

CORING STRUCTURES AND HILBERT C^* -MODULES

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1. INTRODUCTION

S. Baaĵ and G. Skandalis [1] introduced the notion of multiplicative unitaries and they studied Hopf C^* -algebras associated with them. J. M. Vallin introduced the notion of pseudo-multiplicative unitaries and algebraic structures associated with them ([11], [12]). M. Enock and Vallin [2] studied pseudo-multiplicative unitaries and quantum groupoids associated with inclusions of von Neumann algebras. The author introduced a notion of multiplicative unitary operators (MUO) on Hilbert C^* -modules ([8], see also [6] and [7]). It is interesting to study natural algebraic structures associated with MUO's. In this note, we will show the relation between MUO's and coring structures on Hilbert C^* -modules. Coring structures were introduced by M. Sweedler [10] in the purely algebraic framework. Y. Watatani [13] showed that inclusions of C^* -algebras give natural coring structures in the framework of his index theory. In this note, we introduce notions of coring structures on Hilbert C^* -modules and study coring structures associated with MUO's. In Sections 2 and 3, we study coring structures associated with MUO's arising from groupoids and inclusions of C^* -algebras of index finite type in the sense of Watatani. In the case of groupoids, the base algebras are commutative. In the case of inclusions of

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C^* -algebras of index finite type, we do not know any concrete examples of MUO's on infinite-dimensional Hilbert C^* -modules. Therefore it is interesting to study concrete examples of MUO's and the associated coring structures such that the base algebras are not commutative and the Hilbert C^* -modules are infinite-dimensional. In the last section, we study an MUO and the associated coring structures on the Hilbert C^* -module of compact operators. In this case, the Hilbert C^* -module is infinite-dimensional and the base algebra is the C^* -algebra of compact operators.

2. PRELIMINARIES

2.1. Multiplicative operators on Hilbert C^* -modules. Let A be a C^* -algebra, let E be a Hilbert A -module and let ϕ and ψ be $*$ -homomorphisms of A to $\mathcal{L}_A(E)$. We assume that ϕ and ψ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all $a, b \in A$. We define a $*$ -homomorphism $\iota \otimes_\phi \psi$ of A to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$. and define a $*$ -homomorphism $\iota \otimes_\psi \phi$ of A to $\mathcal{L}_A(E \otimes_\psi E)$ by $(\iota \otimes_\psi \phi)(a) = I \otimes_\psi \phi(a)$. Let W be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that W satisfies the following equations;

$$(2.1) \quad W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W,$$

$$(2.2) \quad W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W,$$

$$(2.3) \quad W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W$$

for all $a \in A$. Then we can define following operators;

$$W \otimes_\psi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E),$$

$$I \otimes_{\phi \otimes \iota} W \in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E),$$

$$W \otimes_\phi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E),$$

$$I \otimes_{\psi \otimes \iota} W \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_{\iota \otimes \psi} (E \otimes_\phi E)),$$

$$I \otimes_{\iota \otimes \phi} W \in \mathcal{L}_A(E \otimes_{\iota \otimes \phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E).$$

Since ϕ and ψ commute, there exists an isomorphism Σ_{12} of $E \otimes_{\iota \otimes \psi} (E \otimes_{\phi} E)$ onto $E \otimes_{\iota \otimes \phi} (E \otimes_{\psi} E)$ as Hilbert A -modules such that, for $x_i \in E$ ($i = 1, 2, 3$),

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3).$$

Definition 2.1 ([8]). Let W be an element of $\mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$. Assume that W satisfies the equations (2.1), (2.2) and (2.3). An operator W is said to be multiplicative if it satisfies the pentagonal equation

$$(2.4) \quad (W \otimes_{\phi} I)(I \otimes_{\phi \otimes \iota} W)(W \otimes_{\psi} I) = (I \otimes_{\iota \otimes \phi} W)\Sigma_{12}(I \otimes_{\psi \otimes \iota} W).$$

Example 2.2. Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_{\mathbb{C}}(E) = \mathcal{L}(H)$ is the C^* -algebra of bounded linear operators on H . Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let W be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (2.4) has the following form:

$$(2.5) \quad (W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W).$$

Defin an operator $\widetilde{W}$ by $\widetilde{W} = W\Sigma$. Then W satisfies the pentagonal equation (2.5) if and only if $\widetilde{W}$ satisfies the usual pentagonal equation ;

$$(2.6) \quad \widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{13}.$$

2.2. Coproducts on Hilbert C^* -modules. Let E be a Hilbert A -module and ϕ be a $*$ -homomorphism of A to $\mathcal{L}_A(E)$.

