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A determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2

By

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Abstract

We give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2, which is a generalization of Endô's formulas for the mth cyclotomic field, m an odd integer, and its quadratic extension.

§ 1. Introduction

Let p be an odd prime. For an integer u let $R_p(u)$ and $R'_p(u)$ be the integers such that

$$R_p(u) \equiv u \pmod{p}, \quad 0 \leq R_p(u) < p$$

and

$$R'_p(u) \equiv u \pmod{p}, \quad -\frac{p}{2} < R'_p(u) < \frac{p}{2},$$

respectively. For an integer u coprime to p, let u^{-1} be an integer with $uu^{-1} \equiv 1 \pmod{p}$. We have already obtained a lot of determinant formulas for the pth cyclotomic

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field $\mathbf{Q}(\zeta_p)$, ζ_p a primitive pth root of unity. For example,

$$\det\left(R_p(uv^{-1})\right)_{1\leq u,\,v\leq(p-1)/2}=(-1)^{\frac{p-3}{2}}\,p^{\frac{p-3}{2}}h_p^*, \tag{1.1}$$

$$\det(R_p'(uv^{-1}))_{1\leq u,\,v\leq(p-1)/2}=\begin{cases}2^{\frac{p-1}{\mathrm{ord}_p(2)}-1}p^{\frac{p-3}{2}}h_p^* & \text{if } 2\,|\,\mathrm{ord}_p(2),\\ 0 & \text{otherwise,}\end{cases} \tag{1.2}$$

$$\det\left((-1)^{R_p(uv^{-1})}\right)_{1\leq u,\,v\leq(p-1)/2}=(-1)^{\frac{p-1}{2}}\frac{2^{\frac{p-3}{2}}}{p}\prod_{\chi\in X^-}(1-\chi(2)2)\cdot h_p^*, \tag{1.3}$$

where $\mathrm{ord}_p(2)$ is the order of 2 modulo p, X^- the set of odd characters of the field $\mathbf{Q}(\zeta_p)$ and h_p^* the relative class number of the field $\mathbf{Q}(\zeta_p)$.

The determinant in the formula (1.1) is called Maillet determinant (See [1]) and the one in (1.3) could be called Dem'janenko determinant. The formulas (1.1) and (1.3) are special ones of the generalized formulas in [3], [14], [17] and [18]; and the formula (1.2) a special one of [3], [15] and [17]. Funakura [7] gave, up to sign, a generalized formula of (1.3) for the mth cyclotomic field, m an odd integer.

As a corresponding formula to (1.3), we shall obtain by the formula (2.5) in Corollary 2.3

$$\det\left((-1)^{R_p'(uv^{-1})}\right)_{1\leq u,\,v\leq(p-1)/2}=\begin{cases}-2^{\frac{p-3}{2}}\dfrac{h_{4p}^*}{h_p^*} & \text{if } p\equiv 3 \pmod 4,\\ 0 & \text{otherwise,}\end{cases} \tag{1.4}$$

where h_{4p}^* is the relative class number of the $4p$th cyclotomic field. Kanemitsu and Kuzumaki [15, Corollary 4] have already obtained the formula (1.4), up to sign, under some condition.

The aim of this paper is to give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields with relative degree 2, which is a generalization not only of the formula (1.4) but also of Endô's formulas in [2] and [4]. As does the formula in [14], our determinant formula has a parameter b. By taking $b=fm+1$ (fm : the conductor of "the larger field") we obtain the formula in [2, Theorem 1 for $k=1$] ; by taking $b=2$, the one in [4, Theorem 1]; by taking $b=g$ (g : a primitive root modulo p), the one in Corollary 2.5, in which the elements of determinant are coefficients of some digit expression as in [11].

Our result would be an answer to the inquiry of Kanemitsu and Kuzumaki [15, p.285] about the relation between Tsumura's and the author's generalized Dem'janenko determinants and (generalization of) Endô's determinants S_p, T_p and U_p in [5]. The determinants S_p, T_p and U_p are special ones of the left-hand sides of (2.5), (2.6) and (2.7) for the pth cyclotomic field $\mathbf{Q}(\zeta_p)$, respectively.

§ 2. Results

Let m be an integer with $m \geq 3$ and $m \not\equiv 2 \pmod 4$. For an integer u let $R_m(u)$ and $R'_m(u)$ be the integers such that

$$R_m(u) \equiv u \pmod m, \quad 0 \leq R_m(u) < m$$

and

$$R'_m(u) \equiv u \pmod m, \quad -\frac{m}{2} \leq R'_m(u) < \frac{m}{2},$$

respectively. For an integer u coprime to m, let u^{-1} be an integer such that $uu^{-1} \equiv 1 \pmod m$.

Let K be an imaginary abelian number field of degree $2n = [K : \mathbf{Q}]$ and with conductor m. Let h_K^*, Q_K and w_K be the relative class number of K, the unit index of K and the number of roots of unity in K, respectively.

