

Sato Hyperfunctions and Reproducing Kernels (佐藤超関数と再生核形式)

By

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Abstract

We survey the theory of the microlocal energy method for hyperfunctions and microfunctions developed in [5, 6, 7]; in particular, we introduce the important notions, *positivity* and *quasi-positivity* for hermitian microkernels. The quasi-positivity is based on the good properties of Bergman's reproducing kernels for some conic domains in $T^*\mathbb{C}^n$. Further we introduce some recent result on the Sobolev type 2-forms of order 0 for microfunctions with real analytic parameters obtained by Kaito Yamasaki [11].

§ 1. What is an energy method in the theory of hyperfunctions?

For a Sato hyperfunction $f(x)$, we cannot define $|f(x)|^p$ or L^p -norm $\|f\|_p$ in general. For $p = 2$, however, we can consider a microfunction $f(x)\overline{f(u)}$ as the substitute of $|f(x)|^2$ as introduced in [5, 6]. Further, for a hyperfunction $f(t, x)$ with real analytic parameters $t \in T$, we can consider a hyperfunction

$$E(x, u) := \int_{\overline{T}} f(t, x)\overline{f(t, u)}dt \left(= \int_{\mathbb{R}^m} \text{ext}_{\overline{T}}(f(t, x)\overline{f(t, u)})dt \right)$$

as the L^2 -energy form of f (the precise definition of real analytic parameters $t \in T$ and the meaning of the integration over $\overline{T}$ will be given in the later section). At the same time, in energy arguments we use some inequality (an order relation)

$$k_1(x, u) \ll k_2(x, u) \quad \text{at } (\overset{\circ}{x}, \overset{\circ}{x}; i\overset{\circ}{\eta}, -i\overset{\circ}{\eta})$$

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instead of the equality. In order to illustrate our energy method, we consider the following example: Let $\Omega \subset \mathbb{R}_x^n$ be a bounded domain with real analytic boundary $N = \partial\Omega$, and $f(t, x)$ be a hyperfunction in $\{t \in \mathbb{R}; a < t < b\} \times \Omega$ satisfying the boundary value problem:

$$(1.1) \quad \begin{cases} (\partial_t - \Delta_x)f(t, x) = 0 & \text{in } (a, b) \times \Omega, \\ f(t, x) = 0 & \text{on } (a, b) \times \partial\Omega. \end{cases}$$

We remark here that, since $(a, b) \times \partial\Omega$ is non-characteristic to $\partial_t - \Delta_x$, the boundary value $f|_{(a, b) \times \partial\Omega}$ is well-defined for any hyperfunction solution f . Then, our conclusion is that $f \in \mathcal{A}((a, b) \times \overline{\Omega})$ (real analyticity of f up to the boundary). To prove this, we consider the energy form:

$$(1.2) \quad E(t, s) := \int_{\overline{\Omega}} f(t, x) \overline{f(s, x)} dx.$$

Indeed, this is a well-defined hyperfunction in (t, s) because we have not only an estimate

$$\text{SS}(f) \subset \{(t, x; i\tau, i\xi); x \in \Omega, \xi = 0\},$$

but also the estimates up to the boundary:

$$\begin{aligned} \text{SS}(\text{ext}_{\overline{\Omega}}(f(t, x))) &\subset \{(t, x; i\tau, i\xi); x \in \overline{\Omega}, \xi = 0\} \cup \{(t, x; i\tau, i\lambda d\varphi(x)); \varphi(x) = 0\}, \\ \text{SS}(\text{ext}_{\overline{\Omega}}(f(t, x)\overline{f(s, x)})) &\subset \\ &\{(t, s, x; i\tau_1, i\tau_2, i\xi); x \in \overline{\Omega}, \xi = 0\} \cup \{(t, s, x; i\tau_1, i\tau_2, i\lambda d\varphi(x)); \varphi(x) = 0\}. \end{aligned}$$

Here, $\varphi(x) \in C^\omega(\overline{\Omega})$ satisfies $\Omega = \{\varphi(x) > 0\}$, $d\varphi(x) \neq 0$, and the boundary value theory of hyperfunction solutions permits the extension $\text{ext}_{\overline{\Omega}}(f(t, x)\overline{f(s, x)})$ (roughly, $= \chi_{\overline{\Omega}}(x)f(t, x)\overline{f(s, x)}$) of $f(t, x)\overline{f(s, x)}$ satisfying the above estimates. Therefore (1.2) is a hyperfunction on $(a, b)^2$. Then, we have

$$\begin{aligned} \partial_t E(t, s) &= \int_{\mathbb{R}^n} \text{ext}_{\overline{\Omega}}(\partial_t f(t, x) \cdot \overline{f(s, x)}) dx = \int_{\mathbb{R}^n} \text{ext}_{\overline{\Omega}}(\Delta_x f(t, x) \cdot \overline{f(s, x)}) dx \\ &= - \int_{\mathbb{R}^n} \text{ext}_{\overline{\Omega}}(\nabla_x f(t, x) \cdot \overline{\nabla_x f(s, x)}) dx = \partial_s E(t, s). \end{aligned}$$

Hence, $(\partial_t - \partial_s)E(t, s) = 0$. Since $\partial_t - \partial_s$ is an elliptic operator on

$$\Delta^a(\sqrt{-1}T^*\mathbb{R}) := \{(t, s; i\tau_1, i\tau_2); t = s, \tau_1 = -\tau_2\},$$

we get an estimate:

$$\text{SS}(E(t, s)) \cap \Delta^a(\sqrt{-1}T^*\mathbb{R}) \subset \{\tau_1 = \tau_2 = 0\}.$$

Then, some theorem on the integration of positive hermitian microkernels concludes the real analyticity of the integrand $f(t, x)$ up to the boundary.

