sigma(r) + (1-t)r$. Since $(a \circ \mu_t)(i) \in (N(H) \cap N(K_1))/H$ for $i = 0,1$, $a \circ \mu_t$ is an element of Q . Define $q: Q_0 \times [0,1] \rightarrow Q$ by $q(a,t) = a \circ \mu_t$. Let $\mu: [0,1] \rightarrow C^\infty([0,1], [0,1])$ be a map given by $\mu(t) = \mu_t$. Then it is easy to see that μ is continuous. Note that q is given by the composition

$$\begin{aligned} Q_0 \times [0,1] &\xrightarrow{(1,\mu)} Q_0 \times C^\infty([0,1], [0,1]) \\ &\xrightarrow{\text{comp}} C^\infty([0,1], N(H)/H), \end{aligned}$$

where comp is given by $\text{comp}(a,f) = a \circ f$. By Lemma 2.2 (6), q is continuous. Then $q(Q_0 \times [0,1])$ is contained in Q_0 . Let $q_t: Q_0 \rightarrow Q_0$ be a map given by $q_t(a) = q(a,t)$. Since $\mu_1 = \sigma$, $q_1(Q_0)$ is contained in Q_1 . Thus q is a homotopy between $q_0 = 1_{Q_0}$ and $q_1 = i \circ q_1$. Note that $q_t(Q_1)$ is contained in Q_1 for any t . Then $q: Q_1 \times [0,1] \rightarrow Q_1$ is a homotopy between 1_{Q_1} and $q_1 \circ i$. Therefore Lemma 5.1 follows.

Put $S = L^{-1}(Q_1) \subset \text{Ker } P$. Let $\iota: S \hookrightarrow \text{Ker } P$ be an inclusion.

Lemma 5.2. $\iota: S \hookrightarrow \text{Ker } P$ is a homotopy equivalence.

Proof. Put $a = L(h^{-1})$ for $h \in \text{Ker } P$. Let $h_t: M \rightarrow M$ ($0 \leq t \leq 1$) be a map as follows: h_t is given on $\pi^{-1}((0,1))$ by the composition $\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{\hat{h}_t} G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1))$, where $\hat{h}_t$ is defined by $\hat{h}_t(gH, r) = (g \cdot q_t(a)(r), r)$. $h_t(gK_i) = ga(i) \cdot K_i$ ($i = 0, 1$) for $g \in G$. Here we need the following

Assertion 5.3. h_t is a smooth map for any t .

Proof. By the definition, h_t is smooth on $\pi^{-1}((0,1))$. We shall prove that h_t is smooth on $\pi^{-1}(0)$. Let a_0 be an element of G such that $a_0 H = a(0)$ and $a_0 \in N(H) \cap N(K_0)$. For $[g, 0] \in p_{0,1/2}^{-1}(lK_0)$, $(p_{0,1/2} \circ h)([g, 0]) = \pi_0(ga_0) = \pi_0(a_0) \in a_0 U_0$. Then there exists a neighborhood W_0 of lK_0 in G/K_0 such that $(p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\bar{W}_0))$ is contained in $a_0 U_0$. For $[g, re_0] \in p_{0,1/2}^{-1}(\bar{W}_0)$ and $0 \leq t \leq 1$,

$$\begin{aligned} (p_{0,1/2} \circ h_t)([g, re_0]) &= \bar{\pi}_0(gq_t(a)(r^2)) \\ &= \bar{\pi}_0(ga((1-t)r^2)) \\ &= (p_{0,1/2} \circ h)([g, \sqrt{1-t}re_0]) \end{aligned}$$

which is contained in $(p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\bar{W}_0)) \subset a_0 U_0$. Then $h_t(p_{0,1/2}^{-1}(g\bar{W}_0))$ is contained in $p_{0,1/2}^{-1}(ga_0 U_0)$ for $g \in G$ and $0 \leq t \leq 1$.

Let $\tilde{h}: W_0 \times D_{1/2}(V_0) \rightarrow U_0 \times D_{1/2}(V_0)$ be a map given by $\tilde{h} = \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$ for $g \in G$. Let $\rho_1: U_0 \times D_{1/2}(V_0) \rightarrow U_0$ and $\rho_2: U_0 \times D_{1/2}(V_0) \rightarrow D_{1/2}(V_0)$ be the projections on the first factor and the second factor respectively.