Let G_m be the multiplicative group $(\mathbf{Z}/m\mathbf{Z})^\times$, $\mathbf{Z}$ the ring of integers, and H the subgroup of G_m corresponding to K. For an integer t coprime to m let $\bar{t} = t + m\mathbf{Z} \in G_m$.

Since H does not contain $\overline{-1}$, we can take classes $C_1, C_2, \cdots, C_n$ of G_m/H satisfying

$$G_m/H = \{C_1, -C_1, C_2, -C_2, \ldots, C_n, -C_n\}.$$

Let $\mathcal{R} = \{C_1, C_2, \ldots, C_n\}$ and let $C_1 = H$.

Let X^+ and X^- be the sets of primitive even and odd Dirichlet characters of K, respectively. In the following, except to specify, we assume that the characters we consider are primitive.

Let b be an integer with $b \geq 2$ and $m \nmid b$. Let $m' = m/(m,b)$, $b' = b/(m,b)$, where (m,b) is the greatest common divisor of m and b.

For a character $\chi \in X^-$ let f_χ be the conductor of χ and define $c_\chi(b)$ as

$$c_\chi(b) = \begin{cases} b\prod_{l|m}(1-\overline{\chi}(l)) & \text{if } f_\chi \nmid m', \\ \\ b\prod_{l|m}(1-\overline{\chi}(l)) - \frac{\varphi(m)}{\varphi(m')}\chi(b')\prod_{l|m'}(1-\overline{\chi}(l)) & \text{if } f_\chi \mid m', \end{cases}$$

where l runs over prime numbers, $\overline{\chi}$ is the conjugate character of χ and φ the Euler totient function.

Let $\widetilde{K}$ be the composite of K and a quadratic field $\mathbf{Q}(\sqrt{D})$, where D is the discriminant of the field $\mathbf{Q}(\sqrt{D})$. We assume that D is coprime to m. Let f be the conductor of the field $\mathbf{Q}(\sqrt{D})$ and ψ the quadratic Dirichlet character corresponding to $\mathbf{Q}(\sqrt{D})$.

For a class $C = \bar{c}H$ in G_m/H let

$$T_{C,\psi}^{(b)} = \sum_{\bar{t} \in H} \sum_{d=0}^{f-1} \psi(R_m(ct) + dm)\left(\left[b \cdot \frac{R_m(ct) + dm}{fm}\right] - \frac{b-1}{2}\right),$$

where $[x]$ means the integral part of a rational number x. When $H=\{\overline{1}\}$, we use $T_{c,\psi}^{(b)}$ instead of $T_{C,\psi}^{(b)}$ for $C=\overline{c}H=\{\overline{c}\}$.

Let $h_{\widetilde{K}}^*$, $Q_{\widetilde{K}}$, $w_{\widetilde{K}}$ and $\widetilde{c}_\chi(b)$ be defined for $\widetilde{K}$ as above. Note that we define $\widetilde{c}_\chi(b)$ by using fm instead of m.

Theorem 2.1. *Let K be an imaginary abelian number field of degree $2n$ and with conductor m. Let $\widetilde{K}$, ψ and f be as above. Take an integer b with $b\geq 2$ and $fm\nmid b$. Then we have*

$$\det\left(T_{C_iC_j^{-1},\psi}^{(b)}\right)_{C_i,C_j\in\mathcal{R}}=\prod_{\chi\in X^*}\widetilde{c}_{\chi\psi}(b)\cdot\prod_{\chi\in X^*}\frac{1}{2}B_{1,\chi\psi} \tag{2.1}$$

$$=(-1)^n\prod_{\chi\in X^*}\widetilde{c}_{\chi\psi}(b)\cdot\frac{Q_K\,w_K}{Q_{\widetilde{K}}\,w_{\widetilde{K}}}\cdot\frac{h_{\widetilde{K}}^*}{h_K^*},$$

where X^* is X^+ or X^- according as $\psi(-1)=-1$ or $\psi(-1)=+1$.

When $b=fm+1$, we have the following formula, which is obtained by taking $k=1$ in [2, Theorem 1]:

Corollary 2.2 (cf. [2, Theorem 1]). *Let K be an imaginary abelian number field of degree $2n$ and with conductor m. Let $\widetilde{K}$, ψ and f be as above. Then we have*

$$\det\left(\sum_{\overline{t}\in H}\sum_{d=0}^{f-1}\psi(R_m(c_ic_j^{-1}t)+dm)\left(\frac{R_m(c_ic_j^{-1}t)+dm}{fm}-\frac{1}{2}\right)\right)_{C_i,\,C_j\in\mathcal{R}} \tag{2.2}$$

$$=\prod_{\chi\in X^*}\prod_{l|m}(1-\chi\psi(l))\cdot\prod_{\chi\in X^*}\frac{1}{2}B_{1,\chi\psi}$$

$$=(-1)^n\prod_{\chi\in X^*}\prod_{l|m}(1-\chi\psi(l))\cdot\frac{Q_K\,w_K}{Q_{\widetilde{K}}\,w_{\widetilde{K}}}\cdot\frac{h_{\widetilde{K}}^*}{h_K^*},$$

where $C_i=\overline{c_i}H$, $C_j=\overline{c_j}H$.