Definition 2.3. Let δ be an operator in $\mathcal{L}_A(E, E \otimes_{\phi} E)$. We say that δ is a coproduct of (E, ϕ) if δ satisfies the following equations;

$$(2.7) \quad \delta\phi(a) = (\phi \otimes \iota)(a)\delta \quad \text{for all } a \in A,$$

$$(2.8) \quad (\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta.$$

Suppose that δ is a coproduct for E . For $\xi, \eta \in E$, we define a product $\xi\eta$ in E by $\xi\eta = \delta^*(\xi \otimes_\phi \eta)$. It follows from (2.8) that this product is associative. Then E is an algebra over $\mathbb{C}$. Note that we have $\|\xi\eta\| \leq \|\delta\| \|\xi\| \|\eta\|$.

2.3. Coproducts associated with MUO's. Let E be a Hilbert A -module and let ϕ and ψ be $*$ -homomorphisms of A to $\mathcal{L}_A(E)$ such that ϕ and ψ commute. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be a multiplicative unitary operator (MUO).

For an element ξ_0 of E , we say that ξ_0 has Property E1 if it satisfies the following conditions;

$$(i) \quad W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0.$$

(ii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$\langle \eta, \pi_{\xi_0}(\xi)\zeta \rangle = \langle W(\xi_0 \otimes_\psi \eta), \xi \otimes_\phi \zeta \rangle \quad \text{for every } \eta, \zeta \in E.$$

Fix an element ξ_0 with Property E1. Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\phi E)$ by $\delta(\eta) = W(\xi_0 \otimes_\psi \eta)$. Then we have $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since W satisfies the pentagonal equation, δ is a coproduct of (E, ϕ) .

For an element ξ_0 of E , we say that ξ_0 has Property E2 if it satisfies the following conditions;

$$(i) \quad W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0.$$

(ii) For every $\xi \in E$, there exists an element $\widehat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$\langle \eta, \widehat{\pi}_{\xi_0}(\xi)\zeta \rangle = \langle W^*(\xi_0 \otimes_\phi \eta), \xi \otimes_\psi \zeta \rangle \quad \text{for every } \eta, \zeta \in E.$$

Fix an element ξ_0 with Property E2. Define an operator $\widehat{\delta} = \widehat{\delta}_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\psi E)$ by $\widehat{\delta}(\eta) = W^*(\xi_0 \otimes_\phi \eta)$. Since W satisfies the pentagonal equation, $\widehat{\delta}$ is a coproduct of (E, ψ) .

3. CORING STRUCTURES ON HILBERT C^* -MODULES

Let E be a Hilbert A -module and let ϕ be a $*$ -homomorphism of A to $\mathcal{L}_A(E)$. Note that A itself is a Hilbert A -module with the A -valued inner product $\langle a, b \rangle = a^*b$.

We denote by i the $*$ -homomorphism of A to $\mathcal{L}_A(A)$ defined by $i(a)b = ab$. Then there exists a unitary operator t in $\mathcal{L}_A(E \otimes_i A, E)$ defined by $t(\xi \otimes_i a) = \xi a$. If ϕ is non-degenerate, then there exists a unitary operator t' in $\mathcal{L}_A(A \otimes_\phi E, E)$ such that $t'(a \otimes_\phi \xi) = \phi(a)\xi$.

Definition 3.1. Suppose that ϕ is non-degenerate. Let δ be a coproduct of (E, ϕ) and let Q be an element of $\mathcal{L}_A(E, A)$, such that $Q\phi(a) = aQ$ for $a \in A$.

(1) We say that (E, ϕ, δ, Q) is a right counital A -coring if it satisfies the following equation;

$$t(I_E \otimes_\phi Q)\delta = I_E.$$

Then Q is called a right counit.

(2) We say that (E, ϕ, δ, Q) is a left counital A -coring if it satisfies the following equation;

$$t'(Q \otimes_\phi I_E)\delta = I_E.$$

Then Q is called a left counit.

(3) We say that (E, ϕ, δ, Q) is a counital A -coring if Q is a right and left counit.

Then Q is called a counit.

