

PATHS, TABLEAUX, AND q -CHARACTERS OF QUANTUM AFFINE ALGEBRAS

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1. INTRODUCTION

This paper is based on [15], joint work with T. Nakanishi.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\hat{\mathfrak{g}}$ be the corresponding non-twisted affine Lie algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is the quantized universal enveloping algebra of $\hat{\mathfrak{g}}$. The q -character of $U_q(\hat{\mathfrak{g}})$ was introduced in [9] to study the intricate structure of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$.

Earlier than the introduction of the q -character, the tableaux descriptions of the spectra of the transfer matrices of a vertex model associated to $U_q(\hat{\mathfrak{g}})$ was studied in [5, 11, 13] etc. for $\mathfrak{g}$ of classical type. Since the q -character is designed to be a “universalization” of the family of transfer matrices, one can interpret their results in the context of the q -character. Then, the Jacobi-Trudi determinant $\chi_{\lambda/\mu, a}$ is conjectured to be the q -character of certain finite dimensional representation associated to a skew diagram λ/μ and $a \in \mathbb{C}$ for A_n and B_n . For these cases, the determinant $\chi_{\lambda/\mu, a}$ is described by the tableaux which satisfy certain “horizontal” and “vertical” rules [5, 11]. In contrast, the tableaux description of $\chi_{\lambda/\mu, a}$ for C_n and D_n is known only for the cases when λ/μ is a one-row or one-column diagram.

In this paper, we conjecture that $\chi_{\lambda/\mu, a}$ is the q -character of a finite dimensional representation, and give a summary of the method to give a tableaux description for C_n , using the “paths” method of Gessel-Viennot [10]. For simplicity, the paths method is introduced by applying it for the A_n case. To apply it for the C_n case, some modifications are required. As a result, the tableaux are given by certain horizontal and vertical rules with an “extra” rule. The case for two-row diagrams are given, which is the simplest typical example of it. In conclusion, we give a conjecture of an implicit form of the extra rule in terms of paths.

2. PRELIMINARIES

2.1. Quantum affine algebras. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n , and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. Let $U_q(\hat{\mathfrak{g}})$ be the *quantum affine algebra*, namely, it is the associative algebra generated by $x_i^\pm, k_i^{\pm 1}$ ($i = 0, \dots, n$) with relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$

$$k_i x_j^\pm k_i^{-1} = q^{\pm B_{ij}} x_j^\pm,$$

$$x_i^+ x_j^- - x_j^- x_i^+ = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-C_{ij}} \begin{bmatrix} 1-C_{ij} \\ r \end{bmatrix}_{q_i} (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-C_{ij}-r} = 0, \quad i \neq j.$$

Here, $q \in \mathbb{C}^\times$, $C = (C_{ij})_{0 \leq i, j \leq n}$ is the Cartan matrix of $\hat{\mathfrak{g}}$, and $q_i := q^{r_i}$, where r_i 's are relatively prime integers such that $B = (B_{ij}) = DC$ for $D = \text{diag}(r_1, \dots, r_n)$. We also set

$$\begin{aligned} \begin{bmatrix} s \\ t \end{bmatrix}_q &:= \frac{[s]_q!}{[t]_q! [s-t]_q!}, \\ [s]_q! &:= [s]_q [s-1]_q \dots [1]_q, \quad [s]_q := \frac{q^s - q^{-s}}{q - q^{-1}}. \end{aligned}$$

If we let $q \rightarrow 1$, then $U_q(\hat{\mathfrak{g}})$ becomes the universal enveloping algebra $U(\hat{\mathfrak{g}}')$ of the subalgebra $\hat{\mathfrak{g}}' := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \subset \hat{\mathfrak{g}}$ (c is the center).

2.2. Finite dimensional representations of $U_q(\hat{\mathfrak{g}})$. There is a bijection between the set of the isomorphism classes of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ and the set of n -tuples polynomials $\mathbf{P}(u) = (P_i(u))_{i=1, \dots, n}$ with constant term 1, which are called the *Drinfel'd polynomials* [6, 7]. For any skew diagram λ/μ with its depth $d(\lambda/\mu) \leq n$ ($d(\lambda/\mu) \leq n-1$ for B_n and $d(\lambda/\mu) \leq n-2$ for D_n) and $a \in \mathbb{C}$, let $V(\lambda/\mu, a)$ be the representation that corresponds to the Drinfel'd polynomial

$$\prod_{j=1}^{d(\lambda'/\mu')} P_{\lambda'_j - \mu'_j, a(j)}(u)$$

where $\mathbf{P}_{i,a}(u) = (P_j(u))_{j=1, \dots, n}$ is defined as

$$P_j(u) = \begin{cases} 1 - uq^a, & \text{if } j = i, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$a(j) = a + (2j - \lambda'_j - \mu'_j - 1)\delta.$$