Let ψ_0 be the principal character modulo f and let

$$T_{C,\psi_0}^{(b)}=\sum_{\overline{t}\in H}\sum_{d=0}^{f-1}\psi_0(R_m(ct)+dm)\left(\left[b\cdot\frac{R_m(ct)+dm}{fm}\right]-\frac{b-1}{2}\right)$$

for $C=\overline{c}H$.

We remark that we have already obtained

$$\text{(2.3)} \qquad \det\left(T^{(b)}_{C_iC_j^{-1},\psi_0}\right)_{C_i,C_j\in\mathcal{R}} = \prod_{\chi\in X^-} \tilde{c}_\chi(b)\cdot\prod_{\chi\in X^-}\frac{1}{2}B_{1,\chi}$$

(See [12, (24)]) and

$$\text{(2.4)} \quad \det\left(T^{(b)}_{C_iC_j^{-1},\psi_0}\right)_{C_i,C_j\in\mathcal{R}} \det\left(T^{(b)}_{C_iC_j^{-1},\psi}\right)_{C_i,C_j\in\mathcal{R}} = \prod_{\chi\in\widetilde{X}^-} \tilde{c}_\chi(b)\cdot\prod_{\chi\in\widetilde{X}^-}\frac{1}{2}B_{1,\chi}$$
$$= \prod_{\chi\in\widetilde{X}^-} \tilde{c}_\chi(b)\cdot\frac{h^*_{\widetilde{K}}}{Q_{\widetilde{K}}w_{\widetilde{K}}},$$

where $\widetilde{X}^-$ is the set of odd characters of $\widetilde{K}$. Kučera [16] gave a determinant formula generalizing the formula (2.4) but he did not refer to the formula (2.3).

Endô [6, Theorem] gave the formula (2.4), up to sign, in the case where $K=\mathbf{Q}(\zeta_{p^\mu})$, a p-power-th cyclotomic field, $\widetilde{K}=\mathbf{Q}(\zeta_{p^\mu},\ \sqrt{f})$, $(f,p)=1$ and $b=fp^\mu+1$.

We can not get our result (2.1) "directly" from (2.3) and (2.4), because $\det\left(T^{(b)}_{C_iC_j^{-1},\psi_0}\right)_{C_i,C_j\in\mathcal{R}}$ is equal to zero under some conditions.

In Theorem 2.1, taking $b=fm+1$; $\psi=\chi_4$, $\psi=\chi_4\psi_8$ and $\psi=\psi_8$, we have generalizations of Endô's formulas for the mth cyclotomic field $\mathbf{Q}(\zeta_m)$, m an odd integer, in [2, Theorem 2], where χ_4 is the odd character with conductor 4 and ψ_8 the even character with conductor 8:

Corollary 2.3 (cf. [2, Theorem 2]). *Let K be an imaginary abelian number field of degree $2n$ and with odd conductor m. Then we have*

$$\text{(2.5)} \quad \det\left(\sum_{\bar{t}\in H}(-1)^{R'_m(c_ic_j^{-1}t)}\right)_{C_i,C_j\in\mathcal{R}}$$
$$=\chi_4(m)^n\,2^n\prod_{\chi\in X^+}\chi(2)\cdot\prod_{\chi\in X^+}\prod_{l\mid m}(1-\chi\chi_4(l))\cdot\frac{Q_K\,w_K}{Q_{K(\sqrt{-1})}\,w_{K(\sqrt{-1})}}\cdot\frac{h^*_{K(\sqrt{-1})}}{h^*_K},$$

$$\text{(2.6)} \quad \det\left(\sum_{\bar{t}\in H}T'_m(c_ic_j^{-1}t)\right)_{C_i,C_j\in\mathcal{R}}$$
$$=\chi_4\psi_8(m)^n\prod_{\chi\in X^+}\chi(2)\cdot\prod_{\chi\in X^+}\prod_{l\mid m}(1-\chi\chi_4\psi_8(l))\cdot\frac{Q_K\,w_K}{Q_{K(\sqrt{-2})}\,w_{K(\sqrt{-2})}}\cdot\frac{h^*_{K(\sqrt{-2})}}{h^*_K}$$

and

$$(2.7)\qquad \det\left(\sum_{\bar{t}\in H} U'_m(c_i c_j^{-1} t)\right)_{C_i, C_j\in\mathcal{R}}$$