For $n \geq 2$, we set

$$E^{\otimes_\phi n} = E \otimes_\phi \cdots \otimes_\phi E \quad (n \text{ times}).$$

Let (E, ϕ, δ, Q) be a left or right counital A -coring. We define an element ω of $\mathcal{L}_A(E^{\otimes_\phi 4}, E^{\otimes_\phi 2})$ by

$$\omega = \{t(I_E \otimes_\phi Q) \otimes_\phi I_E\}(I_E \otimes_{\phi \otimes_\iota} \delta^* \otimes_\phi I_E).$$

Then we have $\omega(\omega \otimes_{\phi \otimes_\iota} I) = \omega(I \otimes_{\phi \otimes_\iota} \omega)$. Therefore we can define a product on $E \otimes_\phi E$ by $xy = \omega(x \otimes_{\phi \otimes_\iota} y)$. Then $E \otimes_\phi E$ is an algebra over $\mathbb{C}$. Note that we have

$$(\xi_1 \otimes_\phi \xi_2)(\eta_1 \otimes_\phi \eta_2) = (\xi_1 Q(\xi_2 \eta_1)) \otimes_\phi \eta_2.$$

Definition 3.2. We say that δ and Q are compatible if the following equation holds; $\delta(\xi\eta) = \delta(\xi)\delta(\eta)$ for every $\xi, \eta \in E$.

Example 3.3 ([13]). Let $1 \in A_0 \subset A_1$ be an inclusion of C^* -algebras and let $P_1 : A_1 \rightarrow A_0$ be a faithful positive conditional expectation of index finite type. Let $\{u_i, u_i^*; i = 1, \dots, N\}$ be a quasi-basis of P_1 . Let $E_1 = A_1$ be a Hilbert A_0 -module with the A_0 -valued inner product defined by $\langle a, b \rangle = P_1(a^*b)$. Let $\phi_1 : A_1 \rightarrow \mathcal{L}_{A_0}(E_1)$ be a $*$ -homomorphism defined by $\phi_1(a)b = ab$. We denote by ϕ_0 the restriction of ϕ_1 to A_0 . Define $\delta \in \mathcal{L}_{A_0}(E_1, E_1 \otimes_{\phi_0} E_1)$ by $\delta(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*$. The product on E_1 induced by δ agrees with the product on A_1 . Then $(E_1, \phi_0, \delta, P_1)$ is a compatible counital A -coring.

Example 3.4. Let G be a finite groupoid. Set $A = C(G^{(0)})$ and $E = C(G)$. Then E is a right A -module with the right A -action defined by $(\xi a)(x) = \xi(x)a(s(x))$ for $\xi \in E, a \in A$ and $x \in G$. We define an A -valued inner product of E by

$$\langle \xi, \eta \rangle(u) = \sum_{g \in G_u} \overline{\xi(g)} \eta(g)$$

for $\xi, \eta \in E$ and $u \in G^{(0)}$, where $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Then E is a Hilbert A -module. Define $*$ -homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $(\phi(a)\xi)(x) = a(r(x))\xi(x)$ and $\psi(a) = \xi a$ respectively for $a \in A, \xi \in E$ and $x \in G$. Note that we have $E \otimes_{\psi} E = C(G^2(ss))$ and $E \otimes_{\phi} E = C(G^{(2)})$, where $G^2(ss) = \{(g, h) \in G^2; s(g) = s(h)\}$. Let $W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$ be the MUO defined by $(W\xi)(g, h) = \xi(h, gh)$. Define an element $a_0 \in A$ by $a_0(u) = |G_u|^{-1/2}$ and define an element $\xi_0 \in E$ by $\xi_0(g) = a_0(s(g))$. Then ξ_0 satisfies Properties E1 and E2. Note that we have $\|\xi_0\| = 1$. Define an element $\eta_0 \in E$ by $\eta_0 = \chi_{G^{(0)}} a_0^{-1}$. Define operators $Q_{\eta_0}, Q_{\xi_0} : E \rightarrow A$ by $Q_{\eta_0}(\xi) = \langle \eta_0, \xi \rangle$ and $Q_{\xi_0}(\xi) = \langle \xi_0, \xi \rangle$ respectively. Then $(E, \phi, \delta_{\xi_0}, Q_{\eta_0})$ is a compatible counital A -coring. The product on E induced by δ_{ξ_0} is of the form $\xi\eta = (\xi * \eta)a_0$, where $\xi * \eta$ is the convolution product on $C(G)$. We also have two compatible right counital A -corings $(E, \psi, \delta_{\xi_0}, Q_{\xi_0})$

and $(E, \psi, \delta_{\xi_0}, Q_{\eta_0})$. Two products on $E \otimes_{\psi} E$ associated with above right counital A -corings are different.