The plan of this article is the following: In Section 2, we give a brief introduction to positivity of analytic hermitian kernels. In Section 3 we show the importance of some Bergman reproducing kernels in our theory with respect to the definition of quasi-positivity. Section 4 is devoted to introduce the definition of positivity in microlocal analysis. In Section 5 we give the definition of quasi-positivity of hermitian microkernels by using positive hermitian pseudodifferential operators of infinite order. In Section 6 we introduce K. Yamasaki's result on Sobolev type 2-forms of order 0 for microfunctions with real analytic parameters $t \in T$.

Hereafter we consider hyperfunctions $f(x)$ or $f(t, x)$, where $t = (t_1, \dots, t_m)$ are real analytic parameters. So the variables t, x change the roles from Section 2.

§ 2. Positive analytic hermitian kernels

Definition 2.1. Let X be a set. A $\mathbb{C}$ -valued function $K(x, y)$ on $X \times X$ is said to be a hermitian kernel on X if

$$K(y, x) = \overline{K(x, y)} \quad (\forall x, y \in X).$$

Further a hermitian kernel $K(x, y)$ on X is said to be $K \gg 0$ (positive) $\iff$

For $\forall N \in \mathbb{N}, \forall x_1, \dots, \forall x_N \in X, \forall \xi_1, \dots, \forall \xi_N \in \mathbb{C}$, we have

$$\sum_{j,k=1}^N K(x_j, x_k) \xi_j \overline{\xi_k} \geq 0.$$

Then, we define an order relation for hermitian kernels K_1, K_2 on X by

$$K_1 \gg K_2 \iff K_1 - K_2 \gg 0.$$

It is easy to see that

$$K_1 \gg 0, K_2 \gg 0 \implies K_1 + K_2 \gg 0, \quad K_1 \cdot K_2 \gg 0,$$

where $K_1 \cdot K_2$ is a product as functions on $X \times X$.

Definition 2.2. Let X be a domain of $\mathbb{C}^n$. Then, a hermitian kernel $K(z, w)$ on X is said to be an analytic hermitian kernel on X if $K(z, w)$ is holomorphic in variables $(z, \overline{w})$ on $X \times X^*$, where

$$X^* := \{z \in \mathbb{C}^n; \overline{z} \in X\}.$$

Example 2.3. Put $X = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$, and $\alpha > -1$. Then we have a positive analytic hermitian kernel on X :

$$\{(z - \bar{w})/i\}^{-1-\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{itz} \cdot \overline{e^{itw}} t^\alpha dt \gg 0.$$

The following theorem is the most important for our theory, which is due to several authors; the first statement is due to Krëin [8], Bremermann [2], Sommer-Mehring [10], Meschkowski [9], Donoghue [3, 4], and the second statement is due to Krëin [8], Donoghue [4], Meschkowski [9].

Theorem 2.4. *Let X, X' be domains of $\mathbb{C}^n$ such that $\emptyset \neq X' \subset X$. Let K_1, K_2 be analytic hermitian kernels on X . Then we have the following:*

- i) $K_1 \ll K_2$ on $X' \times X' \implies K_1 \ll K_2$ on $X \times X$.
- ii) *Let K_3 be a hermitian kernel (not necessarily analytic) on X' satisfying $K_1 \ll K_3 \ll K_2$ on $X' \times X'$. Then there exists a unique analytic hermitian kernel $\widetilde{K}_3$ on X satisfying*

$$K_1 \ll \widetilde{K}_3 \ll K_2 \text{ on } X \times X, \quad \widetilde{K}_3|_{X' \times X'} = K_3.$$

Let T be a domain of $\mathbb{R}^m$, and X, X' be domains of $\mathbb{C}^n$ such that $\emptyset \neq X' \subset X$.

Theorem 2.5. *(Corollary 1.14 in [6]). Let $K(z, w; t)$ be a C^ω -function (or C^∞ -function) on $X' \times X' \times T$ satisfying the following i), ii):*

- i) *For $\forall t \in T$, $K(z, w; t)$ is a positive analytic hermitian kernel on X' .*
- ii) *For any compact subset $L \subset X'$, $K(z, w; t)$ is integrable on $L \times L \times T$.*

If

$$E(z, w) := \int_T K(z, w; t) dt$$

extends to $X \times X^$ analytically with respect to $(z, \bar{w})$, then $K(z, w; t)$ extends uniquely to $X \times X \times T$ as a positive analytic hermitian kernel on X with C^ω (or C^∞ resp.) parameters $t \in T$.*

§ 3. Quasi-positivity and Bergman reproducing kernels

Definition 3.1. For a domain $X \subset \mathbb{C}^n$, we set

$$A^2(X) := \{f(z) \in \mathcal{O}(X); \int_X |f(z)|^2 dx dy < \infty\},$$

where $z = x + iy$. It is well-known that $A^2(X)$ is a Hilbert space (Bergman space). Let $\{\varphi_j(z)\}_{j=1}^\infty$ be any completely orthonormal system for $A^2(X)$. Then,

$$K_X(z, w) := \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(w)}$$

is said to be a Bergman kernel of X . Indeed, it is well-known that this series converges locally uniformly on $X \times X$, and that K_X does not depend on the choice of $\{\varphi_j(z)\}_{j=1}^\infty$. It is clear that K_X is a positive analytic hermitian kernel on X .