$$= (-1)^{\frac{m+1}{2}n}\psi_8(m)^n \prod_{\chi\in X^+}\chi(2)\cdot\prod_{\chi\in X^-}\prod_{l|m}(1-\chi\psi_8(l))\cdot\frac{Q_K\,w_K}{Q_{K(\sqrt{2})}\,w_{K(\sqrt{2})}}\cdot\frac{h^*_{K(\sqrt{2})}}{h^*_K},$$

where $C_i = \overline{c_i}H$, $C_j = \overline{c_j}H$ *and for an integer* c

$$T'_m(c) = \begin{cases} (-1)^{\frac{R'_m(c)}{2}} & \text{if } R'_m(c)\equiv 0 \pmod 2,\\ 0 & \text{if } R'_m(c)\equiv 1 \pmod 2 \end{cases}$$

and

$$U'_m(c) = \begin{cases} 0 & \text{if } R'_m(c)\equiv 0 \pmod 2,\\ (-1)^{\frac{R'_m(c)-1}{2}} & \text{if } R'_m(c)\equiv 1 \pmod 2. \end{cases}$$

Endô's formula in [2, Theorem 2] are represented, up to sign, by the form of product of first generalized Bernoulli numbers. Endô [5] has already given such determinants in (2.5), (2.6) and (2.7) for the pth cyclotomic field $\mathbf{Q}(\zeta_p)$.

As introduced in §1, the formula (1.4) is a special case of (2.5) for the pth cyclotomic field $\mathbf{Q}(\zeta_p)$. Here we note that if $p\equiv 3 \pmod 4$, then $\prod_{\chi\in X^+}\chi(2) = \chi_p(2)^{(p-1)\frac{p-3}{4}} = +1$, where χ_p is a Dirichlet character with conductor p and of degree $p-1$.

In Theorem 2.1 taking $K=\mathbf{Q}(\zeta_p)$ and $b=2$, we have formulas in [4]:

Corollary 2.4 ([4, Theorem 1]). *Let K be the pth cyclotomic field $\mathbf{Q}(\zeta_p)$, p an odd prime. Let D be a square-free integer such that $(D,p)=1$ and $D\equiv 1 \pmod 4$. Let $\widetilde{K}$ be the composite of K and the quadratic field $\mathbf{Q}(\sqrt{D})$. Let $\psi(u)$ be the quadratic character corresponding to the field $\mathbf{Q}(\sqrt{D})$. For an integer u with $(u,p)=1$ put*

$$S_u(\psi) = \sum_{\substack{k=1\\ (k,pD)=1,\ k\equiv u \pmod p}}^{(p|D|-1)/2}\psi(k).$$

Then we have: If $D>0$, then

$$(2.8)\qquad \det\left(S_{uv}(\psi)\right)_{1\le u,\,v\le(p-1)/2} = \pm\prod_{i=1}^{(p-1)/2}\left(2-\chi_p^{2i-1}\psi(2)\right)\frac{1}{2}B_{1,\chi_p^{2i-1}\psi}$$

$$= \pm\frac{2p}{Q_{\widetilde{K}}w_{\widetilde{K}}}\prod_{i=1}^{(p-1)/2}\left(2-\psi\chi_p^{2i-1}(2)\right)\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}.$$

If $D<0$, *then*

$$\text{(2.9)}\quad \det\left(S_{uv}(\psi)\right)_{1\le u,\,v\le (p-1)/2} = \pm\,(1-\psi(p))\prod_{i=1}^{(p-1)/2}(2-\chi_p^{2i}\psi(2))\frac{1}{2}B_{1,\chi_p^{2i}\psi}$$
$$= \pm\,(1-\psi(p))\,\frac{2p}{Q_{\widetilde{K}}w_{\widetilde{K}}}\prod_{i=1}^{(p-1)/2}(2-\psi\chi_p^{2i}(2))\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}.$$

In Theorem 2.1 taking $K=\mathbf{Q}(\zeta_p)$ and $b=g$, g a primitive root modulo p, we have a formula corresponding to the one in [11, Corollary 2]:

Corollary 2.5. *Let K be the pth cyclotomic field $\mathbf{Q}(\zeta_p)$, p an odd prime. Let g be a primitive root modulo p with $g\ge 2$ and $g\equiv 1 \pmod 4$. Let $\widetilde{K}$ be the composite of K and the quadratic field $\mathbf{Q}(\sqrt{-1})$. Then for an integer i we have*

$$T^{(g)}_{R_p(g^i),\chi_4} = \sum_{\substack{0\le k\le p-2\\ R_{4p}(g^k)\equiv R_p(g^i)\pmod p}} \chi_4(R_{4p}(g^k))\left[\frac{gR_{4p}(g^k)}{4p}\right] + \sum_{\substack{0\le k\le p-2\\ R_{4p}(-g^k)\equiv R_p(g^i)\pmod p}} \chi_4(R_{4p}(-g^k))\left[\frac{gR_{4p}(-g^k)}{4p}\right]$$
$$= \sum_{\substack{0\le k\le p-2\\ R_{4p}(g^k)\equiv R_p(g^i)\pmod p}} \left[\frac{gR_{4p}(g^k)}{4p}\right] - \sum_{\substack{0\le k\le p-2\\ R_{4p}(-g^k)\equiv R_p(g^i)\pmod p}} \left[\frac{gR_{4p}(-g^k)}{4p}\right]$$

and

$$\det\left(T^{(g)}_{R_p(g^{i-j}),\chi_4}\right)_{0\le i,j\le (p-3)/2} = (-1)^{\frac{p-1}{2}}\frac{1}{4}(1-\chi_4(p))\prod_{\chi\in X^+}(g-\chi(g))\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}.$$