Here, λ' denotes the conjugate of λ . Then, the highest weight of $V(\lambda/\mu, a)$ considered as a $U_q(\mathfrak{g})$ -module is $\sum_{j=1}^{d(\lambda'/\mu')} \omega_{\lambda'_j - \mu'_j}$, where ω_r is the r th fundamental weight.

For the definition of the representation $V(\lambda/\mu, a)$ associated to a skew diagram λ/μ of $d(\lambda/\mu) = n$ for B_n and $d(\lambda/\mu) = n-1, n$ for D_n , see [15].

2.3. The q -characters of quantum affine algebras. The q -character of $U_q(\hat{\mathfrak{g}})$, introduced in [9], is an injective ring homomorphism

$$(2.1) \quad \chi_q : \text{Rep } U_q(\hat{\mathfrak{g}}) \rightarrow U_q(\tilde{\mathfrak{h}})[[u]],$$

where $\text{Rep } U_q(\hat{\mathfrak{g}})$ be the Grothendieck ring of the category of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$, and $U_q(\tilde{\mathfrak{h}})$ is a certain subalgebra of $U_q(\hat{\mathfrak{g}})$. It is defined as a composition of two maps; the map that sends $V \in \text{Rep } U_q(\hat{\mathfrak{g}})$ to the "universal" transfer matrix

$$t_V(u) := \text{Tr}_V(\pi_{V(u)} \otimes \text{id})(\mathcal{R}) \in U_q(\hat{\mathfrak{g}})[[u]],$$

and the projection $U_q(\hat{\mathfrak{g}})[[u]] \rightarrow U_q(\tilde{\mathfrak{h}})[[u]]$. The element $\mathcal{R} \in U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}})$, called the *universal R-matrix*, satisfies the Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}.$$

Sending the second component of $\mathcal{R}$ by a representation $(\pi^{\otimes p}, W^{\otimes p})$, the element $t_V(u)$ becomes the transfer matrix

$$t_V(u; u_1, \dots, u_p) := \text{Tr}_V(R_{V,W}(u - u_1) \dots R_{V,W}(u - u_p)).$$

For the simplest example, the tableaux description (with spectral parameter $a \in \mathbb{C}$) of the q -character for the first fundamental representation $V((1), a)$ is given as follows:

$$(2.2) \quad \chi_q(V((1), a)) = \sum_{i \in I} z_{i,a} = \sum_{i \in I} \boxed{i}_a.$$

Here, the set I is

$$(2.3) \quad I = \begin{cases} \{1, 2, \dots, n, n+1\}, & (A_n) \\ \{1 \prec 2 \prec \dots \prec n \prec 0 \prec \bar{n} \prec \dots \prec \bar{2} \prec \bar{1}\}, & (B_n) \\ \{1 \prec 2 \prec \dots \prec n \prec \bar{n} \prec \dots \prec \bar{2} \prec \bar{1}\}, & (C_n) \\ \{1 \prec 2 \prec \dots \prec \frac{n}{\bar{n}} \prec \dots \prec \bar{2} \prec \bar{1}\}. & (D_n) \end{cases}$$

See [9] for the monomials $z_{i,a}$ occurring in (2.2).

3. THE JACOBI-TRUDI FORMULA FOR THE q -CHARACTERS

For any $a \in \mathbb{C}$, we define $E_a(z, X)$ and $H_a(z, X)$ as follows:

$$(3.1) \quad E_a(z, X) := \begin{cases} \prod_{1 \leq k \leq n+1} (1 + z_{k,a} X) & (A_n) \\ \{\prod_{1 \leq k \leq n} (1 + z_{k,a} X)\} (1 - z_{0,a} X)^{-1} \{\prod_{1 \leq k \leq n} (1 + z_{\bar{k},a} X)\} & (B_n) \\ \{\prod_{1 \leq k \leq n} (1 + z_{k,a} X)\} (1 - z_{n,a} X z_{\bar{n},a} X)^{-1} \{\prod_{1 \leq k \leq n} (1 + z_{\bar{k},a} X)\} & (C_n) \\ \{\prod_{1 \leq k \leq n} (1 + z_{k,a} X)\} (1 - z_{\bar{n},a} X z_{n,a} X)^{-1} \{\prod_{1 \leq k \leq n} (1 + z_{\bar{k},a} X)\} & (D_n) \end{cases}$$

