

A remark on multilinear Fourier multipliers satisfying Besov estimates

By

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Abstract

By using the L^r -based Sobolev space $H_s^r(\mathbb{R}^{Nn})$ with $1 < r \leq 2$ and $s > Nn/r$, Grafakos and Si [6] proved the boundedness of multilinear Fourier multiplier operators. In this paper, we try to replace $H_s^r(\mathbb{R}^{Nn})$ by the Besov space $B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$ as the critical case for their result.

§ 1. Introduction

For $m \in L^\infty(\mathbb{R}^{Nn})$, the N -linear Fourier multiplier operator T_m is defined by

$$\begin{aligned} T_m(f_1, \dots, f_N)(x) \\ = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m(\xi_1, \dots, \xi_N) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N) d\xi_1 \dots d\xi_N \end{aligned}$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. As the classical Coifman-Meyer theorem [1], it is well known that if $m \in C^L(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \leq C_{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)}$$

for all $|\alpha_1| + \dots + |\alpha_N| \leq L$, where L is a sufficiently large natural number, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N \leq \infty$ and $1 < p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$.

Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$(1.1) \quad \text{supp } \Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}, \quad \sum_{k \in \mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

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For $m \in L^\infty(\mathbb{R}^{Nn})$ and $j \in \mathbb{Z}$, we set

$$(1.2) \quad m_j(\xi) = m(2^j \xi_1, \dots, 2^j \xi_N) \Psi(\xi_1, \dots, \xi_N),$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and Ψ is as in (1.1) with $d = Nn$. In order to weaken the regularity condition to assure the boundedness, Tomita [11] gave a Hörmander type theorem for multilinear Fourier multipliers. More precisely, he proved that if $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{H_s^2(\mathbb{R}^{Nn})} < \infty \quad \text{with } s > Nn/2,$$

where H_s^2 is the L^2 -based Sobolev space (see Section 2), then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$. Grafakos and Si [6] removed the condition $1 < p < \infty$, and extended this result as follows (see also Grafakos, Miyachi and Tomita [5], Miyachi and Tomita [7] for the cases where some indices p_j are equal to infinity, and $p_j \leq 1$):

Theorem 1.1 ([6]). *Let $1 < r \leq 2$, $r \leq p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{H_s^r(\mathbb{R}^{Nn})} < \infty \quad \text{with } s > Nn/r,$$

then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The purpose of this paper is to consider the critical case $s = Nn/r$ for Theorem 1.1. We note that

$$H_s^r(\mathbb{R}^{Nn}) \hookrightarrow B_{Nn/r}^{r,1}(\mathbb{R}^{Nn}) \quad \text{if } s > Nn/r,$$

where $B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$ is the L^r -based Besov space (see Section 2), and try to replace $H_s^r(\mathbb{R}^{Nn})$ by $B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$. At least, by the slight modification of the arguments in [4, 6, 11], we have

Theorem 1.2. *Let $1 \leq r < 2$, $r < p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})} < \infty,$$

then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

However, our argument given in this paper does not seem to work for the proof of Theorem 1.2 with $r = 2$, and its case will need a different method. It should be pointed out that we can replace $H_{(n/2)+\epsilon}^2(\mathbb{R}^n)$ by $B_{n/2}^{2,1}(\mathbb{R}^n)$ in the linear case (see Seeger [9]).

For the sake of simplicity, we only treat the (usual) Besov spaces in this paper. However, in Theorem 1.2, we can replace $B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$ by the Besov space of product type $B_{(n/r,\dots,n/r)}^{r,1}(\mathbb{R}^n \times \dots \times \mathbb{R}^n)$ (see Remark 4.1).

Our paper is organized as follows: In Section 2, we give definitions and preliminary lemmas. In Section 3, we give a key estimate used in the proof of Theorem 1.2. In Section 4, we prove Theorem 1.2.

§ 2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |f(y)| dy$$

for locally integrable functions f on $\mathbb{R}^n$.