Remark. With the notation in Corollary 2.5 expand $1/(4p)$ to the basis $1/g$:

$$\frac{1}{4p}=\sum_{k=1}^{\infty}\frac{x(k)}{g^k},\quad x(k)\in\{0,1,\dots,g-1\}.$$

Then we have

$$x(k)=\left[\frac{gR_{4p}(g^{k-1})}{4p}\right]\quad \text{for } k=1,2,\dots$$

(See [11, Theorem 10]).

Example 2.6. We give here an example of Corollary 2.5. Let K, $\widetilde{K}$ and g be as in Corollary 2.5. We take $p=7$, $g=5$. Then $K=\mathbf{Q}(\zeta_7), \widetilde{K}=\mathbf{Q}(\zeta_{28})$ and

$$\begin{aligned}
T^{(5)}_{R_7(5^0),\chi_4} &= \left[\frac{5\cdot 1}{28}\right] - \left[\frac{5\cdot 15}{28}\right] = 0-2 = -2,\\
T^{(5)}_{R_7(5^1),\chi_4} &= \left[\frac{5\cdot 5}{28}\right] - \left[\frac{5\cdot 19}{28}\right] = 0-3 = -3,\\
T^{(5)}_{R_7(5^2),\chi_4} &= \left[\frac{5\cdot 25}{28}\right] - \left[\frac{5\cdot 11}{28}\right] = 4-1 = 3,\\
T^{(5)}_{R_7(5^3),\chi_4} &= \left[\frac{5\cdot 13}{28}\right] - \left[\frac{5\cdot 27}{28}\right] = 2-4 = -2,\\
T^{(5)}_{R_7(5^4),\chi_4} &= \left[\frac{5\cdot 9}{28}\right] - \left[\frac{5\cdot 23}{28}\right] = 1-4 = -3,\\
T^{(5)}_{R_7(5^5),\chi_4} &= \left[\frac{5\cdot 17}{28}\right] - \left[\frac{5\cdot 3}{28}\right] = 3-0 = 3.
\end{aligned}$$

Hence

$$\det\left(T^{(5)}_{R_7(5^{i-j}),\chi_4}\right)_{0\le i,j\le 2} = \det\begin{pmatrix} -2 & 3 & -3\\ -3 & -2 & 3\\ 3 & -3 & -2\end{pmatrix} = -62.$$

On the other hand, letting ζ_u be a primitive uth root of unity, we have

$$\begin{aligned}
(-1)^{\frac{7-1}{2}}\frac{1}{4}(1-\chi_4(7))\prod_{\chi\in X^+}(5-\chi(5))\cdot\frac{h^*_{\widetilde{K}}}{h^*_K} &= -\frac{1}{4}\cdot 2\cdot(5-1)(5-\zeta_3)(5-\zeta_3^2)\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}\\
&= -62\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}.
\end{aligned}$$

Therefore $h^*_{\widetilde{K}} = h^*_K$. Actually, we have already known that $h^*_{\widetilde{K}} = h^*_K = 1$.

§ 3. Proofs of Theorem and Corollaries

To prove Theorem 2.1 we need the following lemma originating from [8].

Lemma 3.1 ([14, Lemma 1]). *Let K be an imaginary abelian number field with conductor m and b an integer with $b\ge 2$ and $m\nmid b$. Then, for an odd character χ of K and for $C=\bar{c}H$ we have*

$$\begin{aligned}
\frac{1}{2}c_\chi(b)B_{1,\overline{\chi}} &= \sum_{C\in\mathcal{R}}\overline{\chi}(c)T^{(b)}_C\\
&= \frac{1}{2}\sum_{\substack{k=1\\(k,m)=1}}^{m}\overline{\chi}(k)\left(\left[b\cdot\frac{R_m(k)}{m}\right]-\frac{b-1}{2}\right),
\end{aligned}$$

where

$$T_C^{(b)} = \sum_{\bar{t}\in H} \left(\left[b \cdot \frac{R_m(ct)}{m} \right] - \frac{b-1}{2} \right).$$

