

Second order complete class in consistent estimators

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$x_1, x_2, \dots, x_n$ は確率密度関数 $f(x, \theta)$ ($\theta \in \Theta \subset \mathbb{R}^p$) をもつ確率変数とする。 Θ は開集合とし, $f(x, \theta)$ は次の条件を満たすものとする,

Condition 1. $f(x, \theta)$ は x, θ に関して可測とする。

Condition 2. $\{x \mid f(x, \theta) > 0\}$ は $\theta \in \Theta$ と独立である。

Condition 3. 各 x に対して, $f(x, \theta)$ は θ に関して 3 次まで偏微分可能で, それらの微係数は θ に関して連続である。

Condition 4. すべての $\theta \in \Theta$ に対して,

$$E_{\theta}(|\log f(x, \theta)|) < \infty$$

かつ $t \left(\frac{\partial}{\partial \theta_1} \log f(x, \theta), \frac{\partial}{\partial \theta_2} \log f(x, \theta), \dots, \frac{\partial}{\partial \theta_p} \log f(x, \theta) \right)$

の共分散行列 $I_{\theta}(x)$ は正値行列である。

Condition 5. 各 $\theta_0 \in \Theta$ に対して, 完備な近傍 $\Theta_0 \ni \theta_0$

および関数 $G(x), H(x)$ が存在して, $\theta \in \Theta_0$ に対して一様に

$$\left| \frac{\partial^i}{\partial \theta_\alpha^i} \log f(x, \theta) \right| \leq G(x), \quad i=1, 2, 3, \alpha=1, 2, \dots, p$$

$$\left| \frac{\partial^4}{\partial \theta_\alpha^4} \log f(x, \theta) \right| \leq H(x)$$

を満たし,かつ

$$\sup_{\theta \in \Theta_0} E_\theta(G^3(x)) < \infty, \quad \sup_{\theta \in \Theta_0} E_\theta(H(x)) < \infty$$

Condition 6. 各 $\theta_0 \in \Theta$ に対して, 完備な近傍 $\Theta_0 \ni \theta_0$

が存在して, $n^{-\frac{1}{2}} \bar{I}_{1,1}^{-\frac{1}{2}}(\theta) L_1(\theta), n^{-\frac{1}{2}} C_{2,2}^{-\frac{1}{2}}(\theta) (L_2(\theta) - n e_{2,1}(\theta))$ は, $\theta \in \Theta_0$ に対して一様に, $o(n^{-\frac{1}{2}})$

まで Edgeworth-展開可能である. ただし

$$l(\theta) = \prod_{i=1}^n f(x, \theta), \quad L_i(\theta) = {}^t \left(\frac{\partial^i}{\partial \theta_1^i} \log l(\theta), \frac{\partial^i}{\partial \theta_2^i} \log l(\theta), \dots, \frac{\partial^i}{\partial \theta_p^i} \log l(\theta) \right), \quad n e_{2,1}(\theta) = E_\theta(L_2(\theta)),$$

$$n C_{2,2}(\theta) = E_\theta \left[(L_2(\theta) - n e_{2,1}(\theta)) {}^t (L_2(\theta) - n e_{2,1}(\theta)) \right]$$

Condition 7. 最尤推定量 $\hat{\theta}_n$ は θ に関して局所一様に $o(n^{-\frac{1}{2}})$ まで Edgeworth-展開可能である.

ここで取扱う一致推定量 θ_n は任意の $b \in R^p$ に対して $P(\sqrt{n} {}^t b (\theta_n - \theta) \leq 0)$ が θ に関して局所一様に $o(n^{-\frac{1}{2}})$ まで展開可能であるものとする. またそのような推

定量の全体を $\mathcal{F}$, $\mathcal{F}$ の中で $o(n^{-\frac{1}{2}})$ まで Edgeworth-展開可能なものの全体を $\mathcal{F}_E$, また特定な $g(\theta, \frac{1}{\sqrt{n}}; b)$ に対して $P(\sqrt{n}tb(\theta_n - \theta) \leq 0) = g(\theta, \frac{1}{\sqrt{n}}; b)$ を満たす $\theta_n \in \mathcal{F}$ の全体を $C(g(\theta, \frac{1}{\sqrt{n}}; b))$ で表わす。また任意の $\theta_n \in \mathcal{F}$ に対して,

$$E_{\theta}(\sqrt{n}(\theta_n - \theta)) = a(\theta) = a_1(\theta) + \frac{1}{\sqrt{n}} a_2(\theta) + o(n^{-\frac{1}{2}})$$

$$\begin{aligned} E_{\theta}(n(\theta_n - \theta - \frac{1}{\sqrt{n}} a(\theta))^t (\theta_n - \theta - \frac{1}{\sqrt{n}} a(\theta))) \\ = C(\theta) = C_1(\theta) + \frac{1}{\sqrt{n}} C_2(\theta) + o(n^{-\frac{1}{2}}) \end{aligned}$$