4. CORING STRUCTURES ASSOCIATED WITH INCLUSIONS OF C^* -ALGEBRAS

Let $1 \in A_0 \subset A_1$ be an inclusion of C^* -algebras and let $P_1 : A_1 \rightarrow A_0$ be a faithful positive conditional expectation of index-finite type with a quasi-basis $\{u_i, u_i^*\}_{i=1}^N$. Let E_1 , ϕ_1 and ϕ_0 be as in Example 3.3. Set $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a $*$ -homomorphism $\phi_2 : A_1 \rightarrow \mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. Define a C^* -algebra A by $A = \mathcal{L}_{A_0}(E_1, \phi_1)$ and a Hilbert A -module E by

$$E = \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2)),$$

that is, E is the set of elements $x \in \mathcal{L}_{A_0}(E_1, E_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. The A -valued inner product on E is defined by $\langle x, y \rangle = x^*y$. We define $*$ -homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes_{\phi_0} I)x$ and $\psi(a)x = (I \otimes_{\phi_0} a)x$ respectively. We suppose that there exists an MUO $W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$ such that $V^*\tilde{V} = W \otimes_i I_{E_1}$, where $V : E \otimes_{\phi} E \otimes_i E_1 \rightarrow E_3$ and $\tilde{V} : E \otimes_{\psi} E \otimes_i E_1 \rightarrow E_3$ are operators defined in [8]. As for sufficient conditions for W to exist, see [7] and [8]. Define an element $x_0 \in E$ by $x_0(\xi) = \xi \otimes_{\phi_0} 1$. Then x_0 satisfies Properties E1 and E2. Note that we have $\|x_0\| = 1$. Define an element $\tilde{y}_0 \in E$ by

$$\tilde{y}_0(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*.$$

Note that we have $\tilde{y}_0^*(\xi \otimes_{\phi_0} \eta) = \xi\eta$, where $\xi\eta$ is the product on A_1 . Define $Q_{x_0}, Q_{\tilde{y}_0} \in \mathcal{L}_A(E, A)$ by $Q_{x_0}(x) = \langle x_0, x \rangle$ and $Q_{\tilde{y}_0}(x) = \langle \tilde{y}_0, x \rangle$ respectively. Then we have the following theorem.

Theorem 4.1. (1) $(E, \phi, \delta_{x_0}, Q_{x_0})$ is a compatible right counital A -coring.

(2) Suppose that there exist elements $(v_i, w_i) \in E \times E$ ($i = 1, \dots, K$) such that

$$\hat{\delta}_{x_0}(\tilde{y}_0) = \sum_{i=1}^K v_i \otimes_{\psi} w_i.$$

Then $(E, \psi, \widehat{\delta}_{x_0}, Q_{\widehat{y}_0})$ is a compatible counital A -coring.

5. CORING STRUCTURES ON THE SET OF COMPACT OPERATORS

Let H be an infinite-dimensional separable Hilbert space. We consider H to be a Hilbert $\mathbb{C}$ -module, in particular the inner product is linear in the second variable. We denote by A the C^* -algebra $\mathcal{K}(H)$ of compact operators on H . Let E be a Hilbert A -module $\mathcal{K}(H, H \otimes H)$. The right action of A on E is defined by $(xa)(\xi) = x(a(\xi))$ for $x \in E$, $a \in A$ and $\xi \in H$ and the A -valued inner product of E is defined by $\langle x, y \rangle = x^*y$ for $x, y \in E$. Define $*$ -homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes I_H)x$ and $\psi(a)x = (I_H \otimes a)x$ for $a \in A$ and $x \in E$ respectively. We denote by F the Hilbert A -module $\mathcal{K}(H, H \otimes H \otimes H)$. The right action of A on F and the A -valued inner product of F are defined by the same formulas as those in E . There exist unitary operators $M \in \mathcal{L}_A(E \otimes_\phi E, F)$ and $\widetilde{M} \in (E \otimes_\psi E, F)$ such that

$$M(x \otimes_\phi y) = (x \otimes I_H)y,$$

$$\widetilde{M}(x \otimes_\psi y) = (I_H \otimes x)y$$

for $x, y \in E$ respectively. Define $W = M^{-1}\widetilde{M}$. Then we have the following:

Theorem 5.1. *The operator W is the unique multiplicative unitary operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$.*

Now we introduce a coring structure on (E, ϕ, ψ) . Recall that an approximate unit $\{u_n\}_{n=1}^\infty$ of A is said to be increasing if $u_n \geq 0$ and $u_{n+1} \geq u_n$ for every n .