The importance of Bergman kernels is based on the following proposition ([6]). For the reader's convenience, we will give a proof.

Proposition 3.2. *Let $K(z, w)$ be an analytic hermitian kernel on X such that*

$$\|K\|_{X \times X} := \sqrt{\iint_{X \times X} |K(z, w)|^2 dx dy du dv} < \infty.$$

Then we have an inequality on $X \times X$: $-\|K\|_{X \times X} \cdot K_X \ll K \ll \|K\|_{X \times X} \cdot K_X$.

Proof. Define a linear operator $T : A^2(X) \ni f \mapsto \int_X K(z, w) f(w) du dv \in A^2(X)$, where $w = u + iv$. Then this is an integral operator of Hilbert-Schmidt type. Therefore there exist a completely orthonormal system $\{\varphi_j\}_{j=1}^\infty$ of $A^2(X)$, and real numbers $\lambda_j \in \mathbb{R}$ ($j = 1, 2, \dots$) such that

$$K(z, w) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(z) \overline{\varphi_j(w)}.$$

Hence we have

$$\sum_{j=1}^{\infty} \lambda_j^2 = \iint_{X \times X} |K(z, w)|^2 dx dy du dv < \infty.$$

In particular, $|\lambda_j| \leq \|K(z, w)\|_{L^2(X \times X)} (\forall j)$. Therefore,

$$-\|K\|_{L^2(X \times X)} K_X \ll K \ll \|K\|_{L^2(X \times X)} K_X.$$

□

Let X, Y be domains of $\mathbb{C}^n$ such that $Y \subset X$, and T be a domain of $\mathbb{R}^m$. Let $p(t, z)$ and $f(t, z)$ be holomorphic functions defined in some neighborhoods of $T \times X$, and $T \times Y (\subset \mathbb{C}^{m+n})$, respectively. Assume that for some constant $M > 0$ we have

$$\operatorname{Re} p(t, z) > 0, \text{ and } \|\log(p(t, z) + \overline{p(t, w)})\|_{X \times X} \leq M \quad (\forall t \in T).$$

Then the analytic hermitian kernel on Y

$$E(z, w) := \int_T (p(t, z) + \overline{p(t, w)}) f(t, z) \overline{f(t, w)} dt$$

is not positive, but the energy argument in Theorem 2.5 is applicable to this $E(z, w)$. This is because we can write $p(t, z) + \overline{p(t, w)} = e^{\log(p(t, z) + \overline{p(t, w)})}$, and so we have

$$e^{M \cdot K_X(z, w)} (p(t, z) + \overline{p(t, w)}) \gg 0 \quad (\forall t \in T).$$

Hence, if $E(z, w)$ extends to an analytic hermitian kernel on X , then $f(t, z)$ extends to a holomorphic function defined in a neighborhood of $T \times X$. Since $e^{M \cdot K_X(z, w)}$ is positive and invertible, we can generalize this argument by introducing a weaker order relation, quasi-positivity “ $\gg_q 0$ ”, than hermitian positivity:

$$K_1 \gg_q K_2 \iff \exists M > 0 \text{ such that } e^{M \cdot K_X}(K_1 - K_2) \gg 0.$$

It is easy to see that “ $\gg_q 0$ ” is an order relation for analytic hermitian kernels on X . However, in the most applications to partial differential equations, we must consider 2-forms of the following type:

$$(3.1) \quad E(z, w) := \int_T (P(t, z, \partial_z) + \overline{P(t, w, \partial_w)}) f(t, z) \overline{f(t, w)} dt,$$

where $P(t, z, \partial_z)$ is an elliptic differential operator with parameters t . Therefore we must consider Bergman kernels in the symbol spaces of analytic pseudo-differential operators. A symbol $P(z, \xi)$ of an analytic pseudo-differential operator $P(z, \partial_z)$ at $(\overset{\circ}{z}; \overset{\circ}{\xi}) \in T^*\mathbb{C}^n$ is a holomorphic function defined in some unbounded domain

$$\Omega_\delta(\overset{\circ}{z}; \overset{\circ}{\xi}) := \{(z; \xi) \in T^*\mathbb{C}^n; |z - \overset{\circ}{z}| < \delta, |\xi/|\xi| - \overset{\circ}{\xi}/|\overset{\circ}{\xi}|| < \delta, |\xi| > \delta^{-1}\}$$

satisfying the following condition for some positive numbers δ, C , and some $\mu \in \mathbb{R}$:

$$|P(z, \xi)| \leq C|\xi|^\mu \quad \text{in } \Omega_\delta(\overset{\circ}{z}; \overset{\circ}{\xi}).$$

