

A Path Integral Preliminary Approach to the FKG Inequality for Yukawa₂ Quantum Field Theory*

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1. By the method used in our previous paper [I1], we construct a countably additive path space measure for the 2-D Euclidean Dirac equation in the polar coordinates to give a path integral representation to its Green's function (For a brief survey, see [I2]). This is a report of trying a preliminary approach with use of the result to give an alternative proof of the FKG inequality for Yukawa₂ quantum field theory obtained by Battle-Rosen [BR], though not yet incomplete.

G.A.Battle and L.Rosen used Vekua-Bers theory of generalized analytic functions to show the FKG inequality for Y_2 QFT. The Y_2 measure is formally given by

$$\nu := \frac{1}{Z} e^{W(\phi)} \prod_{x \in \mathbb{R}^2} d\phi(x)$$

$$W(x) := \frac{1}{2}(\phi, (-\Delta + m_b^2)\phi) + \text{Tr } K - \frac{1}{2} : \text{Tr } K^* K : + \text{Tr } \ln(1 - K)K,$$

with Z is a normalized constant, where

$$K(x, y) := S(x, y)\phi(y)\chi_\Lambda(y), \quad \phi : \text{Boson field (mass : } m_b),$$

$$\chi_\Lambda : \text{indicator function of a square } \Lambda \subset \mathbb{R}^2,$$

and

$$S(x, y) := (-\beta\partial_x + m_f)^{-1}\Gamma, \quad \beta\partial_x = \beta_0\partial_0 + \beta_1\partial_1, x = (x_0, x_1),$$

$$\beta_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \beta_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3,$$

with $m_f \geq 0$ the Fermi mass. They considered the two models

$$a) \Gamma := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \text{ (scalar } Y_2), \quad b) \Gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \text{ (pseudo-scalar } Y_2).$$

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Then FKG enequality (like $\langle fg \rangle \geq \langle f \rangle \langle g \rangle$) holds: $\frac{\delta^2 W}{\delta \phi(x) \delta \phi(y)} \geq 0$, $x \neq y$.

By some heuristic arguments, this is equivalent to showing

$$\text{tr } S'(x, y) S'(y, x) \leq 0, \quad x \neq y.$$

where $S' := (1 - K)^{-1} S$ is the Green's function (vanishing at ∞) for 2D- Euclidean Dirac equation

$$[\Gamma^{-1}(-\beta \partial_x + m_f) - \phi(x) \chi_\Lambda(x)] S'(x, y) = \delta(x - y).$$

Battle and Rosen proved the above inequality for $m_f \geq 0$ in the case a) and for $m_f = 0$ in the case b).

So, the first thing to do is to construct this Green's function.

In [I1], we constructed a countably additive path space measure to give a path integral representent for the Green's function for 3D-Dirac equation in the radial coordinate.

The aim of this talk is to give a preliminary approach to ask whether this method can apply to get the Green's function for the above 2D-Euclidean Dirac equation to show the desired inequality.

Put the 2D-Euclidian operator $L^2(\mathbb{R}^2)^2 \equiv L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ as:

$$\begin{aligned} T_\Gamma &:= \Gamma^{-1}(-\beta \partial_x + m_f) - V(x), \quad V(x) := \phi(x) \chi_\Lambda(x), \\ &= \Gamma^{-1} \left[-\sigma_1 \frac{\partial}{\partial x_0} - \sigma_3 \frac{\partial}{\partial x_1} + m_f \right] - V(x), \quad x = (x_0, x_1) \in \mathbb{R}^2, \\ \beta &= (\beta_0, \beta_1), \quad \beta_0 = \sigma_1, \quad \beta_1 = \sigma_3. \end{aligned}$$

They considered the two models: a) scalar Y_2 model: $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

b) pseudoscalar Y_2 model: $\Gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In this note let us consider only a) the scalar Y_2 model.

2. Since $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have by the polar coordinates $x_0 = r \cos \theta$, $x_1 = r \sin \theta$ ($0 \leq r < \infty$, $0 \leq \theta < 2\pi$),

$$T_\Gamma = -C(\theta) \frac{\partial}{\partial r} - \frac{1}{r} D(\theta) \frac{\partial}{\partial \theta} + m_f - V,$$

where

$$\begin{aligned} C(\theta) &:= \sigma_1 \cos \theta + \sigma_3 \sin \theta = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}, \\ D(\theta) &:= -(\sigma_1 \sin \theta - \sigma_3 \cos \theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}. \end{aligned}$$

We write $\mathbb{R}_+ = (0, \infty)$ and $\overline{\mathbb{R}}_+ = [0, \infty)$.

Making the unitary transformation

$$U(\theta) := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \sin \theta} & \frac{\cos \theta}{\sqrt{1 + \sin \theta}} \\ -\frac{\cos \theta}{\sqrt{1 + \sin \theta}} & \sqrt{1 + \sin \theta} \end{pmatrix},$$

we have

$$U(\theta)T_{\Gamma}U(\theta)^{-1} = \left[- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} + \frac{1}{2r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + m_f - V$$

in $L^2(\mathbb{R}^2)^2 = L^2(\overline{\mathbb{R}}_+ \times [0, 2\pi); r dr d\theta)^2$.