We recall the definitions of Sobolev and Besov spaces. For $1 < r < \infty$ and $s \in \mathbb{R}$, the Sobolev space $H_s^r(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H_s^r} = \|(I - \Delta)^{s/2} f\|_{L^r} < \infty,$$

where $(I - \Delta)^{s/2} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \widehat{f}]$. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be as in (1.1), and set $\Psi_0(\xi) = 1 - \sum_{k=1}^{\infty} \Psi(\xi/2^k)$ and $\Psi_k(\xi) = \Psi(\xi/2^k)$ if $k \geq 1$. Note that $\text{supp } \Psi_0 \subset \{|\xi| \leq 2\}$, $\text{supp } \Psi_k \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ if $k \geq 1$, and $\sum_{k=0}^{\infty} \Psi_k(\xi) = 1$. For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, the Besov space $B_s^{p,q}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B_s^{p,q}} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|\mathcal{F}^{-1}[\Psi_k \widehat{f}]\|_{L^p}^q \right)^{1/q} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|(\mathcal{F}^{-1}\Psi_k) * f\|_{L^p}^q \right)^{1/q} < \infty.$$

We refer to Triebel [12] and the references therein for details on Besov spaces.

The following lemmas will be used later on:

Lemma 2.1 ([3]). *Let $1 < p, q < \infty$. Then*

$$\left\| \left\{ \sum_{k \in \mathbb{Z}} (Mf_k)^q \right\}^{1/q} \right\|_{L^p} \lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p}$$

for all sequences $\{f_k\}_{k \in \mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$.

Lemma 2.2 ([2, Theorem 8.6]). *Let $1 < p < \infty$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp } \psi \subset \{\xi \in \mathbb{R}^n : 1/r \leq |\xi| \leq r\}$ for some $r > 1$. Then*

$$\left\| \left\{ \sum_{k \in \mathbb{Z}} |\psi(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^n),$$

where $\psi(D/2^k)f = \mathcal{F}^{-1}[\psi(\cdot/2^k)\widehat{f}]$.

Let N be a natural number, and let ϕ_0 be a C^∞ -function on $[0, \infty)$ satisfying

$$\phi_0(t) = 1 \quad \text{on} \quad [0, 1/(4N)], \quad \text{supp } \phi_0 \subset [0, 1/(2N)].$$

We also set $\phi_1(t) = 1 - \phi_0(t)$. For $(i_1, i_2, \dots, i_N) \in \{0, 1\}^N$, we define the function $\Phi_{(i_1, i_2, \dots, i_N)}$ on $\mathbb{R}^{Nn} \setminus \{0\}$ by

$$(2.1) \quad \Phi_{(i_1, i_2, \dots, i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|) \dots \phi_{i_N}(|\xi_N|/|\xi|),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2 + \dots + |\xi_N|^2}$. Note that $\Phi_{(0,0,\dots,0)} = 0$. Then we have

Lemma 2.3. *Let $\Phi_{(i_1, \dots, i_N)}$ be the same as in (2.1). Then the following are true:*

(1) For $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$,

$$\sum_{\substack{(i_1, i_2, \dots, i_N) \in \{0, 1\}^N \\ (i_1, i_2, \dots, i_N) \neq (0, 0, \dots, 0)}} \Phi_{(i_1, i_2, \dots, i_N)}(\xi) = 1.$$

(2) For $(i_1, \dots, i_N) \in \{0, 1\}^N$ and $(\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^n \times \dots \times \mathbb{Z}_+^n$,

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \Phi_{(i_1, \dots, i_N)}(\xi)| \leq C_{(i_1, \dots, i_N)}^{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)}$$

for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$.

(3) If $i_j = 1$ for some $1 \leq j \leq N$ and $i_k = 0$ for all $1 \leq k \leq N$ with $k \neq j$, then $\text{supp } \Phi_{(i_1, \dots, i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_k| \leq |\xi_j|/N \text{ for } k \neq j\}$. If $i_j = i_{j'} = 1$ for some $1 \leq j, j' \leq N$ with $j \neq j'$, then $\text{supp } \Phi_{(i_1, \dots, i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_j|/(4N) \leq |\xi_{j'}| \leq 4N|\xi_j|, |\xi_k| \leq 4N|\xi_j| \text{ for } k \neq j, j'\}$.

See [11, Section 5], [4, Lemma 3.1] for the proof of Lemma 2.3.