Put $g' = ga_0$ and put $\tilde{h}^i = \rho_i \circ \tilde{h}$ for $i = 0, 1$. Then $\tilde{h}^i$ is a smooth map and

$$\begin{aligned} \tilde{h}^1(x, rke_0) &= g'^{-1} g \sigma_0(x) k \cdot \bar{\pi}_0(a(r^2)), \\ \tilde{h}^2(x, rke_0) &= \sigma_{0,g'}(g \sigma_0(x) k \cdot \bar{\pi}_0(a(r^2))^{-1} g \sigma_0(x) k a(r^2) \cdot re_0 \end{aligned}$$

for $x \in W_0$ and $k \in K_0$, where $\bar{\pi}_0: G/H \rightarrow G/K_0$ is the natural projection.

Put $\tilde{h}_t^i = \rho_i \circ \phi_{0,g'} \circ h_t \circ \phi_{0,g}^{-1}$ for $i = 0, 1$. Then

$$\tilde{h}_t^1(x, rke_0) = g'^{-1} g \sigma_0(x) k \cdot \bar{\pi}_0(a(\mu_t(r^2))),$$

$\tilde{h}_t^2(x, rke_0) = \sigma_{0,g}, (g\sigma_0(x)k \cdot \bar{\pi}_0(a(\mu_t(r^2)))^{-1} g\sigma_0(x)ka(\mu_t(r^2))) \cdot re_0$
for $x \in W_0$ and $k \in K_0$.

Since $\sigma(r^2) = 0$ for $r \leq 1/2$, $\mu(r^2, t) = (1-t)r^2$ for $0 \leq r \leq 1/2$. Then
 $\tilde{h}_t^1(x, v) = \tilde{h}^1(x, \sqrt{1-t} v)$ for $0 \leq t \leq 1$ and $\tilde{h}_t^2(x, v) = 1/\sqrt{1-t} \tilde{h}^2(x, \sqrt{1-t} v)$
for $0 \leq t < 1$. Thus $\tilde{h}_t^1$ ($0 \leq t \leq 1$) and $\tilde{h}_t^2$ ($0 \leq t < 1$) are smooth maps.

By the Taylor formula (c.f. J. Dieudonné [5, Chapter VIII, (8, 14, 3)]), we have

$$\tilde{h}^2(x, v) = \tilde{h}^2(x, 0) + \left(\int_0^1 (D\tilde{h}^2)(x, \zeta v) d\zeta \right) v,$$

where $D\tilde{h}^2$ is the derivative of $\tilde{h}^2$. Since $\tilde{h}^2(x, 0) = 0$,

$$\tilde{h}_t^2(x, v) = \left(\int_0^1 (D\tilde{h}^2)(x, \sqrt{1-t} \zeta v) d\zeta \right) v \quad \text{for } 0 \leq t < 1.$$

Then $\tilde{h}_1^2(x, v) = \lim_{t \rightarrow 1} \tilde{h}_t^2(x, v) = (D\tilde{h}^2)(x, 0)v$, and $\tilde{h}_1^2$ is a smooth map. Therefore h_t is smooth on $\pi^{-1}(0)$ for any $0 \leq t \leq 1$. Similarly we can prove that h_t is smooth on $\pi^{-1}(1)$, and Assertion 5.3 follows.

Proof of Lemma 5.2 continued. Let $\bar{q}: \text{Ker } P \times [0, 1] \rightarrow \text{Ker } P$ be a map defined by $\bar{q}(h, t) = h_t$. By Assertion 5.3, h_t and h_t^{-1} are smooth maps, and $\bar{q}$ is a well defined map. Next we shall prove that $\bar{q}$ is continuous. Let W_i be a neighborhood of lK_i in G/K_i satisfying $\bar{W}_i \subset U_i$ for $i = 0, 1$. Put $O = \{h \in \text{Ker } P; h(p_{i, 1/2}^{-1}(\bar{W}_i)) \subset p_{i, 1/2}^{-1}(U_i) \text{ for } i = 0, 1\}$. Then O is an open neighborhood of l_M in $\text{Ker } P$. For $h \in O$, $g \in G$ and $0 \leq t \leq 1$, $h_t(p_{i, 1/2}^{-1}(g\bar{W}_i))$ is contained in $p_{i, 1/2}^{-1}(gU_i)$ ($i = 0, 1$). Let W_2 be an open neighborhood of lH in G/H satisfying $\bar{W}_2 \subset U_2$. Let C be one of the sets $\{p_{i, 1/2}^{-1}(g\bar{W}_i) \mid i = 0, 1, g \in G\}$, $\alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5]) \mid g \in G\}$. By Lemma 2.3, it is sufficient to show that the composition $\bar{q}_C: O \times [0, 1] \xrightarrow{\bar{q}} \text{Ker } P \xrightarrow{j_C^*} C^\infty(C, M)$ is continuous for any C , where $j_C: C \hookrightarrow M$ is an inclusion map.