Since $\mu > 0$ in general, $P(z, \xi)$ is not in $L^2(\Omega; dv(z, \xi))$, where $dv(z, \xi)$ is the Lebesgue measure on $\mathbb{C}_z^n \times \mathbb{C}_\xi^n \simeq \mathbb{R}^{4n}$. So we must consider Bergman kernels with some weight, for example, $(|\xi| + 1)^{-\beta}$ ($\beta > 0$). Though it is difficult to calculate the Bergman kernel for $\Omega_\delta(\overset{\circ}{z}; \overset{\circ}{\xi})$ with such a weight, we can find some Bergman kernel satisfying the equivalent conditions. For that purpose, we have the following procedure:

Step1. Take a real n -dimensional vector subspace L of $\mathbb{C}^n$ such that $\overset{\circ}{\xi} \in L \subset L + \sqrt{-1}L = \mathbb{C}^n$.

Step2. Choose $\mathbb{R}$ -linearly independent n elements $\xi^1, \dots, \xi^n$ of $L \cap \{\xi \in \mathbb{C}^n; |\xi|/|\xi| - \xi/|\xi| < \delta\}$ such that

$$\xi \in \{s_1 \xi^1 + \dots + s_n \xi^n; s_1, \dots, s_n > 0\}.$$

Step3. Take $\theta^1, \dots, \theta^n \in \mathbb{C}^n$ such that $\langle \xi^k, \theta^\ell \rangle := \sum_{j=1}^n \xi_j^k \theta_j^\ell = \delta_{k\ell}$.

Step4. Choose a large integer $N > 0$ such that for a large $\lambda > 0$ we have

$$\begin{aligned} \lambda \xi \in U_N &:= \bigcap_{j=1}^n \{\xi \in \mathbb{C}^n; |\arg(\langle \xi, \theta^j \rangle - N)| < \pi/(2N)\} \\ &\subset \{\xi \in \mathbb{C}^n; |\xi|/|\xi| - \xi/|\xi| < \delta, |\xi| > \delta^{-1}\}. \end{aligned}$$

Hereafter, we use z^* instead of $\bar{z}$. Put $\rho(\xi) := |1 + \langle \xi, \xi^* \rangle|^{-2n-\sigma}$ ($\sigma < 1$), and

$$\Omega := \{z \in \mathbb{C}^n; |z - \bar{z}| < r\} \times U_N.$$

Then, the Bergman kernel of Ω with respect to $\rho(\xi) dz dz^* d\xi d\xi^*$ is given by

$$\begin{aligned} K_\Omega^\rho(z, \xi, w, \eta) &= C_1 \left(1 - \frac{\langle z - \bar{z}, w^* - \bar{z}^* \rangle}{r^2} \right)^{-n-1} \\ &\times \prod_{j=1}^n \left[\frac{(\xi_j' \eta_j'^*)^{N-1} \{(1 + \langle \xi, \xi^* \rangle)(1 + \langle \eta^*, \xi^* \rangle)\}^{1+\sigma/(2n)}}{(\xi_j'^N + \eta_j'^{*N})^2} \right], \end{aligned}$$

where $\xi_j' = \langle \xi, \theta^j \rangle - N$, $\eta_j' = \langle \eta, \theta^j \rangle - N$ (see Theorem 3.11 in [6]). The most important property of this kernel is the following growth order estimate which is a key for our definition of quasi-positivity for micro hermitian kernels:

$$\begin{aligned} |\partial_{(z, w^*)} K_\Omega^\rho| &\leq C_2 \min\{|\xi|^\sigma, |\eta|^\sigma\} \quad (0 \leq \sigma < 1), \\ |\partial_{(z, w^*)} K_\Omega^\rho| &\leq C_2 (|\xi| + |\eta|)^\sigma \quad (\sigma < 0), \\ |\partial_{(\xi, \eta^*)} K_\Omega^\rho| &\leq C_3 (|\xi| + |\eta|)^{\sigma-1} \quad (\sigma < 1). \end{aligned}$$

Indeed, such a growth order property is indispensable to prove the exponential calculus of pseudo-differential operators with symbol $\exp(M \cdot K_\Omega^\rho(z, \xi, w^*, \eta^*))$ (cf. Aoki [1]).

§ 4. Positivity for hermitian microkernels

We consider hermitian kernels $K(x, u)$ as microfunctions on the anti-diagonal set:

$$\Delta^a(\sqrt{-1}T^*\mathbb{R}_x^n) = \{(x, u; i\xi, i\eta) \in \sqrt{-1}T^*(\mathbb{R}_x^n \times \mathbb{R}_u^n); x = u, \xi + \eta = 0\}.$$

Definition 4.1. Let $k(x, u)$ be a germ at $\overset{\circ}{p} = (\overset{\circ}{x}, \overset{\circ}{x}; i\overset{\circ}{\xi}, -i\overset{\circ}{\xi})$ ($\overset{\circ}{\xi} \neq 0$) of microfunctions in $(x, u) \in \mathbb{R}_x^n \times \mathbb{R}_u^n$. Then, $k(x, u)$ is said to be a hermitian microkernel at $\overset{\circ}{p}$ if

$$k(x, u) = \overline{k(u, x)}.$$

Further, $k(x, u) \gg 0$ at $\overset{\circ}{p}$ (positive as a hermitian microkernel at $\overset{\circ}{p}$) if there exist a small $r > 0$, some open convex cones $\Gamma_1, \dots, \Gamma_N$ in $\mathbb{R}^n$, and some positive analytic hermitian kernel $K_j(z, w)$ on

$$D_j := \{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r, \text{Im } z \in \Gamma_j\}$$

for $j = 1, \dots, N$ such that

$$k(x, u) = \sum_{j=1}^N [K_j(x + i0\Gamma_j, \overline{u - i0\Gamma_j})] \quad \text{at } \overset{\circ}{p}.$$