$$(3.2) \quad H_a(z, X) := \begin{cases} \prod_{1 \leq k \leq n+1} (1 - z_{k,a} X)^{-1} & (A_n) \\ \{\prod_{1 \leq k \leq n} (1 - z_{\bar{k},a} X)^{-1}\} (1 + z_{0,a} X) \{\prod_{1 \leq k \leq n} (1 - z_{k,a} X)^{-1}\} & (B_n) \\ \{\prod_{1 \leq k \leq n} (1 - z_{\bar{k},a} X)^{-1}\} (1 - z_{n,a} X z_{\bar{n},a} X)^{-1} \{\prod_{1 \leq k \leq n} (1 - z_{k,a} X)^{-1}\} & (C_n) \\ \{\prod_{1 \leq k \leq n} (1 - z_{\bar{k},a} X)^{-1}\} (1 - z_{\bar{n},a} X z_{n,a} X)^{-1} \{\prod_{1 \leq k \leq n} (1 - z_{k,a} X)^{-1}\} & (D_n) \end{cases}$$

where $\overleftarrow{\prod}_{1 \leq k \leq n} A_k = A_1 \dots A_n$ and $\overrightarrow{\prod}_{1 \leq k \leq n} A_k = A_n \dots A_1$, and

$$(3.3) \quad z_{i,a} z_{j,a'} = z_{j,a'} z_{i,a}, \quad X z_{i,a} = z_{i,a-2\delta} X, \quad i, j \in I; a, a' \in \mathbb{C},$$

where δ is

$$(3.4) \quad \delta = \begin{cases} 1, & (A_n, C_n, D_n) \\ 2, & (B_n) \end{cases}$$

Then we have

$$(3.5) \quad H_a(z, X) E_a(z, -X) = E_a(z, -X) H_a(z, X) = 1.$$

It has been observed in [8, 12] (see also [11, 13]) that $e_{i,a}$ is the q -character of the r th fundamental representation $V((1^r), a)$ of $U_q(\hat{\mathfrak{g}})$ for $1 \leq r \leq n$ ($r \neq n$ for B_n , $r \neq n-1, n$ for D_n), while $h_{r,a}$ is the q -character of the r th “symmetric” power of the first fundamental representation of $U_q(\hat{\mathfrak{g}})$ for any $r \geq 1$. The tableaux description is given as follows:

$$e_{r,a} = \sum_{i_1, \dots, i_r \in I; (\mathbf{V})} \prod_{l=1}^r z_{i_l, a+2(1-l)} = \sum_{i_1, \dots, i_r \in I; (\mathbf{V})} \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_r \\ \hline \end{array} \begin{array}{l} a \\ a-2 \\ \\ a-2r+2 \end{array},$$

$$h_{r,a+2r-2} = \sum_{i_1, \dots, i_r \in I; (\mathbf{H})} \prod_{l=1}^r z_{i_l, a+2(l-r)} = \sum_{i_1, \dots, i_r \in I; (\mathbf{H})} \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_r \\ \hline a & a+2 & & a+2r-2 \end{array}.$$

The rules $(\mathbf{V})$ and $(\mathbf{H})$ for $i_1, \dots, i_r \in I$, called the *vertical* and *horizontal* rules, are given as follows [5, 11, 13]: for any $k = 1, \dots, r$,

$$(3.6) \quad \begin{array}{ll} (\mathbf{V}) & i_k < i_{k+1}, & (A_n) \\ & i_k \prec i_{k+1} \text{ or } i_k = i_{k+1} = 0, & (B_n) \\ & i_k \prec i_{k+1} \text{ and } [(i_k, i'_k) = (c, \bar{c}) \Rightarrow k' - k \leq n - c], & (C_n) \\ & i_k \prec i_{k+1} \text{ or } (i_k, i_{k+1}) = (n, \bar{n}) \text{ or } (i_k, i_{k+1}) = (\bar{n}, n). & (D_n) \end{array}$$