First consider the case $C = p_{0, 1/2}^{-1}(g\bar{W}_0)$. Let $v_1: C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0) \times [0, 1] \rightarrow C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0)$ be a map given by $v_1(f, t)(x, v) = f(x, \sqrt{1-t} v)$. Let $v_2: C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \xrightarrow{\times [0, 1]} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0))$ be a map given by $v_2(f, t)(x, v) = \left(\int_0^1 (Df)(x, \sqrt{1-t} \zeta v) d\zeta \right) (v)$. It is easy

to see that v_1 and v_2 are continuous. Note that $\bar{q}_C$ is the composition

$$\begin{aligned}
0 \times [0,1] &\xrightarrow{(j_C^*, 1)} C^\infty(C, p_{0,1/2}^{-1}(gU_0)) \times [0,1] \\
&\xrightarrow{((\phi_{0,g})^* \circ (\phi_{0,g})^*, 1)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)) \times [0,1] \\
&\xrightarrow{((\rho_1)_*, (\rho_2)_*, 1)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \\
&\quad \times [0,1] \\
&\xrightarrow{v} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \\
&\xrightarrow{\kappa} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)) \\
&\xrightarrow{(\phi_{0,g}^{-1})^* \circ (\phi_{0,g}^{-1})^*} C^\infty(C, p_{0,1/2}^{-1}(gU_0)) \hookrightarrow C^\infty(C, M).
\end{aligned}$$

Here v is given by $v(f_1, f_2, t) = (v_1(f_1, t), v_2(f_2, t))$ and κ is the map defined in Lemma 2.2 (5). Then $\bar{q}_C$ is continuous by Lemma 2.2.

Next consider the case $C = \alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5])$. Let $m: N(H)/H \times G/H \rightarrow G/H$ be a map defined by $m(nH, gH) = (gn)H$ and $p_2: G/H \times [1/5, 4/5] \rightarrow [0,1]$ be a map given by $p_2(gH, r) = r$. Let $\delta: Q_0 \rightarrow Q_0$ be a map given by $\delta(a) = a^{-1}$. Then the map $\bar{q}_C$ is the composition

$$\begin{aligned}
0 \times [0,1] &\xrightarrow{(L, 1)} Q_0 \times [0,1] \xrightarrow{\delta \circ q} Q_0 \xrightarrow{p_2^*} C^\infty(G/H \times [1/5, 4/5], N(H)/H) \\
&\xrightarrow{(1_{G/H \times [1/5, 4/5]})^*} C^\infty(G/H \times [1/5, 4/5], N(H)/H \times G/H \times [1/5, 4/5]) \\
&\xrightarrow{m_*} C^\infty(G/H \times [1/5, 4/5], G/H \times [1/5, 4/5]) \\
&\xrightarrow{(\alpha \circ j_C)^* \circ (\alpha^{-1})_*} C^\infty(C, \alpha^{-1}(G/H \times [1/5, 4/5])) \hookrightarrow C^\infty(C, M),
\end{aligned}$$

which is continuous because L and q are continuous.

Similarly as in the case $C = p_{0,1/2}^{-1}(g\bar{W}_0)$, we can see that $\bar{q}_C$ is continuous in the case $C = p_{1,1/2}^{-1}(g\bar{W}_1)$. Thus $\bar{q}$ is continuous.

Since $q_1(Q_0) \subset Q_1$, $\bar{q}_1(\text{Ker } P) \subset S$. Therefore $\bar{q}$ is a homotopy between

$\bar{q}_0 = 1_{\text{Ker } P}$ and $\bar{q}_1 = 1 \circ \bar{q}_1$. Since $q(Q_1 \times [0,1]) \subset Q_1$, $\bar{q}(S \times [0,1]) \subset S$.

Then $\bar{q}: S \times [0,1] \rightarrow S$ is a homotopy between 1_S and $\bar{q}_1 \circ 1$. Thus 1

is a homotopy equivalence, and this completes the proof of Lemma 5.2.

§6. Proof of Theorem.

In this section, we shall see that $L: S \rightarrow Q_1$ is an isomorphism between topological groups, and we shall prove our Theorem.

Proposition 6.1. $L: S \rightarrow Q_1$ is an isomorphism between topological groups.

Before the proof of Proposition 6.1, we begin with some lemmas. For any topological subgroup K of G , K^0 denotes the identity component of K .

Lemma 6.2. For any $a \in N(K_0)^0 \cap N(H)$, there exist $a' \in N(H^0) \cap K_0^0$ and $n \in \text{Cent}(K_0^0)$ such that $a = n \cdot a'$, where $\text{Cent}(K_0^0)$ is the centralizer of K_0^0 in G .