Here, we used the notation $K_j(z, \bar{w})$ because $K_j(z, \bar{w})$ is holomorphic in (z, w) on $D_j \times D_j^*$. Further, for two hermitian microkernels $k_1(x, u), k_2(x, u)$ we define an order relation:

$$k_1(x, u) \gg k_2(x, u) \text{ at } \overset{\circ}{p} \iff k_1(x, u) - k_2(x, u) \gg 0 \text{ at } \overset{\circ}{p}.$$

Theorem 4.2. The relation $k_1 \gg k_2$ at $\overset{\circ}{p}$ satisfies the axioms of order relations; in particular, “ $k \gg 0$ at $\overset{\circ}{p}$ and $-k \gg 0$ at $\overset{\circ}{p}$ ” implies “ $k = 0$ at $\overset{\circ}{p}$ ”.

Example 4.3.

$$\delta(x - u) \gg 0 \quad \text{on} \quad \Delta^a(\sqrt{-1}T^*\mathbb{R}_x^n).$$

Definition 4.4. Let $T \subset \mathbb{R}^m$ be a bounded domain, and $\varphi(t) \in C^\omega(\bar{T})$ be a real-valued function satisfying

$$T = \{\varphi(t) > 0\}, \quad \text{and } \varphi(t) = 0, \nabla\varphi(t) \neq 0 \text{ on } \partial T.$$

Let $(\overset{\circ}{t}, \overset{\circ}{x})$ be a point of $\bar{T} \times \mathbb{R}^n$. For a small $r > 0$, a hyperfunction $f(t, x)$ defined on $\{(t, x) \in T \times \mathbb{R}^n; |t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\}$ is said to have real analytic parameters $t \in T$ at $(\overset{\circ}{t}, \overset{\circ}{x})$ if the following i) and ii) are satisfied (also see the remark below):

- i) When $\overset{\circ}{t} \in \partial T$, f is mild on $\partial T \times \mathbb{R}^n$ from $T \times \mathbb{R}^n$. When $\overset{\circ}{t} \in T$, $\{|t - \overset{\circ}{t}| < r\} \subset T$.
- ii) The extension $\text{ext}_{\bar{T}}(f)$ of f to $\{|t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\}$ with support in $\bar{T}$ satisfies

$$\begin{aligned} & \text{SS}(\text{ext}_{\bar{T}}(f)) \cap \{(t, x; i\tau, i\xi); \xi = 0, |t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\} \\ & \subset \sqrt{-1}T_{\partial T \times \mathbb{R}^n}^*(\mathbb{R}^{m+n}) := \{(t, x; i\lambda\nabla\varphi(t), 0); t \in \partial T, \lambda \in \mathbb{R}\}. \end{aligned}$$

Remark. The above definition for $f(t, x)$ is equivalent to the following condition: There exist some $r' > 0$, some open convex cones $\Gamma_1, \dots, \Gamma_N$ in $\mathbb{R}^n$, and some holomorphic function $F_j(\tilde{t}, z)$ defined in

$$\left\{ (\tilde{t}, z) \in \mathbb{C}^{m+n}; \text{dis}(\tilde{t}, T) < r', |z - \overset{\circ}{x}| < r', \text{Im } z \in \Gamma_j, \right. \\ \left. (-\varphi(\text{Re } \tilde{t}))_+ + |\text{Im } \tilde{t}| < r' |\text{Im } z| \right\}$$

($j = 1, \dots, N$) such that

$$f(t, x) = \sum_{j=1}^N F_j(t, x + i0\Gamma_j) \quad \text{in } \{t \in T, |x - \overset{\circ}{x}| < r'\}.$$

Here, $(s)_+ = s$ ($s \geq 0$), $= 0$ ($s < 0$). Further, let $Y(s)$ be the Heaviside function. Then,

$$\text{ext}_{\overline{T}}(f(t, x)) = \sum_{j=1}^N F_j(t, z) Y(\varphi(t))|_{\text{Im } z \rightarrow 0 \cdot \Gamma_j}.$$

§ 5. Positive hermitian pseudodifferential operators and quasi-positivity

We denote by z, w the complexifications of x, u , and by ξ, η the symbols for ∂_z, ∂_w . Further, we use the notation z^*, ξ^* for the complex conjugate of z, ξ ; for example, $\overset{\circ}{z}^*$. Therefore, the coordinates of $T^*(\mathbb{C}_z^n \times \mathbb{C}_w^n)$ are given as $(z, w; \xi, \eta)$, and the hermitian pseudo-differential operators are defined on the hermitian diagonal set of $T^*(\mathbb{C}_z^n \times \mathbb{C}_w^n)$:

$$\Delta^h(T^*\mathbb{C}^n) := \{(z, w; \xi, \eta) \in T^*(\mathbb{C}_z^n \times \mathbb{C}_w^n); w = z^*, \eta = \xi^*\}.$$