We make one more unitary transformation W of the $r dr$ -measure space to the dr -measure space:

$$W : L^2(\mathbb{R}^2) \equiv L^2(\overline{\mathbb{R}}_+ \times [0, 2\pi); r dr d\theta)^2 \ni f \mapsto r^{1/2} f \in L^2(\overline{\mathbb{R}}_+ \times [0, 2\pi); dr d\theta)^2$$

to get

$$WU(\theta)T_{\Gamma}U(\theta)^{-1}W^{-1} = \left[- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} - \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + m_f - V.$$

Then we multiply $r^{1/2}$ from the left and the right and then multiply the factor i to put

$$\begin{aligned} H_{sc}(rV) &:= i r^{1/2} WU(\theta)T_{\Gamma}U(\theta)^{-1}W^{-1} r^{1/2} \\ &= \left[-i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + i(m_f - V)r. \end{aligned}$$

Since the operator $-i \frac{\partial}{\partial \theta}$ is a selfadjoint operator in $L^2([0, 2\pi); d\theta)$ having as the spectrum consisting of only the eigenvalues $\{k\}_{k \in \mathbb{Z}}$ with eigenfunctions $\{\frac{e^{ik\theta}}{\sqrt{2\pi}}\}_{k \in \mathbb{Z}}$, our L^2 space $L^2(\overline{\mathbb{R}}_+ \times [0, 2\pi); dr d\theta)^2$ admits the direct sum decomposition:

$$L^2(\overline{\mathbb{R}}_+ \times [0, 2\pi); dr d\theta)^2 = \sum_{k \in \mathbb{Z}} \oplus \left(L^2(\overline{\mathbb{R}}_+; dr)^2 \otimes \left[\frac{e^{ik\theta}}{\sqrt{2\pi}} \right] \right).$$

Then we have

$$\begin{aligned} H_{sc}(rV) &= \sum_{k \in \mathbb{Z}} \oplus H_{sc}(k), \\ H_{sc}(k) &:= \left[-i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + i(m_f - V)r. \end{aligned}$$

We want to find a path integral representation for the Green's function for this operator having a singularity at $r = 0$.

For each fixed $k \in \mathbb{Z}$, put the free part of $H_{sc}(k)$ to be equal to

$$H_0(k) := -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is an operator in $L^2(\overline{\mathbb{R}_+}; dr)^2$. We can show that $H_0(k)$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}_+)^2$, which is a non-trivial result. Therefore the Cauchy problem for it :

$$\begin{aligned} \frac{\partial}{\partial t} \psi(r, t) &= -i H_0(k) \psi(r, t), \quad t \in \mathbb{R}, \\ \psi(r, 0) &= g(r), \quad t = 0, \end{aligned}$$

is L^2 well-posed. In other words, we can solve it in the space $L^2(\overline{\mathbb{R}_+}; dr)^2$.

Crucial is that this Cauchy problem is even L^∞ well-posed. Namely, we have the following lemma.

Lemma. There exists a unique solution $\psi(r, t) = (e^{-itH_0(k)}g)(r)$ which satisfies

$$\|\psi(\cdot, t)\|_\infty = \|e^{-itH_0(k)}g\|_\infty \leq e^{|t|(|k|+1/2)} \|g\|_\infty.$$

By the method in [I1] based on this lemma, we can construct a 2×2 -matrix-distribution-valued countably additive path space measure $\mu_{t,0}^k$ on the space $C([0, t] \rightarrow \overline{\mathbb{R}_+})$ of the continuous paths $R : [0, t] \rightarrow \overline{\mathbb{R}_+}$ which represents the solution of the above Cauchy problem: for every pair of f and g in $C_0^\infty(\mathbb{R}_+)^2$,

$$\begin{aligned} (f, \psi(\cdot, t)) &= \int_0^\infty \overline{f(r)} (e^{-itH_{sc}(k)}g)(r) dr = \int_0^\infty \int_0^\infty \overline{f(r)} e^{-itH_{sc}(k)}(r, \rho) g(\rho) dr d\rho \\ &= \int_{C([0, t] \rightarrow \overline{\mathbb{R}_+})} \langle \overline{f(R(t))}, d\mu_{t,0}^k(R) g(R(0)) \rangle e^{\int_0^t (m_f - V(R(s))R(s)) ds}. \end{aligned}$$

Hence, supposing that we can get the inverse of the operator $H_{sc}(k)$ as $H_{sc}(k)^{-1} = i \int_0^\infty e^{-itH_{sc}(k)} dt$ by the Laplace transform, we have the following path integral representation for its Green's function, which is a little formally expressed, suppressing the use of test functions:

$$\begin{aligned} H_{sc}(k)^{-1}(r, \rho) \\ = i \int_0^\infty dt \int_{C([0, t] \rightarrow \overline{\mathbb{R}_+}), R(0)=\rho, R(t)=r} r^{1/2} \rho^{1/2} e^{\int_0^t (m_f - V(R(s))R(s)) ds} d\mu_{t,0}^k(R). \end{aligned}$$