Proof. Since $N(K_0)^0$ is a compact connected Lie group, there exist a torus group T and a simply connected semi-simple compact Lie group G' such that $\hat{N}_0 = T \times G'$ is a finite covering group of $N(K_0)^0$ (c.f. L. Pontrjagin [8, §64]). Let $q_0: \hat{N}_0 \rightarrow N(K_0)^0$ be the covering projection. Put $\hat{K}_0 = q_0^{-1}(K_0^0)$. Since K_0^0 is a normal subgroup of $N(K_0)^0$, $\hat{K}_0$ is a normal subgroup of $\hat{N}_0$. Then $\hat{K}_0^0$ is also a normal subgroup of $\hat{N}_0$. Here we need the following:

Assertion 6.3. There exists a closed normal subgroup K'_0 of $\hat{N}_0$ such that $\hat{N}_0$ is isomorphic to the product group $\hat{K}_0^0 \times K'_0$ as a Lie group.

Proof. There exist simply connected simple Lie groups G_i ($1 \leq i \leq r$) such that $G' = G_1 \times \dots \times G_r$. Since $\hat{K}_0^0$ is a compact connected Lie group, there exist simply connected simple Lie groups K_j ($1 \leq j \leq s$) and a torus group T' such that $\tilde{K}_0 = T' \times K_1 \times \dots \times K_s$ is a finite covering of $\hat{K}_0^0$. Let $p_0: \tilde{K}_0 \rightarrow \hat{K}_0^0$ be the covering projection. Let $\rho_i: \hat{N}_0 = T \times G_1 \times \dots \times G_r \rightarrow G_i$ be a projection on the direct factor G_i ($1 \leq i \leq r$). Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$, $\rho_i(\hat{K}_0^0)$ is a normal subgroup of G_i .

Since G_i is a simple Lie group, $\rho_i(\hat{K}_0^0) = G_i$ or $\{1\}$. If $\rho_i(\hat{K}_0^0) = G_i$, $\rho_i(p_0(K_j))$ is a normal subgroup of G_i . Thus $\rho_i(p_0(K_j)) = G_i$ or $\{1\}$, for $1 \leq i \leq r$, $1 \leq j \leq s$.

Put $\rho'_i = \rho_i \circ p_0$. If $\rho'_i(K_{j_1}) = \rho'_i(K_{j_2})$ ($j_1 \neq j_2$), then $\rho'_i(g_1) \cdot \rho'_i(g_2) = \rho'_i(g_1 \cdot g_2) = \rho'_i(g_2 \cdot g_1) = \rho'_i(g_2) \cdot \rho'_i(g_1)$ for $g_1 \in K_{j_1}$, $g_2 \in K_{j_2}$. Then $\rho'_i(K_{j_1})$ is a commutative normal subgroup of G_i , and $\rho'_i(K_{j_1}) = \{1\}$. If $\rho'_i(K_j) = G_i$, then $\rho'_i(T')$ is a normal subgroup of G_i , hence $\rho'_i(T') = \{1\}$. Therefore, if $\rho'_i(K_j) = G_i$, then $\rho'_i(T') = \{1\}$ and $\rho'_i(K_n) = \{1\}$ for $n \neq j$.

Assume $\rho'_{i_1}(K_j) = G_{i_1}$ and $\rho'_{i_2}(K_j) = G_{i_2}$ for $i_1 \neq i_2$. Let $\rho': \tilde{K}_0 \rightarrow G_{i_1} \times G_{i_2}$ be a map defined by $\rho'(k) = (\rho'_{i_1}(k), \rho'_{i_2}(k))$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$ and $\rho'(\tilde{K}_0) = \rho'(K_j)$, $\rho'(K_j)$ is a normal subgroup of $G_{i_1} \times G_{i_2}$. Then, for $x, y \in K_j$, there exists $k \in K_j$ such that $(\rho'_{i_1}(x), 1) \rho'(y) (\rho'_{i_1}(x)^{-1}, 1) = \rho'(k)$. Then $\rho'_{i_1}(xyx^{-1}) = \rho'_{i_1}(x) \rho'_{i_1}(y) \rho'_{i_1}(x)^{-1} = \rho'_{i_1}(k)$ and $\rho'_{i_2}(y) = \rho'_{i_2}(k)$. Since K_j , G_{i_n} ($n=1, 2$) are simply connected simple Lie groups, $\rho'_{i_n}: K_j \rightarrow G_{i_n}$ is an isomorphism between the Lie groups. Thus $xyx^{-1} = k = y$ for any $x, y \in K_j$, and K_j must be a commutative Lie group, which is a contradiction since K_j is a simple Lie group.