Definition 5.1. Let $\overset{\circ}{p} = (\overset{\circ}{z}, \overset{\circ}{z}^*; \overset{\circ}{\xi}, \overset{\circ}{\xi}^*)$ be a point of $\Delta^h(T^*\mathbb{C}^n)$ ($|\overset{\circ}{\xi}| = 1$). Then,

$$P = \sum_{j,k=0}^{\infty} P_{jk}(z, w, \xi, \eta)$$

is said to be a formal symbol at $\overset{\circ}{p}$ of product hermitian pseudo-differential operators if there exist some positive numbers r, d, A ($0 < A < 1$) such that conditions i)~iii) hold for any j, k :

i) P_{jk} is holomorphic on $V_j \times V_k^*$, where

$$V_j := \{(z; \xi) \in \mathbb{C}^n; |z - \overset{\circ}{z}| < r, |(\xi/|\xi|) - \overset{\circ}{\xi}| < r, (j+1)d < |\xi|\}.$$

ii) For any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ (independent of j, k) such that

$$|P_{jk}(z, w, \xi, \eta)| \leq C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } V_j \times V_k^*.$$

iii) $P_{jk}(z, w, \xi, \eta) = \overline{P_{kj}(w^*, z^*, \eta^*, \xi^*)}$ on $V_j \times V_k^*$.

Concerning 0-equivalence class, we have the following definition: $\sum_{j,k} P_{jk} \sim 0$ if

$$\left| \sum_{j=0}^s \sum_{k=0}^t P_{jk}(z, w, \xi, \eta) \right| \leq C'_\epsilon \exp(-\alpha \min\{s, t\} + \epsilon(|\xi| + |\eta|))$$

holds on $V_s \times V_t^*$ for some $\alpha > 0$, every $s, t \geq 0$, and any $\epsilon > 0$ with some $C'_\epsilon > 0$.

Further, $P = \sum_{j,k=0}^\infty P_{jk}(z, w, \xi, \eta) \gg 0$ at $\overset{\circ}{p} \iff$

For $\forall S(\geq 1), \forall J(\geq 0), \forall (z_{s,j}, \xi_{s,j}; \lambda_{s,j}) \in V_j \times \mathbb{C}$ ($s = 1, \dots, S, j = 0, 1, \dots, J$), we have

$$\sum_{s,t=1}^S \sum_{j,k=0}^J P_{jk}(z_{s,j}, z_{t,k}^*, \xi_{s,j}, \xi_{t,k}^*) \lambda_{s,j} \lambda_{t,k}^* \geq 0.$$

Proposition 5.2. Any formal symbol $\sum_{j,k} P_{jk}$ of product hermitian pseudo-differential operators is equivalent to some simple symbol $P' = P'_{00}$ ($P'_{jk} = 0$ for $\forall (j, k) \neq (0, 0)$) of product hermitian pseudo-differential operator. In particular, $\sum_{j,k} P_{jk}$ is identified with a usual pseudo-differential operator $P(w, z, \partial_z, \partial_w)$ at $\overset{\circ}{p} = (\overset{\circ}{z}, \overset{\circ}{z}^*; \overset{\circ}{\xi}, \overset{\circ}{\xi}^*) \in \Delta^h(T^*\mathbb{C}^n)$. Further, the operator product $P \circ Q$ of $P = \sum_{j,k} P_{jk}$ and $Q = \sum_{j,k} Q_{jk}$ is defined by

$$(P \circ Q)_{jk} := \sum_{\substack{j=|\alpha|+\ell+\ell' \\ k=|\beta|+m+m'}} \frac{1}{\alpha! \beta!} \partial_\xi^\alpha \partial_\eta^\beta P_{\ell m} \cdot \partial_z^\alpha \partial_w^\beta Q_{\ell' m'}.$$

In particular, the product of two formal symbols of product hermitian pseudo-differential operators is a formal symbol of product hermitian pseudo-differential operators. Further, if $P = \sum_{j,k=0}^\infty P_{jk}(z, w, \xi, \eta) \gg 0$ at $\overset{\circ}{p}$ and $Q = \sum_{j,k=0}^\infty Q_{jk}(z, w, \xi, \eta) \gg 0$ at $\overset{\circ}{p}$, then $P \circ Q \gg 0$ at $\overset{\circ}{p}$.