なる記号を用いる。以上の条件の下で次の結果を得る。

定理 1. $\theta_n \in C(g(\theta, \frac{1}{\sqrt{n}}; b))$ に対して,

$$\begin{aligned} P(-t_1 \leq \sqrt{n}tb(\theta_n - \theta) \leq t_2) \\ \leq F(t_2; \theta, \frac{1}{\sqrt{n}}; b; g(\theta, \frac{1}{\sqrt{n}}; b)) - F(-t_1; \theta, \frac{1}{\sqrt{n}}; b; g(\theta, \frac{1}{\sqrt{n}}; b)) \\ + o(n^{-\frac{1}{2}}) \\ (\forall t_1, t_2 \geq 0, \forall b \in R^p) \end{aligned}$$

ただし

$$\begin{aligned} F(t; \theta, \frac{1}{\sqrt{n}}; b; g(\theta, \frac{1}{\sqrt{n}}; b)) &= \Phi(u(g_{00}(\theta, b)) + t I_{1,1}^{\frac{1}{2}}(\theta, \xi)) \\ &+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta, b)) + t I_{1,1}^{\frac{1}{2}}(\theta, \xi)) \left[\{g_{10}(\theta, b) + t g_{01}(\theta, \xi)\} \right. \\ &\quad / \phi(u(g_{00}(\theta, b))) \} + t \{ (J_{1,1,1}(\theta, \xi) u(g_{00}(\theta, b))) / \\ &\quad 6 I_{1,1}(\theta, \xi) \} + t^2 \{ (3 J_{2,1}(\theta, \xi) + 2 J_{1,1,1}(\theta, \xi)) / \\ &\quad 6 I_{1,1}(\theta, \xi) \} \} \end{aligned}$$

$$I_{\alpha, \beta}(\theta) = E_{\theta} \left(\frac{\partial}{\partial \theta_{\alpha}} \log f(x, \theta) \cdot \frac{\partial}{\partial \theta_{\beta}} \log f(x, \theta) \right), \quad I_{1,1}(\theta) = [I_{\alpha, \beta}(\theta)]$$

$$J_{\alpha, \beta, r}(\theta) = E_{\theta} \left(\frac{\partial^2}{\partial \theta_{\alpha} \partial \theta_{\beta}} \log f(x, \theta) \cdot \frac{\partial}{\partial \theta_r} \log f(x, \theta) \right)$$

$$J_{\alpha, \beta, r}(\theta) = E_{\theta} \left(\frac{\partial}{\partial \theta_{\alpha}} \log f(x, \theta) \cdot \frac{\partial}{\partial \theta_{\beta}} \log f(x, \theta) \cdot \frac{\partial}{\partial \theta_r} \log f(x, \theta) \right)$$

$$I_{1,1}(\theta, \xi) = \sum_{\alpha, \beta=1}^p I_{\alpha, \beta}(\theta) \xi_{\alpha} \xi_{\beta} = {}^t \xi I_{1,1}(\theta) \xi$$

$$J_{2,1}(\theta, \xi) = \sum_{\alpha, \beta, r=1}^p J_{\alpha, \beta, r}(\theta) \xi_{\alpha} \xi_{\beta} \xi_r$$

$$J_{1,1,1}(\theta, \xi) = \sum_{\alpha, \beta, r=1}^p J_{\alpha, \beta, r}(\theta) \xi_{\alpha} \xi_{\beta} \xi_r$$

$$g(\theta, \frac{1}{\sqrt{n}}; b) = g_{00}(\theta, b) + \frac{1}{\sqrt{n}} g_{10}(\theta, b) + o(n^{-\frac{1}{2}})$$

$$g_{01}(\theta; b, \xi) = \sum_{\alpha=1}^p \frac{\partial}{\partial \theta_{\alpha}} g_{00}(\theta, b) \xi_{\alpha}$$

$$u(g_{00}(\theta, b)) = \Phi^{-1}(g_{00}(\theta, b)), \quad {}^t b \xi = 1$$

$\theta_n \in \mathcal{F}_E$ の cumulants が次式で与えられるとしよう。

$$K_1(\sqrt{n} \theta_{n\alpha}) = a_{1\alpha}(\theta) + \frac{1}{\sqrt{n}} a_{2\alpha}(\theta) + o(n^{-\frac{1}{2}})$$