Thus we may assume that $\rho'_j(K_j) = G_j$ and $\rho'_i(K_j) = \{1\}$ ($i \neq j$) for $1 \leq j \leq s$, $1 \leq i \leq r$. For $i > s$, $\rho_i(\hat{K}_0^0) = \rho'_i(\tilde{K}_0) = \rho'_i(T')$ which is a commutative normal subgroup of G_i , hence $\rho'_i(T') = \{1\}$. Then $p_0(T')$ is a subgroup of T , and there exists a torus subgroup S of T such that $T = p_0(T') \times S$. Put $K' = S \times G_{s+1} \times \dots \times G_r$. Then $\hat{N}_0 = \hat{K}_0^0 \times K'$, and Assertion 6.3 follows.

Proof of Lemma 6.2 continued. By Assertion 6.3, there exists a closed normal subgroup K'_0 of $\hat{N}_0$ such that $\hat{N}_0 = \hat{K}_0^0 \times K'_0$. Since K_0^0 is a connected group, $q_0(\hat{K}_0^0) = K_0^0$. Then $N(K_0^0) = q_0(\hat{N}_0) = q_0(\hat{K}_0^0) \cdot q_0(K'_0)$

$= K_0^0 \cdot q_0(K'_0)$. Note that $q_0(K'_0)$ is contained in $\text{Cent}(K_0^0)$. Thus, for $a \in N(K_0^0) \cap N(H)$, there exists $a' \in K_0^0$ and $n \in \text{Cent}(K_0^0)$ such that $a = a' \cdot n$. Since $N(H) \subset N(H^0)$ and $H^0 \subset K_0^0$, $H^0 = aH^0 a^{-1} = a' n H^0 n^{-1} a'^{-1} = a' H^0 a'^{-1}$. Thus $a' \in N(H^0)$ and Lemma 6.2 follows.

For $a \in Q_1$, we define a map $h: M \rightarrow M$ as follows:

$$\begin{aligned} h(\alpha^{-1}(gH, r)) &= \alpha^{-1}(ga(r)^{-1}, r) \quad \text{for } (gH, r) \in G/H \times (0, 1), \\ h([g, 0]) &= [ga(i)^{-1}, 0] \quad \text{for } [g, 0] \in \pi^{-1}(i) \quad (i = 0, 1). \end{aligned}$$

Lemma 6.4. h is a smooth map.

Proof. Choose $a_0 \in (N(H) \cap N(K_0))^0 \subset N(H)^0 \cap N(K_0)^0$ such that $a(0)^{-1} = a_0 H$. There exists a neighborhood W_0 of $1K_0$ in G/K_0 such that $\pi_0^{-1}(W_0) \cdot a_0$ is contained in $a_0 \cdot \pi_0^{-1}(U_0)$. Since $a(r) = a(0)$ for $0 \leq r \leq 1/4$, $h(p_{0,1/2}^{-1}(gW_0))$ is contained in $p_{0,1/2}^{-1}(ga_0 U_0)$. Let $\tilde{h}_1: W_0 \times D_{1/2}(V_0) \rightarrow U_0$ be a map given by the composition $\rho_1 \circ \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$, and let $\tilde{h}_2: W_0 \times D_{1/2}(V_0) \rightarrow D_{1/2}(V_0)$ be a map given by the composition $\rho_2 \circ \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$. Note that

$$\begin{aligned} (h \circ \phi_{0,g}^{-1})(x, rke_0) &= h([g\sigma_0(x)k, re_0]) \\ &= h(\alpha^{-1}((g\sigma_0(x)k)H, r^2)) \\ &= \alpha^{-1}(g\sigma_0(x)ka_0 H, r^2) \\ &= [g\sigma_0(x)ka_0, re_0] \end{aligned}$$

for $x \in W_0$, $k \in K_0$, $0 < r \leq 1/2$. Since $a_0 \in N(K_0)$, $ka_0 K_0 = a_0 K_0$. Then

$$\tilde{h}_1(x, v) = a_0^{-1} \sigma_0(x) a_0 K_0 \quad \text{for } (x, v) \in W_0 \times D_{1/2}(V_0), \text{ and}$$

$$\tilde{h}_2(x, rke_0) = \sigma_{0,ga_0}(g\sigma_0(x)a_0 K_0)^{-1} g\sigma_0(x)ka_0 \cdot re_0 \quad \text{for}$$

$x \in W_0$, $k \in K_0$, $0 \leq r \leq 1/2$. Thus $\tilde{h}_1$ is a smooth map and $\tilde{h}_2$ is smooth on

$W_0 \times (D_{1/2}(V_0) - 0)$. We shall prove that $\tilde{h}_2$ is smooth on $W_0 \times 0$, hence h is smooth on $\pi^{-1}(0)$. This is trivial in the case $\dim S(V_0) = 0$.