Example 5.3. Let $P(z, \xi)$ be a symbol at $(\overset{\circ}{z}; \overset{\circ}{\xi}) \in T^*\mathbb{C}^n$ of analytic pseudo-differential operator. Put

$$P^*(w, \eta) := \overline{P(w^*, \eta^*)}$$

Then,

$$P(z, \xi) + P^*(w, \eta), \quad P(z, \xi)P^*(w, \eta)$$

are examples of symbols at $p = (\overset{\circ}{z}, \overset{\circ}{z}^*; \overset{\circ}{\xi}, \overset{\circ}{\xi}^*) \in \Delta^h(T^*\mathbb{C}^n)$ of product hermitian pseudo-differential operators. Further, $P(z, \xi)P^*(w, \eta) \gg 0$ at p . Another non-trivial example of positive product hermitian pseudo-differential operators is

$$Q(z, w, \partial_z, \partial_w) := (P(z, \partial_z) + P^*(w, \partial_w))^{-1} \gg 0,$$

where $P(z, \xi)$ is a simple symbol of analytic pseudo-differential operators satisfying the following estimate for some $m > 0$ and some $C > 0$:

$$C^{-1}|\xi|^m \leq \operatorname{Re} P(z, \xi) \leq |P(z, \xi)| \leq C|\xi|^m \quad (|\xi| \rightarrow \infty)$$

in a conic neighborhood of $(\overset{\circ}{z}; \overset{\circ}{\xi}) \in T^*\mathbb{C}^n$.

As we explained in Section 3, in order to treat an operator $P(t, x, \partial_x) + P^*(t, u, \partial_u)$ we must introduce a special type of positive product hermitian pseudo-differential operators; for example, $\exp(M \cdot K_\Omega^\rho(z, \xi, w^*, \eta^*))$ in Section 3.

Definition 5.4. Let $\overset{\circ}{p} = (\overset{\circ}{t}, \overset{\circ}{z}, \overset{\circ}{z}^*; \overset{\circ}{\xi}, \overset{\circ}{\xi}^*)$ be a point $\mathbb{R}^m \times \Delta^h(T^*\mathbb{C}^n)$ ($|\overset{\circ}{\xi}| = 1$). A simple symbol $P(t, z, w, \xi, \eta)$ is said to be a symbol at $\overset{\circ}{p}$ with real analytic parameters t of restricted hermitian pseudo-differential operators with growth order σ if P is holomorphic in

$$V(r) := \{(t, z, w, \xi, \eta) \in \mathbb{C}^{m+4n}; |t - \overset{\circ}{t}| < r, |z - \overset{\circ}{z}| < r, |w - \overset{\circ}{z}^*| < r, \\ |(\xi/|\xi|) - \overset{\circ}{\xi}| < r, |(\eta/|\eta|) - \overset{\circ}{\xi}^*| < r, |\xi| > r^{-1}, |\eta| > r^{-1}\}$$

such that $P(t, z, w, \xi, \eta) = \overline{P(t^*, w^*, z^*, \eta^*, \xi^*)}$ and

$$(5.1) \quad \begin{cases} |\nabla_{(z, w)} P| \leq C \cdot \min\{|\xi|^\sigma, |\eta|^\sigma\}, \\ |\nabla_{(\xi, \eta)} P| \leq C \cdot (|\xi| + |\eta|)^{\sigma-1}. \end{cases}$$

Here, $0 < \sigma < 1/2$, and C, r are some positive constants. Further, a symbol

$$\exp(P(t, z, w, \xi, \eta))$$

with a restricted hermitian symbol P is said to be a simple symbol at $\overset{\circ}{p}$ with real analytic parameters t of exponential restricted hermitian pseudo-differential operators with growth order σ . Indeed, it is easy from (5.1) to obtain an estimate

$$|P(t, z, w, \xi, \eta)| \leq M(|\xi|^\sigma + |\eta|^\sigma) \quad \text{on } V(r)$$

with some $M > 0$. Hence, $\exp(P)$ is a symbol of product hermitian pseudo-differential operators.

Example 5.5.

$$P := A(t, z, w) \cdot \frac{(\xi_1 \eta_1)^{1+(\sigma/2)}}{\xi_1^2 + \eta_1^2} \quad \text{at } (\overset{\circ}{t}, \overset{\circ}{z}, \overset{\circ}{z}^*; dz_1 + dw_1),$$

is a symbol of positive restricted hermitian pseudo-differential operators with growth order σ , where $A(t, z, w)$ is a holomorphic function in a neighborhood of $(\overset{\circ}{t}, \overset{\circ}{z}, \overset{\circ}{z}^*)$ such that for any real fixed t , $A(t, z, w^*)$ is a positive analytic hermitian kernel in z, w .

Definition 5.6. Let $\overset{\circ}{p} = (\overset{\circ}{x}, \overset{\circ}{x}; i\overset{\circ}{\xi}, -i\overset{\circ}{\xi})$ be a point of $\Delta^a(\sqrt{-1}T^*\mathbb{R}^n) (|\overset{\circ}{\xi}| = 1)$. Then a hermitian microkernel $k(x, u)$ is said to be quasi-positive, $k(x, u) \gg_q 0$ at $\overset{\circ}{p}$, if there exists a symbol $P(z, w, \xi, \eta)$ at $\overset{\circ}{p}$ of positive restricted hermitian pseudo-differential operators with growth order $\sigma < 1/2$ such that $:\exp(P(z, w, \xi, \eta)) : k(x, u) \gg 0$ at $\overset{\circ}{p}$. Namely,

$$k(x, u) \gg_q 0 \iff \exists P \text{ (restricted, positive) s. t. } :\exp(P(z, w, \xi, \eta)) : k(x, u) \gg 0.$$

Here, $: Q :$ means the pseudo-differential operator defined by the symbol Q . It is known that the quasi-positivity satisfies the axioms of order relations for hermitian microkernels (Theorem 2.7 in [7]).