§ 3. Key estimate

In this section, we prove the following lemma which plays an essential role in the proof of Theorem 1.2:

Lemma 3.1. *Let $1 \leq r \leq 2$. Then*

$$|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \lesssim \|m\|_{B_{Nn/r}^{r,1}} M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}$$

for all $j \in \mathbb{Z}$, $m \in B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$ and $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $\{\Psi_k\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{R}^{Nn})$ be a sequence of functions which appeared in the definition of Besov spaces. Then

$$\begin{aligned} & T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x) \\ &= \int_{\mathbb{R}^{Nn}} 2^{Njn} \mathcal{F}^{-1} m(2^j(x - y_1), \dots, 2^j(x - y_N)) f_1(y_1) \dots f_N(y_N) dy \\ &= (2\pi)^{-Nn} \sum_{k=0}^\infty \int_{\mathbb{R}^{Nn}} 2^{Njn} \Psi_k(2^j(y_1 - x), \dots, 2^j(y_N - x)) \\ &\quad \times \widehat{m}(2^j(y_1 - x), \dots, 2^j(y_N - x)) f_1(y_1) \dots f_N(y_N) dy, \end{aligned}$$

where $y = (y_1, \dots, y_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Let r' be the conjugate exponent of r . Since

$$\text{supp } \Psi_k \subset \{y \in \mathbb{R}^{Nn} : |y| \leq 2^{k+1}\} \subset \{y \in \mathbb{R}^{Nn} : |y_j| \leq 2^{k+1}, j = 1, \dots, N\},$$

we have by Hölder's inequality

$$\begin{aligned} & \left| \int_{\mathbb{R}^{Nn}} 2^{Njn} \Psi_k(2^j(y_1 - x), \dots, 2^j(y_N - x)) \widehat{m}(2^j(y_1 - x), \dots, 2^j(y_N - x)) \right. \\ & \quad \times f_1(y_1) \dots f_N(y_N) dy \Big| \\ & \leq 2^{Njn} \left(\int_{\mathbb{R}^{Nn}} \left| \Psi_k(2^j(y_1 - x), \dots, 2^j(y_N - x)) \widehat{m}(2^j(y_1 - x), \dots, 2^j(y_N - x)) \right|^{r'} dy \right)^{1/r'} \\ & \quad \times \left(\int_{|2^j(y_1 - x)| \leq 2^{k+1}} |f_1(y_1)|^r dy_1 \right)^{1/r} \dots \left(\int_{|2^j(y_N - x)| \leq 2^{k+1}} |f_N(y_N)|^r dy_N \right)^{1/r} \\ & = 2^{N(k+1)n/r} \left(\int_{\mathbb{R}^{Nn}} \left| \Psi_k(y_1, \dots, y_N) \widehat{m}(y_1, \dots, y_N) \right|^{r'} dy \right)^{1/r'} \\ & \quad \times \left(\frac{1}{2^{(k-j+1)n}} \int_{|y_1 - x| \leq 2^{k-j+1}} |f_1(y_1)|^r dy_1 \right)^{1/r} \\ & \quad \times \dots \times \left(\frac{1}{2^{(k-j+1)n}} \int_{|y_N - x| \leq 2^{k-j+1}} |f_N(y_N)|^r dy_N \right)^{1/r} \\ & \leq 2^{N(k+1)n/r} \|\Psi_k \widehat{m}\|_{L^{r'}} M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}. \end{aligned}$$

It follows from the Hausdorff-Young inequality that $\|\Psi_k \widehat{m}\|_{L^{r'}} \lesssim \|\mathcal{F}^{-1}[\Psi_k \widehat{m}]\|_{L^r}$, and consequently

$$\begin{aligned} |T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \\ \lesssim \left(\sum_{k=0}^{\infty} 2^{Nkn/r} \|\mathcal{F}^{-1}[\Psi_k \widehat{m}]\|_{L^r} \right) M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}. \end{aligned}$$

This completes the proof. $\square$

§ 4. Proof of Theorem 1.2

In this section, we use the following notation to distinguish linear and multilinear Fourier multiplier operators: For $\varphi \in L^\infty(\mathbb{R}^n)$, the (linear) Fourier multiplier operator $\varphi(D)$ is defined by $\varphi(D)f = \mathcal{F}^{-1}[\varphi \widehat{f}]$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We also use the following notation: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\text{supp } \varphi$ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ for which $\text{supp } \widetilde{\psi}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$.