Proof of Theorem 2.1. Recall that for a class $C = \bar{c}H$ in G_m/H we define

$$T_{C,\psi}^{(b)} = \sum_{\bar{t}\in H} \sum_{d=0}^{f-1} \psi(R_m(ct)+dm) \left(\left[b \cdot \frac{R_m(ct)+dm}{fm} \right] - \frac{b-1}{2} \right).$$

Let $\psi(-1) = (-1)^{k'}$, $k' = 0$ or 1. Then we have

$$T_{-C,\psi}^{(b)} = (-1)^{k'+1} T_{C,\psi}^{(b)}.$$

First we consider the case where $\psi(-1) = -1$. Since $T_{-C,\psi}^{(b)} = T_{C,\psi}^{(b)}$, it follows from the group determinant (cf. for example, [19, p.71]) that

$$\det(T_{C_iC_j^{-1},\psi}^{(b)})_{C_i,C_j\in\mathcal{R}} = \prod_{\chi\in X^+} \sum_{\bar{c}H\in\mathcal{R}} \chi(c) T_{\bar{c}H,\psi}^{(b)}.$$

Since

$$\sum_{\bar{c}H\in\mathcal{R}} \chi(c) T_{\bar{c}H,\psi}^{(b)} = \frac{1}{2} \sum_{\substack{k=1\\(k,fm)=1}}^{fm} \chi\psi(k) \left(\left[\frac{bk}{fm} \right] - \frac{b-1}{2} \right),$$

noting that the assumption of b, we obtain by Lemma 3.1

$$\sum_{\bar{c}H\in\mathcal{R}} \chi(c) T_{\bar{c}H,\psi}^{(b)} = \frac{1}{2} \tilde{c}_{\overline{\chi\psi}}(b) B_{1,\chi\psi}.$$

Therefore we have

$$\begin{aligned}\det(T_{C_iC_j^{-1},\psi}^{(b)})_{C_i,C_j\in\mathcal{R}} &= \prod_{\chi\in X^+} \tilde{c}_{\chi\psi}(b) \cdot \prod_{\chi\in X^+} \frac{1}{2} B_{1,\chi\psi}\\ &= (-1)^n \prod_{\chi\in X^*} \tilde{c}_{\chi\psi}(b) \cdot \frac{Q_K\, w_K}{Q_{\tilde{K}}\, w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.\end{aligned}$$

Next we consider the case where $\psi(-1) = +1$. Taking some odd character χ_1 of

K, we have, letting $C_i = \overline{c_i}H$, $C_j = \overline{c_j}H$,

$$\begin{aligned}\det(T^{(b)}_{C_iC_j^{-1},\psi})_{C_i,C_j\in\mathcal{R}} &= \det(\chi_1(c_ic_j^{-1})T^{(b)}_{C_iC_j^{-1},\psi})_{C_i,C_j\in\mathcal{R}} \\ &= \prod_{\chi\in X^+}\sum_{\overline{a}H\in\mathcal{R}} \chi_1\chi(a)T^{(b)}_{\overline{a}H,\psi} \\ &= \prod_{\chi\in X^-}\sum_{\overline{a}H\in\mathcal{R}} \chi(a)T^{(b)}_{\overline{a}H,\psi} \\ &= \prod_{\chi\in X^-}\widetilde{c}_{\chi\psi}(b)\cdot\prod_{\chi\in X^-}\frac{1}{2}B_{1,\chi\psi} \\ &= (-1)^n\prod_{\chi\in X^*}\widetilde{c}_{\chi\psi}(b)\cdot\frac{Q_K\,w_K}{Q_{\widetilde{K}}\,w_{\widetilde{K}}}\cdot\frac{h^*_{\widetilde{K}}}{h^*_K}.\end{aligned}$$

This completes the proof. □

Corollary 2.2 is easily proved by Theorem 2.1, because when $b = fm+1$, it holds that

$$\widetilde{c}_{\chi\psi}(b) = fm\prod_{l|fm}(1-\overline{\chi}\psi(l)) = fm\prod_{l|m}(1-\overline{\chi}\psi(l)).$$

To prove Corollary 2.3 we need the following two lemmas.

Lemma 3.2 (cf. [13, Lemma 2]). *For an integer c with $(c,m) = 1$, we define the permutation σ_c on $\mathcal{R} = \{C_1, C_2, \dots, C_n\}$ up to "$\pm$" by*

$$\overline{c}\,C_i = \pm C_{\sigma_c(i)} \quad \textit{for } i = 1,2,\dots,n.$$

Then we have

$$\operatorname{sgn}\sigma_c = \prod_{\chi\in X^+}\chi(c), \tag{3.1}$$

where $\operatorname{sgn}\sigma_c = +1$ *or* -1 *according as* σ_c *is even or odd.*

In [13, Lemma 2] we have shown that

$$(-1)^{N_c}\operatorname{sgn}\sigma_c = \prod_{\chi\in X^-}\chi(c),$$

where N_c is the number of the "minus cosets" $-C_{\sigma_c(i)}$ in the set $\{\overline{c}\,C_i = \pm C_{\sigma_c(i)}\,;\, i = 1,2,\dots,n\}$. We can prove the identity (3.1) by taking $\prod_{\chi\in X^+}\chi(c)$ instead of $\prod_{\chi\in X^-}\chi(c)$ in the proof of [13, Lemma 2]. (The right hand $\delta_{ij}\zeta_{g_i}$ of the equation on page 22, line 13 from the top of [13] should be $\zeta_{g_i}^{\delta_{ij}}$.)