Let $\xi_{a_0,g}: W_0 \rightarrow G$ be a map given by $\xi_{a_0,g}(x) = \sigma_{0,ga_0}(g\sigma_0(x)a_0 K_0)^{-1} g\sigma_0(x)$. Then $\xi_{a_0,g}$ is a smooth map. By Lemma 6.2, there

exist $a'_0 \in N(H^0) \cap K_0^0$ and $n \in \text{Cent}(K_0^0)$ such that $a_0 = na'_0$. Then $\tilde{h}_2(x, rke_0) = \xi_{a_0, g}(x) kna'_0 \cdot rke_0 = \xi_{a_0, g}(x) nka'_0 \cdot re_0$ for $x \in W_0$, $k \in K_0^0$ and $0 \leq r \leq 1/2$. Note that $N(H^0) \cap K_0^0 = N(H^0, K_0^0)$.

Assertion 6.5. For $a \in N(H^0, K_0^0)$, let $\psi_a: D(V_0) \rightarrow D(V_0)$ be a map defined by $\psi_a(rke_0) = rkae_0$ for $0 \leq r \leq 1$, $k \in K$. Then ψ_a is a diffeomorphism. Moreover, let $\psi: N(H^0, K_0^0) \rightarrow \text{Diff}^\infty(D(V_0))$ be a map given by $\psi(a) = \psi_a$, then ψ is continuous.

Proof. If $\dim S(V_0) = 0$, then $K_0^0 \subset H$ and $\psi_a = 1_{D(V_0)}$. In this case, the proof is trivial. We assume $\dim S(V_0) > 0$. Since $S(V_0) = K_0/H$ is connected, K_0^0 acts transitively on $S(V_0)$. Let L be the ineffective kernel of the action $K_0^0 \times S(V_0) \rightarrow S(V_0)$. Put $\bar{K} = K_0^0/L$ and $\bar{H} = (H \cap K_0^0)/L$. Then $\bar{K}$ acts transitively and effectively on $S(V_0)$ and $\bar{H}$ is an isotropy subgroup of this action. By Lemma 4.3, $\bar{K}$, $\bar{H}$ and $N(\bar{H}, \bar{K})/\bar{H}$ are G , H and $N(H)/H$ in Lemma 4.3, respectively. Hence $\bar{H}$ is connected. Since the identity component of $H \cap K_0^0$ is H^0 , $\bar{H} = H^0 \cdot L/L$. For $a \in N(H^0, K_0^0)$, the left coset aL is an element of $N(\bar{H}, \bar{K})$. Then a defines an element $\tilde{a} \in N(\bar{H}, \bar{K})/\bar{H}$. Note that $\psi_a(rke_0) = rkae_0 = rk\tilde{a}e_0$ for $0 \leq r \leq 1$, $k \in K_0^0$.

Consider the case $\bar{K} = \text{SU}(n)$ ($n \geq 2$), $\bar{H} = \text{SU}(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = \text{U}(1)$. In this case, $V_0 = \mathbb{C}^n$ and $\text{U}(1)$ acts on V_0 as a scalar multiplication. Thus $\psi_a(rke_0) = \tilde{a} \cdot rke_0$ for $rke_0 \in D(V_0)$. Hence ψ_a is a diffeomorphism. It is easy to see that ψ is continuous.

Next consider the case $\bar{K} = \text{Sp}(n)$ ($n \geq 1$), $\bar{H} = \text{Sp}(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = \text{Sp}(1)$. In this case, $V_0 = \mathbb{H}^n$ and $\text{Sp}(1)$ acts on V_0 as a scalar multiplication on the right. Then $\psi_a(v) = v \cdot \tilde{a}$ for $v \in D(V_0)$, hence ψ_a is a diffeomorphism and ψ is continuous. Similarly we can see that ψ_a is a diffeomorphism and ψ is continuous in the other cases, and Assertion 6.5 follows.

Proof of Lemma 6.4 continued. Since $\tilde{h}_2(x, v) = \xi_{a_0, g}(x) n \cdot \psi_{a_0}(v)$,

by Assertion 6.5, $\tilde{h}_2$ is a smooth map. Thus $\tilde{h}_1$ and $\tilde{h}_2$ are smooth maps, hence h is smooth on $\pi^{-1}(0)$. Similarly we can see that h is smooth on $\pi^{-1}(1)$. By the definition, h is smooth on $\pi^{-1}((0,1))$, and this completes the proof of Lemma 6.4.