Proof of Theorem 1.2. Let $1 \leq r < 2$, $r < p_1, \dots, p_N < \infty$, $1/p_1 + \dots + 1/p_N = 1/p$, and let $m \in L^\infty(\mathbb{R}^{Nn})$ satisfy $\sup_{j \in \mathbb{Z}} \|m_j\|_{B_{Nn/r}^{r,1}} < \infty$, where m_j is defined by (1.2). It follows from Lemma 2.3 (1) that

$$\begin{aligned} (4.1) \quad m(\xi) &= \sum_{\substack{(i_1, i_2, \dots, i_N) \in \{0,1\}^N \\ (i_1, i_2, \dots, i_N) \neq (0,0, \dots, 0)}} \Phi_{(i_1, i_2, \dots, i_N)}(\xi) m(\xi) \\ &= \sum_{\substack{(i_1, i_2, \dots, i_N) \in \{0,1\}^N \\ (i_1, i_2, \dots, i_N) \neq (0,0, \dots, 0)}} m_{(i_1, i_2, \dots, i_N)}(\xi). \end{aligned}$$

Estimate for $m_{(1,0, \dots, 0)}$. We first consider the case where $(i_1, \dots, i_N)$ satisfies $\#\{j : i_j = 1\} = 1$, and may assume without loss of generality that $i_1 = 1$. This means $m_{(i_1, i_2, \dots, i_N)} = m_{(1,0, \dots, 0)}$, and we simply write m instead of $m_{(1,0, \dots, 0)}$. By Lemma 2.3 (3),

$$(4.2) \quad \text{supp } m \subset \{\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n : |\xi_i| \leq |\xi_1|/N, i = 2, \dots, N\}.$$

Let ψ be as in (1.1) with $d = n$. Note that

$$\|g\|_{L^p} \lesssim \|g\|_{\mathcal{H}^p} \approx \left\| \left(\sum_{j \in \mathbb{Z}} |\psi(D/2^j)g|^2 \right)^{1/2} \right\|_{L^p}$$

for appropriate functions g , where $\mathcal{H}^p$ is the Hardy space (e.g. [6, Lemma 2.4]). Then

$$(4.3) \quad \|T_m(f_1, \dots, f_N)\|_{L^p} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |\psi(D/2^j)T_m(f_1, \dots, f_N)|^2 \right)^{1/2} \right\|_{L^p}.$$

It follows from (4.2) that if $(\xi_1, \dots, \xi_N) \in \text{supp } m$ then $|\xi_1 + \dots + \xi_N| \approx |\xi_1|$ and $|\xi_i| \lesssim |\xi_1|$ for $2 \leq i \leq N$, and we can find functions $\varphi \in \mathcal{A}_0$ and $\tilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned} m(\xi)\psi((\xi_1 + \dots + \xi_N)/2^j) \\ = m(\xi)\psi((\xi_1 + \dots + \xi_N)/2^j)\tilde{\psi}(\xi_1/2^j)^2\varphi(\xi_2/2^j)\dots\varphi(\xi_N/2^j), \end{aligned}$$

where we have used the fact $\text{supp } \psi \subset \{\eta \in \mathbb{R}^n : 1/2 \leq |\eta| \leq 2\}$. Hence, setting

$$m_{(j)}(\xi) = m(2^j\xi)\psi(\xi_1 + \dots + \xi_N)\tilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N),$$

we see that

$$\begin{aligned} & \psi(D/2^j)T_m(f_1, \dots, f_N)(x) \\ &= \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m(\xi)\psi((\xi_1 + \dots + \xi_N)/2^j) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \dots \widehat{f_N}(\xi_N) d\xi \\ &= \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m_{(j)}(\xi/2^j) (\tilde{\psi}(\xi_1/2^j) \widehat{f_1}(\xi_1)) \widehat{f_2}(\xi_2) \dots \widehat{f_N}(\xi_N) d\xi \\ &= T_{m_{(j)}(\cdot/2^j)}(\tilde{\psi}(D/2^j)f_1, f_2, \dots, f_N)(x). \end{aligned}$$