Lemma 3.3 ([2, Proof of Theorem 2]). *Assume that m is an odd integer. For an integer c with $(c,m)=1$ let c' an integer such that $(c',m)=1$ and $2c' \equiv c \pmod m$. Then, for any integers c_i, c_j coprime to m, we have*

$$\sum_{d=0}^{3}\chi_4(R_m(c_ic_j'^{-1})+dm)\left(\frac{R_m(c_ic_j'^{-1})+dm}{4m}-\frac{1}{2}\right)=-\chi_4(m)\cdot\frac{1}{2}\cdot(-1)^{R_m'(c_ic_j^{-1})},$$

$$\sum_{d=0}^{7}\chi_4\psi_8(R_m(c_ic_j'^{-1})+dm)\left(\frac{R_m(c_ic_j'^{-1})+dm}{8m}-\frac{1}{2}\right)=-\chi_4\psi_8(m)T_m'(c_ic_j^{-1})$$

and

$$\sum_{d=0}^{7}\psi_8(R_m(c_ic_j'^{-1})+dm)\left(\frac{R_m(c_ic_j'^{-1})+dm}{8m}-\frac{1}{2}\right)=(-1)^{\frac{m-1}{2}}\psi_8(m)U_m'(c_ic_j^{-1}).$$

Proof of Corollary 2.3. By Lemmas 3.3 and 3.2 and by Corollary 2.2 we have

$$(-1)^n\chi_4(m)^n\cdot\frac{1}{2^n}\cdot\det\left(\sum_{\bar t\in H}(-1)^{R_m'(c_ic_j^{-1}t)}\right)_{C_i,C_j\in\mathcal{R}}$$

$$=\det\left(\sum_{\bar t\in H}\sum_{d=0}^{3}\chi_4(R_m(c_ic_j'^{-1}t)+dm)\left(\frac{R_m(c_ic_j'^{-1}t)+dm}{4m}-\frac{1}{2}\right)\right)_{C_i,C_j\in\mathcal{R}}$$

$$=\prod_{\chi\in X^+}\chi(2)\cdot\det\left(\sum_{\bar t\in H}\sum_{d=0}^{3}\chi_4(R_m(c_ic_j^{-1}t)+dm)\left(\frac{R_m(c_ic_j^{-1}t)+dm}{4m}-\frac{1}{2}\right)\right)_{C_i,C_j\in\mathcal{R}}$$

$$=\prod_{\chi\in X^+}\chi(2)\cdot(-1)^n\prod_{\chi\in X^+}\prod_{l\mid m}(1-\chi\chi_4(l))\cdot\frac{Q_K\,w_K}{Q_{K(\sqrt{-1})}\,w_{K(\sqrt{-1})}}\cdot\frac{h^*_{K(\sqrt{-1})}}{h^*_K},$$

where $C_i=\overline{c_i}H$, $C_j=\overline{c_j}H$. Hence we have obtained the first formula (2.5).

In the same way as above we can prove the second and third formulas (2.6) and (2.7). □

Proof of Corollary 2.4. If $\psi(-1)=-1$, then we have $T^{(b)}_{-C,\psi}=-T^{(b)}_{C,\psi}$ and

$$\det\left(T^{(b)}_{C_iC_j^{-1},\psi}\right)_{C_i,C_j\in\mathcal{R}}=(-1)^{\frac{n-\delta}{2}+\delta'}\det\left(T^{(b)}_{C_iC_j,\psi}\right)_{C_i,C_j\in\mathcal{R}}.$$

If $\psi(-1)=+1$, then we have $T^{(b)}_{-C,\psi}=T^{(b)}_{C,\psi}$ and

$$\det\left(T^{(b)}_{C_iC_j^{-1},\psi}\right)_{C_i,C_j\in\mathcal{R}}=(-1)^{\frac{n-\delta}{2}}\det\left(T^{(b)}_{C_iC_j,\psi}\right)_{C_i,C_j\in\mathcal{R}}.$$

Here δ is the number of the cosets $C_i \in \mathcal{R}$ whose square C_i^2 are H or $-H$, and δ' the number of cosets $C_i \in \mathcal{R}$ whose inverses are not contained in $\mathcal{R}$, i.e., $C_i^{-1} = -C_{\sigma(i)}$ for some $\sigma(i) \in \{1, 2, \ldots, n\}$ (As for the signs of the two identities of determinants just above, see [9, Proposition 2]).