$$K_2(\sqrt{n} \theta_{n\alpha}, \sqrt{n} \theta_{n\beta}) = C_{1\alpha\beta}(\theta) + \frac{1}{\sqrt{n}} C_{2\alpha\beta}(\theta) + o(n^{-\frac{1}{2}})$$

$$K_3(\sqrt{n} \theta_{n\alpha}, \sqrt{n} \theta_{n\beta}, \sqrt{n} \theta_{nr}) = \frac{1}{\sqrt{n}} d_{2\alpha\beta r}(\theta) + o(n^{-\frac{1}{2}})$$

$$K_i(\sqrt{n} \theta_{n\alpha_1}, \sqrt{n} \theta_{n\alpha_2}, \dots, \sqrt{n} \theta_{n\alpha_i}) = o(n^{-\frac{1}{2}}) \quad (i \geq 4)$$

$$1 \leq i \leq p \quad \theta_n = {}^t (\theta_{n1}, \theta_{n2}, \dots, \theta_{np}).$$

そのとき

$$\begin{aligned} \bar{g}(\theta, \frac{1}{\sqrt{n}}; b) &= P(\sqrt{n} {}^t b (\theta_n - \theta) \leq 0) \\ &= \Phi(-{}^t b a_1(\theta) / (C_{1\alpha\beta}(\theta) b_{\alpha} b_{\beta})^{\frac{1}{2}}) \\ &\quad - \frac{1}{\sqrt{n}} \phi(-{}^t b a_1(\theta) / (C_{1\alpha\beta}(\theta) b_{\alpha} b_{\beta})^{\frac{1}{2}}) \left[\frac{{}^t b a_2(\theta)}{(C_{1\alpha\beta}(\theta) b_{\alpha} b_{\beta})^{\frac{1}{2}}} \right. \\ &\quad \left. - \{ C_{2\alpha\beta}(\theta) b_{\alpha} b_{\beta} \cdot {}^t b a_1(\theta) / 2 (C_{1\alpha\beta}(\theta) b_{\alpha} b_{\beta})^{\frac{3}{2}} \} \right] \end{aligned}$$

$$\{d_{2\alpha\beta r}(\theta) b_\alpha b_\beta b_r / 6 (C_{1\alpha\beta}(\theta) b_\alpha b_\beta)\} \{(({}^t b a_1(\theta))^2 / C_{1\alpha\beta}(\theta) b_\alpha b_\beta) - 1\} + o(n^{-\frac{1}{2}})$$

従って $b = I_{1,1}^{\frac{1}{2}}(\theta) b'$ ($\|b'\| = 1$) に對して,

$$F(t; \theta, \frac{1}{\sqrt{n}}; I_{1,1}^{\frac{1}{2}}(\theta) b', I_{1,1}^{-\frac{1}{2}}(\theta) b'; \bar{g}(\theta, \frac{1}{\sqrt{n}}; I_{1,1}^{\frac{1}{2}}(\theta) b'))$$

$$= \bar{\Phi}(-{}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_1(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') + t)$$

$$- \frac{1}{\sqrt{n}} \phi(-{}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_1(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') + t) \left[{}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_2(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') \right.$$

$$- \frac{1}{2} {}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_1(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') \times {}^t b' I_{1,1}^{\frac{1}{2}}(\theta) C'_2(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b')$$

$$\times I_{1,1}^{\frac{1}{2}}(\theta) b' + \frac{1}{6} d'_{2\alpha\beta r}(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') (I_{1,1}^{\frac{1}{2}}(\theta) b')_\alpha (I_{1,1}^{\frac{1}{2}}(\theta) b')_\beta$$

$$\times (I_{1,1}^{\frac{1}{2}}(\theta) b')_r \{({}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_1(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b'))^2 - 1\}$$

$$+ t \{ {}^t b' I_{1,1}^{\frac{1}{2}}(\theta) A'(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') I_{1,1}^{-\frac{1}{2}}(\theta) b'$$

$$+ \frac{1}{3} {}^t b' I_{1,1}^{\frac{1}{2}}(\theta) a'_1(\theta, I_{1,1}^{\frac{1}{2}}(\theta) b') (3 J_{2,1}(\theta, I_{1,1}^{-\frac{1}{2}}(\theta) b')$$

$$+ 2 J_{1,1,1}(\theta, I_{1,1}^{-\frac{1}{2}}(\theta) b')) \}$$

$$+ t^2 \{ 3 J_{2,1}(\theta, I_{1,1}^{-\frac{1}{2}}(\theta) b') + 2 J_{1,1,1}(\theta, I_{1,1}^{-\frac{1}{2}}(\theta) b') \}$$