By Lemma 3.1,

$$\begin{aligned} & |T_{m_{(j)}(\cdot/2^j)}(\tilde{\psi}(D/2^j)f_1, f_2, \dots, f_N)(x)| \\ & \lesssim \|m_{(j)}\|_{B_{Nn/r}^{r,1}} M(|\tilde{\psi}(D/2^j)f_1|^r)(x)^{1/r} M(|f_2|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}, \end{aligned}$$

and consequently

$$\begin{aligned} (4.4) \quad & \left(\sum_{j \in \mathbb{Z}} |\psi(D/2^j)T_m(f_1, \dots, f_N)(x)|^2 \right)^{1/2} \\ & \lesssim \left(\sup_{k \in \mathbb{Z}} \|m_{(k)}\|_{B_{Nn/r}^{r,1}} \right) \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_1|^r)(x)^{2/r} \right)^{1/2} \\ & \quad \times M(|f_2|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}. \end{aligned}$$

Since $1 \leq r < 2$ and $r < p_1, \dots, p_N < \infty$, we see that $1 < 2/r, p_1/r, \dots, p_N/r < \infty$. Then, it follows from Hölder's inequality, Lemmas 2.1 and 2.2 that

$$\begin{aligned}
 (4.5) \quad & \left\| \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{1/2} M(|f_2|^r)^{1/r} \dots M(|f_N|^r)^{1/r} \right\|_{L^p} \\
 & \leq \left\| \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{1/2} \right\|_{L^{p_1}} \|M(|f_2|^r)^{1/r}\|_{L^{p_2}} \dots \|M(|f_N|^r)^{1/r}\|_{L^{p_N}} \\
 & = \left\| \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{r/2} \right\|_{L^{p_1/r}}^{1/r} \|M(|f_2|^r)^{1/r}\|_{L^{p_2/r}} \dots \|M(|f_N|^r)^{1/r}\|_{L^{p_N/r}} \\
 & \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.
 \end{aligned}$$

Thus, by (4.3)-(4.5),

$$\|T_m(f_1, f_2, \dots, f_N)\|_{L^p} \lesssim \left(\sup_{j \in \mathbb{Z}} \|m_{(j)}\|_{B_{Nn/r}^{r,1}} \right) \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.$$

Recall that $m(\xi) = m_{(1,0,\dots,0)}(\xi)$, and

$$m_{(j)}(\xi) = m(2^j \xi) \Phi_{(1,0,\dots,0)}(2^j \xi) \psi(\xi_1 + \dots + \xi_N) \tilde{\psi}(\xi_1) \varphi(\xi_2) \dots \varphi(\xi_N).$$

Let us prove

$$(4.6) \quad \sup_{j \in \mathbb{Z}} \|m_{(j)}\|_{B_{Nn/r}^{r,1}} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{B_{Nn/r}^{r,1}},$$

where m_j is defined by (1.2). Once this is proved, we have the desired estimate:

$$\|T_{m_{(1,0,\dots,0)}}(f_1, f_2, \dots, f_N)\|_{L^p} \lesssim \left(\sup_{j \in \mathbb{Z}} \|m_j\|_{B_{Nn/r}^{r,1}} \right) \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.$$

Let Ψ be as in (1.1) with $d = Nn$. Since $\text{supp } \Psi(\cdot/2^\ell) \subset \{2^{\ell-1} \leq |\xi| \leq 2^{\ell+1}\}$, $\text{supp } \tilde{\psi}(\xi_1) \varphi(\xi_2) \dots \varphi(\xi_N) \subset \{2^{-j_0} \leq |\xi| \leq 2^{j_0}\}$ for some $j_0 \in \mathbb{N}$ and $B_{Nn/r}^{r,1}(\mathbb{R}^{Nn})$ is a multiplication algebra (Triebel [12, Theorem 2.8.3]), we have

$$\begin{aligned}
 \|m_{(j)}(\xi)\|_{B_{Nn/r}^{r,1}} & \leq \sum_{\ell=-j_0}^{j_0} \|m_{(j)}(\xi) \Psi(\xi/2^\ell)\|_{B_{Nn/r}^{r,1}} \\
 & \lesssim \sum_{\ell=-j_0}^{j_0} \|m(2^j \xi) \Psi(\xi/2^\ell)\|_{B_{Nn/r}^{r,1}} \\
 & \quad \times \|\Phi_{(1,0,\dots,0)}(2^j \xi) \psi(\xi_1 + \dots + \xi_N) \tilde{\psi}(\xi_1) \varphi(\xi_2) \dots \varphi(\xi_N)\|_{B_{Nn/r}^{r,1}}.
 \end{aligned}$$