§ 6. K. Yamasaki's Sobolev type 2-form of order 0

Let $T \subset \mathbb{R}^m$ be a bounded domain with real analytic boundary ∂T . Let $f(t, x)$ be a hyperfunction on $T \times \{|x - \overset{\circ}{x}| < r\}$, which have real analytic parameters $t \in T$ at any point of $\overline{T} \times \{\overset{\circ}{x}\}$. For $\mu > 0$, a hermitian microkernel

$$\int_{\mathbb{R}^m} ((-\Delta_x)^\mu + (-\Delta_u)^\mu) \text{ext}_{\overline{T}}(f(t, x) \overline{f(t, u)}) dt,$$

in (x, u) is almost identified with Sobolev type 2-form of order μ with respect to x, u . However, concerning t , it is only an L^2 -form. Though we can treat a more general 2-form like (3.1) by using quasi-positivity, we cannot treat the following type 2-form:

$$(6.1) \quad E(x, u) := \int_T (P(t, x, \partial_t, \partial_x) + \overline{P(s, u, \partial_s, \partial_u)}) f(t, x) \overline{f(s, u)}|_{t=s} dt,$$

where $P(t, x, \partial_t, \partial_x)$ is a pseudo-differential operator including ∂_t . Even in such a case, if P is of finite order with respect to ∂_t , we treat E by using microlocal energy methods for vector-valued functions developed in [7]. Indeed, in that case, we consider all the derivatives $\partial_t^\alpha f(t, x)$ as independent hyperfunctions. However, such a method cannot be applied to the case :

$$P = 1 + \sum_{\ell=1}^{\infty} \partial_t^\ell \partial_x^{-\ell}$$

at $(0, 0; 0, i) \in \sqrt{-1}T^*(\mathbb{R}_t \times \mathbb{R}_x)$ because P is not of finite order concerning ∂_t . To overcome this difficulty, K. Yamasaki [11] introduced a Sobolev-type 2-form of order 0.

Definition 6.1. Let s, u be the copies of t, x , respectively. For positive numbers C_1, C_2 , and constants $p, q \geq 0$, we define a micro-differential operator Λ (a fundamental

microdifferential operator) in the variables $(t, x, s, u) \in \mathbb{R}^{m+n+m+n}$ of order 0 (consider in $\{\xi_n \neq 0, \eta_n \neq 0\}$):

$$\Lambda(\partial_t, \partial_{x_n}, \partial_s, \partial_{u_n}) := \sum_{j \geq 0, I \geq 0} \left(\frac{\Gamma((p+1)j + q|I| + 1)}{\Gamma(pj + q|I| + 1)} \right)^2 C_1^{2j} C_2^{2|I|} \partial_t^I \partial_{x_n}^{-j-|I|} \partial_s^I \partial_{u_n}^{-j-|I|},$$

where $j \in \mathbb{N}_+$, $I = (i_1, \dots, i_m) \in \mathbb{N}_+^m$ ($\mathbb{N}_+ = \{0, 1, 2, \dots\}$).

Definition 6.2. Let $f(t, x)$ and $g(t, x)$ be hyperfunctions on $T \times \{|x - \overset{\circ}{x}| < r\}$, which have real analytic parameters $t \in T$ at any point of $\bar{T} \times \{\overset{\circ}{x}\}$. Then, we can introduce an inner product form of f, g as follows:

$$E[f, g](x, u) := \int_{\mathbb{R}^m} \text{ext}_{\bar{T}} [\Lambda(\partial_t, \partial_{x_n}, \partial_s, \partial_{u_n})(f(t, x) \overline{g(s, u)})]_{t=s} dt.$$

Hence, our order 0 Sobolev type 2-form over $\bar{T}$ of $f(t, x)$ is defined as

$$E[f, f](x, u).$$

Let $A(t, x, \partial_t, \partial_x)$ be any 0-th order analytic pseudo-differential operator. Then, our aim is to get the following inequality:

$$(6.2) \quad E[Af, g](x, u) + E[g, Af](x, u) \ll_q C_A (E[f, f](x, u) + E[g, g](x, u))$$

with a positive constant C_A depending only on A . Kaito Yamasaki's main result [11] is the following:

Theorem 6.3. Put $C_1 = C_2 C_3$, and

$$(6.3) \quad \begin{cases} p \geq 0, & q \geq 1, & p - q \geq -1, \\ C_2 > \frac{2^{m+2}}{\lambda}, & C_3 > \frac{2^{m+3}}{\lambda}, & C_2 C_3 > \max\{\frac{8}{\lambda}, \frac{16(m+n)}{\lambda^2}\}. \end{cases}$$

Then, for

$$C_A = 2^{2m+1} N(A; \lambda)$$

we have an inequality (6.2), where $N(A; \lambda)$ is the formal norm of A due to Boutet de Monvel. Namely, $N(A; \lambda)$ is a formal power series of λ , and so $\lambda > 0$ should be taken small enough such that $N(A; \lambda)$ is convergent. Further, the condition for p, q is the necessary and sufficient to get (6.2).

Remark. In the definition of $E[f, g]$, we need some stronger assumption on the singular spectrum of f, g for a given C_2 than the assumption ii) in Definition 4.4. Such a condition is fulfilled if C_2 is sufficiently small.

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