Therefore by Theorem 2.1 we obtain, in the both cases where $\psi(-1) = \pm 1$,

$$\begin{aligned} \det \left(T^{(b)}_{C_iC_j,\psi} \right)_{C_i, C_j \in \mathcal{R}} &= \pm \prod_{\chi \in X^*} \widetilde{c}_{\chi\psi}(b) \cdot \prod_{\chi \in X^*} \frac{1}{2} B_{1,\chi\psi} \\ &= \pm \prod_{\chi \in X^*} \widetilde{c}_{\chi\psi}(b) \cdot \frac{Q_K\, w_K}{Q_{\widetilde{K}}\, w_{\widetilde{K}}} \cdot \frac{h^*_{\widetilde{K}}}{h^*_K}. \end{aligned}$$

In our case where $K = \mathbf{Q}(\zeta_p), H = \{\overline{1}\}$ and $b = 2$, we have $T^{(2)}_{C,\psi} = -S_c(\psi)$ for $C = \overline{c}H$. Hence, calculating $\widetilde{c}_{\chi\psi}(2)$ we have the desired formula. □

Corollary 2.5 immediately follows from Theorem 2.1 by taking $b = g$. Here we only note that $Q_K = 1$ and $Q_{\widetilde{K}} = 2$ and that the multiplicative group $(\mathbf{Z}/4p\mathbf{Z})^\times$ constitutes of $R_{4p}(g^k)$ modulo $4p$ and $R_{4p}(-g^k)$ modulo $4p$ for all $k = 0, 1, \ldots, p-2$.

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References

[1] Carlitz, L. and Olson, F.-R., Maillet's determinant, *Proc. Amer. Math. Soc.*, **6** (1955), 265–269.

[2] Endô, A., The Maillet Determinant Associated with a Dirichlet Character, preprint.

[3] Endô, A., A generalization of the Maillet determinant and the Demyanenko determinant, *Abh. Math. Sem. Univ. Hamburg*, **73** (2003), 181–193.

[4] Endô, A., The Relative Class Number of Certain Imaginary Abelian Number Fields of Odd Conductors, *Proc. Japan Acad.*, **72**, Ser. A (1996), 64–68.

[5] Endô, A., The Relative Class Numbers of Certain Imaginary Abelian Number Fields and Determinants, *J. Number Theory*, **34** (1990), 13–20.

[6] Endô, A., The Relative Class Numbers of Certain Imaginary Abelian Number Fields, *Abh. Math. Sem. Univ. Hamburg*, **58** (1988), 237–243.

[7] Funakura, T., On Kronecker's limit formula for Dirichlet series with periodic coefficients, *Acta Arith.*, **55** (1990), 59–73.

[8] Girstmair, K., Class Number Factors and Distributions of Residues, *Abh. Math. Sem. Univ. Hamburg*, **67** (1997), 65–104.

[9] Girstmair, K., The relative class numbers of imaginary cyclic fields of degrees 4, 6, 8 and 10, *Math. Comp.*, **61** (1993), 881–887.

[10] Hasse, H., *Über die Klassenzahl abelsher Zahlkörper*, Akademie Verlag: Berlin, 1952. Reprinted with an introduction by J. Martinet: Springer-Verlag: Berlin, 1985.

[11] Hirabayashi, M., Generalizations of Girstmair's formulas, *Abh. Math. Sem. Univ. Hamburg*, **75** (2005), 83–95.

[12] Hirabayashi, M., A formula of Girstmair and a remark on the relative class number formula, *Proceedings of the 2003 Nagoya Conference "On Yokoi-Chowla Conjecture and Related Topics"*, Furukawa Total Pr. Co. (2004), 63–76.

[13] Hirabayashi, M., A generalization of Maillet and Demjanenko determinants for the cyclotomic $\mathbf{Z}_p$-extension, *Abh. Math. Sem. Univ. Hamburg*, **71** (2001), 15–27.

[14] Hirabayashi, M., A generalization of Maillet and Demjanenko determinants, *Acta Arith.*, **83** (1998), 391–397.

[15] Kanemitsu, S. and Kuzumaki, T., On a generalization of the Maillet determinant, *Proc. Number Theory Conf. Eger 1997*, de Gruyter (1998), 271–287.

[16] Kučera, R., Formulae for the relative class number of an imaginary abelian field in the form of a product of determinants, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **10** (2002), 79–83.

[17] Kučera, R., Formulae for the relative class number of an imaginary abelian field in the form of a determinant, *Nagoya Math. J.*, **163** (2001), 167–191.

[18] Tsumura, H., On Demyanenko's matrix and Maillet's determinant for imaginary abelian number fields, *J. Number Theory*, **60** (1996), 391–397 .

[19] Washington, L.-C., *Introduction to Cyclotomic Fields*, Second Edition Springer-Verlag, 1991.