By a change of variables,

$$\begin{aligned} \|m(2^j \xi) \Psi(\xi/2^\ell)\|_{B_{Nn/r}^{r,1}} &\lesssim (2^{-\ell})^{-Nn/r} (\max\{1, 2^{-\ell}\})^{Nn/r} \|m(2^{j+\ell} \xi) \Psi(\xi)\|_{B_{Nn/r}^{r,1}} \\ &\lesssim \sup_{j \in \mathbb{Z}} \|m(2^j \xi) \Psi(\xi)\|_{B_{Nn/r}^{r,1}} = \sup_{j \in \mathbb{Z}} \|m_j\|_{B_{Nn/r}^{r,1}} \end{aligned}$$

for all $|\ell| \leq j_0$ (see, for example, [8, Proposition 2.1.3/3], [10, Proposition 1.1]). On the other hand, by Lemma 2.3 (2),

$$\left| \partial_\xi^\alpha \left(\Phi_{(1,0,\dots,0)}(2^j \xi) \psi(\xi_1 + \dots + \xi_N) \tilde{\psi}(\xi_1) \varphi(\xi_2) \dots \varphi(\xi_N) \right) \right| \leq C_\alpha \chi_{\{2^{-j_0} \leq |\xi| \leq 2^{j_0}\}}(\xi)$$

for all α and j , and consequently

$$\sup_{j \in \mathbb{Z}} \|\Phi_{(1,0,\dots,0)}(2^j \xi) \psi(\xi_1 + \dots + \xi_N) \tilde{\psi}(\xi_1) \varphi(\xi_2) \dots \varphi(\xi_N)\|_{B_{Nn/r}^{r,1}} < \infty.$$

Combining these estimates, we have (4.6).

Estimate for $m_{(1,1,i_3,\dots,i_N)}$. We next consider the case where $(i_1, \dots, i_N)$ satisfies $\#\{j : i_j = 1\} \geq 2$, and may assume without loss of generality that $i_1 = i_2 = 1$. This means $m_{(i_1,i_2,i_3,\dots,i_N)} = m_{(1,1,i_3,\dots,i_N)}$, where $i_3, \dots, i_N \in \{0, 1\}$. We simply write m instead of $m_{(1,1,i_3,\dots,i_N)}$ as before. By Lemma 2.3 (3),

$$(4.7) \quad \text{supp } m \subset \{|\xi_1|/(4N) \leq |\xi_2| \leq 4N|\xi_1|, |\xi_i| \leq 4N|\xi_1|, i = 3, \dots, N\}.$$

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (1.1) with $d = n$. By (4.7), we can find $\varphi \in \mathcal{A}_0$ and $\tilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$m(\xi) \psi(\xi_1/2^j) = m(\xi) \psi(\xi_1/2^j) \tilde{\psi}(\xi_1/2^j) \tilde{\psi}(\xi_2/2^j)^2 \varphi(\xi_3/2^j) \dots \varphi(\xi_N/2^j).$$

Hence, setting

$$m_{(j)}(\xi) = m(2^j \xi) \psi(\xi_1) \tilde{\psi}(\xi_2) \varphi(\xi_3) \dots \varphi(\xi_N),$$

we see that

$$\begin{aligned} &T_m(f_1, \dots, f_N)(x) \\ &= \sum_{j \in \mathbb{Z}} \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m(\xi) \psi(\xi_1/2^j) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N) d\xi \\ &= \sum_{j \in \mathbb{Z}} \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m_{(j)}(\xi/2^j) (\tilde{\psi}(\xi_1/2^j) \widehat{f_1}(\xi_1)) (\tilde{\psi}(\xi_2/2^j) \widehat{f_2}(\xi_1)) \\ &\quad \times \widehat{f_3}(\xi_3) \dots \widehat{f_N}(\xi_N) d\xi \\ &= \sum_{j \in \mathbb{Z}} T_{m_{(j)}(\cdot/2^j)}(\tilde{\psi}(D/2^j) f_1, \tilde{\psi}(D/2^j) f_2, f_3, \dots, f_N)(x). \end{aligned}$$