に對して

$$a'_1(\theta, b) = \bar{g}^{\frac{1}{2}}(\theta, b) a_1(\theta), \quad a'_2(\theta, b) = \bar{g}^{\frac{1}{2}}(\theta, b) a_2(\theta)$$

$$C'_1(\theta, b) = \bar{g}(\theta, b) C_1(\theta), \quad C'_2(\theta, b) = \bar{g}(\theta, b) C_2(\theta)$$

$$d'_{2\alpha\beta r}(\theta, b) = \bar{g}^{\frac{3}{2}}(\theta, b) d_{2\alpha\beta r}(\theta)$$

$$A'(\theta, b) = {}^t (\frac{\partial}{\partial \theta_1} a'_1(\theta, b), \frac{\partial}{\partial \theta_2} a'_1(\theta, b), \dots, \frac{\partial}{\partial \theta_p} a'_1(\theta, b))$$

$$\bar{g}(\theta, b) = {}^t b I_{1,1}^{-1}(\theta) b / {}^t b C_1(\theta) b.$$

さらに

$$C''_2(\theta, b) = C'_2(\theta, b) - 2 A'(\theta, b) I_{1,1}^{-1}(\theta)$$

$$d_2''(\theta, b) = {}^t(d_{21}''(\theta, b), d_{22}''(\theta, b), \dots, d_{2p}''(\theta, b))$$

$$d_{2r}''(\theta, b) = \sum_{\alpha, \beta} d_{2\alpha\beta r}''(\theta, b) b_\alpha b_\beta, \quad r=1, 2, \dots, p$$

$$\begin{aligned} d_{2\alpha\beta r}''(\theta, b) &= d_{2\alpha\beta r}'(\theta, b) + (J_{\eta_1, \eta_2, \eta_3}(\theta) + J_{\eta_1, \eta_3, \eta_2}(\theta) \\ &\quad + J_{\eta_2, \eta_3, \eta_1}(\theta) + 2 J_{\eta_1, \eta_2, \eta_3}(\theta)) I_{\eta_1, \alpha}^{-1}(\theta) \\ &\quad \times I_{\eta_2, \beta}^{-1}(\theta) I_{\eta_3, r}^{-1}(\theta) \end{aligned}$$

と置き,

$$\begin{aligned} \theta_n' &= \hat{\theta}_n + \frac{1}{\sqrt{n}} a_1'(\hat{\theta}_n, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b') + \frac{1}{n} \left[a_2'(\hat{\theta}_n, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b') - \bar{a}_2(\hat{\theta}_n) \right. \\ &\quad - \frac{1}{2} a_1'(\hat{\theta}_n, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b') \cdot {}^t b' I_{1,1}^{-\frac{1}{2}}(\theta) C_2''(\theta, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b') I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b' \\ &\quad \left. + \frac{1}{6} d_2''(\hat{\theta}_n, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b') \left\{ ({}^t b' I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) a_1'(\hat{\theta}_n, I_{1,1}^{-\frac{1}{2}}(\hat{\theta}_n) b'))^2 - 1 \right\} \right] \end{aligned}$$

と置くと、

$$\begin{aligned} F(t; \theta, \frac{1}{\sqrt{n}}; I_{1,1}^{-\frac{1}{2}}(\theta) b', I_{1,1}^{-\frac{1}{2}}(\theta) b'; \bar{f}(\theta, \frac{1}{\sqrt{n}}; I_{1,1}^{-\frac{1}{2}}(\theta) b')) \\ = P(\sqrt{n} {}^t b' I_{1,1}^{-\frac{1}{2}}(\theta) (\theta_n' - \theta) \leq t) + o(n^{-\frac{1}{2}}) \quad (\forall t \in R) \end{aligned}$$