Definition 5.2. Let δ be a coproduct of (E, ϕ) . For $n = 1, 2, \dots$, let Q_n be an element of $\mathcal{L}_A(E, A)$ such that $Q_n(\phi(a)x) = aQ_n(x)$ for $a \in A$ and $x \in E$ and let $\{u_n\}_{n=1}^\infty$ be an increasing approximate unit of A such that $u_1 \neq 0$ and $u_n \neq u_{n+1}$ for every n . Then $(\delta, \{Q_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty)$ is called a coring structure on (E, ϕ, ψ) if it

satisfies the following equations for every n ;

$$t(I_E \otimes_\phi Q_n)\delta = t'(Q_n \otimes_\phi I_E)\delta = \psi(u_n),$$

$$Q_n\psi(u_n) = Q_n.$$

Then $\{Q_n\}$ is called an approximate counit.

Let T be an element of $\mathcal{L}(H, H \otimes H)$. We will say that T has Property D if it satisfies the following conditions:

- (i) $(T \otimes I_H)T = (I_H \otimes T)T$.
- (ii) There exists a family $\{K_n\}_{n=1}^\infty$ of mutually orthogonal non-trivial finite-dimensional subspaces of H such that $H = \bigoplus_{n=1}^\infty K_n$ and there exists a complete orthonormal basis $\{e_{k_{n-1}+1}, \dots, e_{k_n}\}$ of K_n for $n = 1, 2, \dots$, where $k_0 = 0$, such that, if we set $\lambda_{j,\ell}^i = \langle e_j \otimes e_\ell, T e_i \rangle$, then $\{\lambda_{j,\ell}^i\}$ satisfies the following conditions;
 - (a) for $i = k_{n-1} + 1$, $\lambda_{i,i}^i \neq 0$ and $\lambda_{j,\ell}^i = \lambda_{\ell,j}^i = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ ($m = 0, 1, 2, \dots$) except for $j = \ell = i$,
 - (b) if $\dim K_n \geq 2$, for $i = k_{n-1} + 2, \dots, k_n$,

$$\lambda_{i,k_{n-1}+1}^i = \lambda_{k_{n-1}+1,i}^i = \lambda_{k_{n-1}+1,k_{n-1}+1}^{k_{n-1}+1},$$

and $\lambda_{j,\ell}^i = \lambda_{\ell,j}^i = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ ($m = 0, 1, 2, \dots$) except for $(j, \ell) = (i, k_{n-1} + 1)$.

Then we have the following theorem:

Theorem 5.3. *There exists a one-to-one correspondence between the set of coring structures $(\delta, \{Q_n\}, \{u_n\})$ on (E, ϕ, ψ) and the set of elements $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$ which satisfy Property D. The correspondence is given as follows: If $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$*

has Property D, set

$$H_n = \bigoplus_{i=1}^n K_i,$$

$$\xi_n = \sum_{i=1}^n \eta_i \in H_n, \quad \text{where } \eta_i = (\lambda_{k_{i-1}+1, k_{i-1}+1}^{k_{i-1}+1})^{-1} e_{k_{i-1}+1} \in K_i,$$

define $f_n \in H^*$ by $f_n(\xi) = \langle \xi_n, \xi \rangle$, then $u_n \in \mathcal{K}(H)$ is the projection onto H_n and δ and Q_n are given by the following equations;

$$\delta(x) = M^{-1}(I_H \otimes T)x,$$

$$Q_n(x) = (I_H \otimes f_n)x.$$

Question. Suppose that T has Property D. Does T determine $\{K_n\}$ and $\{e_{k_{n-1}+1}\}$ uniquely?

The following theorem shows the relation between the coring structures and the multiplicative unitary operator W defined above:

Theorem 5.4. Let $(\delta, \{Q_n\}, \{u_n\})$ be a coring structure on (E, ϕ, ψ) and let T be the operator which corresponds to $(\delta, \{Q_n\}, \{u_n\})$ by Theorem 5.3. Put $x_n = Tu_n$. Then x_n is an element of E and satisfies Property E1. Let δ_n be the coproduct of (E, ϕ) defined by

$$\delta_n(x) = W(x_n \otimes_\psi x).$$

Then the following equation holds;

$$\delta = \lim_{n \rightarrow \infty} \delta_n$$

with respect to the strict topology on $\mathcal{L}_A(E, E \otimes_\phi E)$.

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