It follows from Lemma 3.1 and Schwarz's inequality that

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}} |T_{m(j)}(\tilde{\psi}(D/2^j)f_1, \tilde{\psi}(D/2^j)f_2, f_3, \dots, f_N)(x)| \\
& \lesssim \sum_{j \in \mathbb{Z}} \|m(j)\|_{B_{Nn/r}^{r,1}} M(|\tilde{\psi}(D/2^j)f_1|^r)(x)^{1/r} M(|\tilde{\psi}(D/2^j)f_2|^r)(x)^{1/r} \\
& \quad \times M(|f_3|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r} \\
& \leq \left(\sup_{k \in \mathbb{Z}} \|m(k)\|_{B_{Nn/r}^{r,1}} \right) \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_1|^r)(x)^{2/r} \right)^{1/2} \\
& \quad \times \left(\sum_{j \in \mathbb{Z}} M(|\tilde{\psi}(D/2^j)f_2|^r)(x)^{2/r} \right)^{1/2} M(|f_3|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}.
\end{aligned}$$

The rest of the proof is similar to that of $m_{(1,0,\dots,0)}$, and we omit it.

We end this paper by giving the following remark:

Remark 4.1. Let $\{\Psi_k\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence of functions appearing in the definition of Besov spaces. For $1 \leq r \leq \infty$ and $s_1, \dots, s_N \in \mathbb{R}$, the Besov space of product type $B_{(s_1, \dots, s_N)}^{r,1}(\mathbb{R}^n \times \dots \times \mathbb{R}^n)$ is defined by the norm

$$\|f\|_{B_{(s_1, \dots, s_N)}^{r,1}} = \sum_{k_1, \dots, k_N=0}^\infty 2^{(k_1 s_1 + \dots + k_N s_N)} \|\mathcal{F}^{-1}[\Psi_{k_1}(\xi_1) \dots \Psi_{k_N}(\xi_N) \widehat{f}(\xi)]\|_{L^r(\mathbb{R}^{Nn})},$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Note that if $s_1, \dots, s_N > 0$, then

$$B_{s_1 + \dots + s_N}^{r,1}(\mathbb{R}^{Nn}) \hookrightarrow B_{(s_1, \dots, s_N)}^{r,1}(\mathbb{R}^n \times \dots \times \mathbb{R}^n).$$

In the same way as in the proof of Lemma 3.1, we can prove

$$\begin{aligned}
& |T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \\
& \lesssim \sum_{k_1, \dots, k_N=0}^\infty 2^{(k_1 + \dots + k_N)n/r} \left(\int_{\mathbb{R}^{Nn}} |\Psi_{k_1}(y_1) \dots \Psi_{k_N}(y_N) \widehat{m}(y)|^{r'} dy \right)^{1/r'} \\
& \quad \times \left(\frac{1}{2^{(k_1-j+1)n}} \int_{|y_1-x| \leq 2^{k_1-j+1}} |f_1(y_1)|^r dy_1 \right)^{1/r} \\
& \quad \times \dots \times \left(\frac{1}{2^{(k_N-j+1)n}} \int_{|y_N-x| \leq 2^{k_N-j+1}} |f_N(y_N)|^r dy_N \right)^{1/r}.
\end{aligned}$$

As a result,

$$|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \lesssim \|m\|_{B_{(n/r, \dots, n/r)}^{r,1}} M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}$$

for $1 \leq r \leq 2$. Then, in the case $1 \leq r < 2$, by using this estimate instead of Lemma 3.1, we can prove Theorem 1.2 with $B_{Nn/r}^{r,1}$ replaced by $B_{(n/r, \dots, n/r)}^{r,1}$. It should be mentioned that dilation and multiplication properties of Besov spaces were used in the proof of Theorem 1.2. See [10, Proposition 1.1, Theorem 1.4] for their properties of Besov spaces of product type.

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