任意の $b \in R^p$ に対しては,

$$\begin{aligned} \theta_n' &= \hat{\theta}_n + \frac{1}{\sqrt{n}} a_1'(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-\frac{1}{2}} b) + \frac{1}{n} \left[a_2'(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-\frac{1}{2}} b) \right. \\ &\quad - \bar{a}_2(\hat{\theta}_n) - \frac{1}{2} a_1'(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-\frac{1}{2}} b) \times ({}^t b I_{1,1}^{-1}(\theta) b)^{-1} \\ &\quad \times {}^t b C_2''(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-\frac{1}{2}} b) b \\ &\quad \left. + \frac{1}{6} d_2''(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-\frac{1}{2}} b) \right. \\ &\quad \left. \times \left\{ ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-1} ({}^t b a_1'(\hat{\theta}_n, ({}^t b I_{1,1}^{-1}(\theta) b)^{-\frac{1}{2}} b))^2 - 1 \right\} \right] \end{aligned}$$

ただし $\bar{a}_2(\theta)$ は,

$$E_0(\sqrt{n}(\hat{\theta}_n - \theta)) = \frac{1}{\sqrt{n}} \bar{a}_2(\theta) + o(n^{-\frac{1}{2}})$$

を満たす。

$C_1(\theta) = I_{1,1}^{-1}(\theta)$ ならば

$$a_1'(\theta, b) = a_1(\theta), \quad a_2'(\theta, b) = a_2(\theta), \quad C_2'(\theta, b) = C_2(\theta)$$

$$a_2''(\theta, b) = 0, \quad A'(\theta, b) = A(\theta)$$

であり, $C_1(\theta) \neq I_{1,1}^{-1}(\theta)$ ならば, 少なくとも 1 つは 0 でない

$$t_1, t_2 \text{ に対して, } \theta_n'' = \hat{\theta}_n + \frac{1}{\sqrt{n}} a_1'(\theta, ({}^t b I_{1,1}^{-1}(\theta) b)^{-\frac{1}{2}} b)$$

と置くと,

$$P(-t_1 \leq \sqrt{n} {}^t b (\theta_n - \theta) \leq t_2) < P(-t_1 \leq \sqrt{n} {}^t b (\theta_n'' - \theta) \leq t_2) + o(1)$$

が成り立つことに注意して次の結果が得られる.

定理 2. 各 $\theta_n \in \mathcal{F}_E$ に対して, $a_1(\theta, b), a_2'(\theta)$

および半正値行列 $D(\theta)$ が存在して, $\theta_n' = \hat{\theta}_n + \frac{1}{\sqrt{n}} a_1(\hat{\theta}_n, b)$

$$+ \frac{1}{n} a_2'(\hat{\theta}_n) - \frac{1}{n} ({}^t b I_{1,1}^{-1}(\hat{\theta}_n) b)^{-1} ({}^t b D(\hat{\theta}_n) b) a_1(\theta)$$

は θ に関して局所一様に,

$$P(-t_1 \leq \sqrt{n} (\theta_n - \theta) \leq t_2) \leq P(-t_1 \leq \sqrt{n} {}^t b (\theta_n' - \theta) \leq t_2) + o(n^{-\frac{1}{2}})$$

$$(\forall b \in R^p, \forall t_1, t_2 \geq 0, \text{ 少なくとも 1 つは正})$$

が成り立つ.

定理 3. $\theta_n \in \mathcal{F}_E$ に対して, $a(\theta) = o(1)$ が成り立つ

ならば, $a_2'(\theta)$ が存在して, $\theta_n' = \hat{\theta}_n + \frac{1}{n} a_2'(\hat{\theta}_n)$

は, θ に関して局所一様に,

$$P(-t_1 \leq \sqrt{n} {}^t b (\theta_n - \theta) \leq t_2) \leq P(-t_1 \leq \sqrt{n} {}^t b (\theta_n' - \theta) \leq t_2) + o(n^{-\frac{1}{2}})$$

($\forall b \in R, \forall t_1, t_2 \geq 0$, 少なくとも1つは正)

が成り立つ.

定理 4. $\theta_n \in \mathcal{F}_E$ に対して, $C(\theta) = I_{1,1}^{-1}(\theta) + \frac{1}{\sqrt{n}} (A(\theta) I_{1,1}^{-1}(\theta) + I_{1,1}^{-1}(\theta) A(\theta)) + o(n^{-\frac{1}{2}})$ が成り立つならば, $a_2'(\theta)$ が存在して, $\theta_n' = \hat{\theta}_n + \frac{1}{\sqrt{n}} a_1(\hat{\theta}_n) + \frac{1}{n} a_2'(\hat{\theta}_n)$ は, θ に関して局所一様に

$$P_\theta(\sqrt{n}(\theta_n - \theta) \in B) = P_\theta(\sqrt{n}(\theta_n' - \theta) \in B) + o(n^{-\frac{1}{2}})$$

($\forall B \in \mathcal{B}'$)

が成り立つ.