Let $\hat{L}(a)$ be a smooth map $h: M \rightarrow M$ in Lemma 6.4, for $a \in Q_1$. Since $\hat{L}(a^{-1}) = \hat{L}(a)^{-1}$, h is a diffeomorphism of M . By the definition, h is an equivariant map. Thus we have a map $\hat{L}: Q_1 \rightarrow \text{Diff}_G^\infty(M)$. Note that $\hat{L}$ is an abstract group homomorphism.

Lemma 6.6. $\hat{L}: Q_1 \rightarrow \text{Diff}_G^\infty(M)$ is continuous.

Proof. Let W_i be a neighborhood of $1K_i$ in G/K_i such that $\bar{W}_i \subset U_i$ ($i=0,1$), and let W_2 be a neighborhood of $1H$ in G/H such that $\bar{W}_2 \subset U_2$. Put $A_i = \{n \in N(K_i)^0; n^{-1}\bar{W}_i n \subset U_i\}$. Then A_i is an open neighborhood of the identity in $N(K_i)^0$. Let $q_i: \hat{N}_i \rightarrow N(K_i)^0$ be a finite covering such that $\hat{N}_i$ is a direct product $T_i \times G'_i$. Here T_i is a torus group and G'_i is a simply connected semi-simple compact Lie group. Put $\hat{K}_i = q_i^{-1}(K_i^0)$. By Assertion 6.3, there exists a closed normal subgroup K'_i of $\hat{N}_i$ such that $\hat{N}_i = \hat{K}_i^0 \times K'_i$. Let s_i be a smooth local cross section of q_i defined on an open neighborhood B_i of the identity in $N(K_i)^0$. Since $\pi_2: (N(H) \cap N(K_i))^0 \rightarrow ((N(H) \cap N(K_i))/H)^0$ is a fibration, there exists a smooth local cross section t_i of π_2 defined on an open neighborhood E_i of $1H$ such that $t_i(E_i) \subset A_i \cap B_i$.

Put $O = \{a \in Q_1; a(i)^{-1} \in E_i \text{ } (i=0,1)\}$. Then O is an open neighborhood of the identity. Since $\hat{L}$ is a group homomorphism, it is enough to show ~~Lemma 6.6~~ that $\hat{L}$ is continuous on O . Let C denote one of the sets $\{p_{i,1/2}^{-1}(g\bar{W}_i) \text{ } (i=0,1, g \in G), \alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5]) \text{ } (g \in G)\}$. By Lemma 2.3, if $\hat{L}_C: O \xrightarrow{\hat{L}} \text{Diff}_G^\infty(M)^0 \xrightarrow{j_C^*} C^\infty(C, M)$ is continuous for any C , then $\hat{L}$ is continuous, where $j_C: C \hookrightarrow M$ is an inclusion map.

First consider the case $C = p_{0,1/2}^{-1}(g\bar{W}_1)$. Let $\beta_1: \hat{N}_0 = \hat{K}_0^0 \times K'_0 \rightarrow \hat{K}_0^0$ and $\beta_2: \hat{N}_0 \rightarrow K'_0$ be the projection on the first factor and the second factor respectively. Let L_1 be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{(\xi_g, q_0 \circ \beta_2 \circ s_0)} C^\infty(\bar{W}_0, G) \times \text{Cent}(K_0^0) \xrightarrow{m} C^\infty(\bar{W}_0, G).$$

Here r , ξ and m are given by $r(a) = a(0)^{-1}$, $\xi_g(a_0)(x) = \xi_{a_0, g}(x)$ and $m(f, n)(x) = f(x) \cdot n$, respectively. Put $a_0 = (t_0 \circ r)(a)$ for $a \in 0$.

Then $\pi_0(\xi_{g, a_0}(x)) = \pi_0(a_0^{-1})$ for $w \in \bar{W}_0$ and $\pi_0((q_0 \circ \beta_2 \circ s_0)(a_0)) = \pi_0(a_0)$.

Therefore $L_1(a) \in K_0$ for any $a \in 0$, and $L_1(0) \subset C^\infty(\bar{W}_0, K_0)$. Let L_2 be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{q_0 \circ \beta_1 \circ s_0} N(H^0, K_0^0) \xrightarrow{\psi} \text{Diff}^\infty(D_{1/2}(V_0)).$$

By Assertion 6.5, L_2 is continuous. Let L_3 be the composition

$$\begin{aligned} 0 &\xrightarrow{(L_1, L_2)} C^\infty(\bar{W}_0, K_0) \times \text{Diff}^\infty(D_{1/2}(V_0)) \\ &\xrightarrow{(\rho_1^*, \rho_2^*)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), K_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), K_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{\mu_*} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)), \end{aligned}$$

where μ is given by $\mu(k, v) = k \cdot v$, and κ is the map in Lemma 2.2. Then

L_3 is continuous, and $L_3(a) = \tilde{h}_2$. Let $\gamma: A_0 \rightarrow C^\infty(\bar{W}_0, U_0)$ be a map defined by $\gamma(a_0)(x) = a_0^{-1} \sigma_0(x) a_0 K_0$. γ is a restriction map to A_0 of a map $\bar{\gamma}: N(K_0) \rightarrow C^\infty(G/K_0, G/K_0)$ given by $\bar{\gamma}(n)(gK_0) = n^{-1}gnK_0$.