3. We have

$$T_\Gamma^{-1} = i r^{1/2} W U(\theta) H_{sc}(rV)^{-1} U(\theta)^{-1} W^{-1} r^{-1/2}.$$

Here, if we use the polar coordinates for $x = (x_0, x_1)$, $y = (y_0, y_1) \in \mathbb{R}^2$

$$\begin{aligned} x_0 &= r \cos \theta, \quad x_1 = r \sin \theta \quad (0 \leq r < \infty, \quad 0 \leq \theta < 2\pi), \\ y_0 &= r' \cos \theta', \quad y_1 = r' \sin \theta' \quad (0 \leq r' < \infty, \quad 0 \leq \theta' < 2\pi), \end{aligned}$$

we may write the integral kernel of the operator $H_{sc}(rV)^{-1}$ as

$$\begin{aligned} & H_{sc}(rV)^{-1}(r, \theta; r', \theta') \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} H_{sc}(k)^{-1}(r, r') e^{-ik(\theta - \theta')} \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-ik(\theta - \theta')} i \int_{R(0)=r', R(t)=r} r^{1/2} r'^{1/2} e^{\int_0^\infty (m_f - V(R(s))) R(s) ds} d\mu_{t,0}^{(k)}(R). \end{aligned}$$

Then

$$\begin{aligned} & \text{tr} \left[T_\Gamma^{-1}(r, \theta; r', \theta') T_\Gamma^{-1}(r', \theta'; r, \theta) \right] \\ &= -\text{tr} \left[r^{-1/2} W U(\theta) H_{sc}(rV)^{-1}(r, \theta; r', \theta') U(\theta')^{-1} W^{-1} r^{-1/2} \right. \\ & \quad \left. \times r^{-1/2} W U(\theta') H_{sc}(rV)^{-1}(r', \theta'; r, \theta) U(\theta)^{-1} W^{-1} r^{-1/2} \right] \\ &= -\text{tr} \left[r r' H_{sc}(rV)^{-1}(r, \theta; r', \theta') H_{sc}(rV)^{-1}(r', \theta'; r, \theta) \right] \\ &= -r r' \text{tr} \left[\left(\sum_{k \in \mathbb{Z}} H_{sc}(k)^{-1}(r, r') \frac{e^{-ik(\theta - \theta')}}{2\pi} \right) \left(\sum_{\ell \in \mathbb{Z}} H_{sc}(\ell)^{-1}(r', r) \frac{e^{-i\ell(\theta' - \theta)}}{2\pi} \right) \right] \\ &= -\frac{r r'}{(2\pi)^2} \text{tr} \left[\sum_{k, \ell \in \mathbb{Z}} H_{sc}(k)^{-1}(r, r') H_{sc}(\ell)^{-1}(r', r) e^{-i(k - \ell)(\theta - \theta')} \right] \\ &= -\frac{r r'}{(2\pi)^2} \text{tr} \sum_{k, \ell \in \mathbb{Z}} a_{k\ell} e^{-i(k - \ell)(\theta - \theta')}. \end{aligned}$$

Here we seem to have

$$\begin{aligned} a_{k\ell} &:= i \int_0^\infty e^{-itH_{sc}(k)}(r, r') dt (-i) \int_0^\infty e^{iuH_{sc}(\ell)}(r', r) du \\ &= \int_0^\infty dt \int_0^\infty du \\ & \quad \times \int_{C([0,t] \rightarrow \overline{\mathbb{R}_+}), R_1(0)=r', R_1(t)=r} e^{\int_0^t (m_f - V(R_1(s))) R_1(s) ds} d\mu_{t,0}^k(R_1) \\ & \quad \times \int_{C([0,u] \rightarrow \overline{\mathbb{R}_+}), R_2(0)=r', R_2(u)=r} e^{\int_u^0 (m_f - V(R_2(s))) R_2(s) ds} d\mu_{0,u}^\ell(R_2) \\ &= \int_0^\infty dt \int_0^\infty du \int_{C([0,t] \rightarrow \overline{\mathbb{R}_+}), R_1(0)=r', R_1(t)=r} \int_{C([0,u] \rightarrow \overline{\mathbb{R}_+}), R_2(0)=r', R_2(u)=r} \\ & \quad \times e^{\int_0^t (m_f - V(R_1(s))) R_1(s) ds - \int_0^u (m_f - V(R_2(s))) R_2(s) ds} d\mu_{t,0}^k(R_1) d^t \mu_{u,0}^\ell(R_2), \end{aligned}$$

where ${}^t \mu_{u,0}^\ell$ is the transposed of the 2×2 -matrix-distribution valued-measure $\mu_{0,u}^\ell$.

Then the problem is to show in the case a) that

$$\text{tr} \sum_{k, \ell \in \mathbb{Z}} a_{k\ell} e^{-i(k - \ell)(\theta - \theta')} \geq 0.$$

But our the argument is stopped here, and will be discussed elsewhere.

References

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