Since $\bar{\gamma}$ is a continuous map, γ is continuous. Let L_4 be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \xrightarrow{\gamma} C^\infty(\bar{W}_0, U_0) \xrightarrow{\rho_1^*} C^\infty(\bar{W}_0 \times D_{1/2}, U_0).$$

Then L_4 is continuous and $L_4(h) = \tilde{h}_1$. L_C is the composition

$$\begin{aligned} 0 &\xrightarrow{(L_4, L_3)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{(\phi_{0, g})^* (\phi_{0, g})_*} C^\infty(C, p_{0,1/2}^{-1}(gU_0)) \hookrightarrow C^\infty(C, M). \end{aligned}$$

Thus L_C is continuous.

Now consider the case $C = \alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5])$. Let $m: g\bar{W}_2 \times N(H)/H$

$\rightarrow G/H$ be a map defined by $m(gH, nH) = gnH$, and let $\rho: G/H \times [1/5, 4/5] \rightarrow [1/5, 4/5]$ be the projection on the second factor. Then $\hat{L}_C$ is given by the composition

$$\begin{aligned} & 0 \xrightarrow{i^* \circ \delta_*} C^\infty([1/5, 4/5], N(H)/H) \\ & \xrightarrow{(1_{g\bar{W}_2})^*} C^\infty(g\bar{W}_2 \times [1/5, 4/5], g\bar{W}_2 \times N(H)/H) \\ & \xrightarrow{m_*} C^\infty(g\bar{W}_2 \times [1/5, 4/5], G/H) \\ & \xrightarrow{\rho_\#} C^\infty(g\bar{W}_2 \times [1/5, 4/5], G/H \times [1/5, 4/5]) \\ & \xrightarrow{\alpha^* \circ (\alpha^{-1})_*} C^\infty(C, \alpha^{-1}(G/H \times [1/5, 4/5])) \hookrightarrow C^\infty(C, M), \end{aligned}$$

where $i: [1/5, 4/5] \hookrightarrow [0, 1]$ is the inclusion map and $\delta: N(H)/H \rightarrow N(H)/H$ is a map given by $\delta(a) = a^{-1}$. By Lemma 2.2, $\hat{L}_C$ is continuous.

We can see that L_C is continuous in the case $C = p_{1, 1/2}^{-1}(g\bar{W}_1)$ similarly as in the case $C = p_{0, 1/2}^{-1}(g\bar{W}_0)$, and this completes the proof of Lemma 6.6.

Proof of Proposition 6.1. From Lemma 6.6, $\hat{L}(Q_1)$ is contained in $\text{Diff}_G^\infty(M)_0$. Then, by the definition, $\hat{L}(Q_1)$ is contained in S , and $\hat{L} = L^{-1}$. Combining Lemma 4.5, Proposition 4.6 and Lemma 6.6, $\hat{L}: S \rightarrow Q_1$ is an isomorphism between topological groups, and this completes the proof of Proposition 6.1.

Proof of Theorem. By Corollary 3.6, $\text{Diff}_G^\infty(M)_0$ has the same homotopy type as $\text{Ker } P$. Combining Lemma 5.1, Lemma 5.2 and Proposition 6.1, $\text{Ker } P$ has the same homotopy type as Q_0 . Note that Q_0 has the same homotopy type as the path space $\Omega(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(K_1))/H)_0$. This completes the proof of our Theorem.

§7. Concluding remarks.

From our Theorem, we have the following:

Corollary 7.1. (1) If $K_0 = K_1 = G$, then $\text{Diff}_G^\infty(M)_0$ has the same homotopy type as $(N(H)/H)^0$.

(2) If $N(H)/H$ is a finite group, then $\text{Diff}_G^\infty(M)_0$ is contractible.

Remark 7.2. In K. Abe and K. Fukui [1], we have proved that $\text{Diff}_G^\infty(M)_0$ is perfect if M is a G -manifold with one orbit type and $\dim M/G \geq 1$. But, by using Proposition 3.1, we can see that $\text{Diff}_G^\infty(M)_0$ is not perfect in the case $M/G = [0,1]$.

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