$$(3.7) \quad \begin{array}{ll} (\mathbf{H}) & i_k \leq i_{k+1}, & (A_n) \\ & i_k \preceq i_{k+1} \text{ and } i_k = i_{k+1} \neq 0, & (B_n) \\ & [i_k \preceq i_{k+1} \text{ or } (i_k, i_{k+1}) = (\bar{n}, n)] & \\ & \text{and } (i_k, i_{k+1}, i_{k+2}) \neq (\bar{n}, \bar{n}, n), (\bar{n}, n, n), & (C_n) \\ & i_k \preceq i_{k+1} \text{ and } n \text{ and } \bar{n} \text{ do not appear simultaneously.} & (D_n) \end{array}$$

Due to the relation (3.5), it holds that [14]

$$(3.8) \quad \det(h_{\lambda_i - \mu_j - i + j, a + 2(\lambda_i - i)\delta})_{1 \leq i, j \leq l} = \det(e_{\lambda'_i - \mu'_j - i + j, a - 2(\mu'_j - j + 1)\delta})_{1 \leq i, j \leq l'}$$

for any partitions (λ, μ) , where l and l' are any non-negative integers such that $l \geq d(\lambda)$, $d(\mu)$ and $l' \geq d(\lambda')$, $d(\mu')$. For any skew diagrams λ/μ , let $\chi_{\lambda/\mu, a}$ denote the determinant on the left or right hand side of (3.8). We call it the *Jacobi-Trudi determinant* of $U_q(\hat{\mathfrak{g}})$ associated to λ/μ and $a \in \mathbb{C}$. Note that $\chi_{(r), a} = h_{r, a}$ and $\chi_{(1^r), a} = e_{r, a}$.

We remark that the determinant (3.8) appears in [5] for A_n and [11] for B_n in the context of transfer matrices.

We conjecture that

- Conjecture 3.1.** (1) If $\mathfrak{g}$ is of type A_n or B_n and λ/μ is a skew diagram of $d(\lambda) \leq n$, then $\chi_{\lambda/\mu, a} = \chi_q(V(\lambda/\mu, a))$.
- (2) If $\mathfrak{g}$ is of type C_n or D_n and λ/μ is a skew diagram of $d(\lambda) \leq n$ when $\mathfrak{g}$ is of type C_n (resp. $d(\lambda) \leq n-1$ when $\mathfrak{g}$ is of type D_n), then $\chi_{\lambda/\mu, a}$ is the q -character of certain (not necessarily irreducible) representation V of $U_q(\hat{\mathfrak{g}})$ which has $V(\lambda/\mu, a)$ as a subquotient; furthermore, if $\mu = \phi$, then $\chi_{\lambda, a} = \chi_q(V(\lambda, a))$.

Remark 3.2. An analogue of Conjecture 3.1 is true for the representations of Yangian $Y(\mathfrak{sl}_n)$, which can be proved [2] using the results in [3, 4].

Remark 3.3. The representation V in Conjecture 3.1 for C_n and D_n do not coincide with the irreducible representation $V(\lambda/\mu, a)$, in general. The case $(\lambda, \mu) = ((3, 1), (2))$ is a counter-example for C_2 and for D_4 , which can be shown from the singularities of the R -matrices (see [1] for example).

In the following sections, we give the tableaux description by applying the path method of Gessel-Viennot [10] (see also [16]). First, we introduce the method for the A_n case in Section 4, and then we refer to the modification of this method to apply it for C_n in Section 5.

4. TABLEAUX DESCRIPTION OF TYPE A_n

The method was originally introduced to derive the well-known semistandard tableaux description of the Schur function from the (original) Jacobi-Trudi determinant. For A_n , this method works out without any modification (except for inserting the spectral parameters).

During this section, I is of type A_n in (2.3).

4.1. Paths description. In this subsection, we give a paths description of $\chi_{\lambda/\mu, a}$ (3.8) in terms of the method of Gessel-Viennot.

A *path* p in the lattice $\mathbb{Z} \times \mathbb{Z}$ is a sequence of steps $(s_1, s_2, \dots)$ such that each step s_i is of unit length with the northward (N) or eastward (E) direction. For example, see Figure 1.

Let p be any path such that the initial point is at height 0 and the final point is at height n , and set $E(p) := \{s \in p \mid s \text{ is an eastward step}\}$. The maps

$$L_a^1 : E(p) \rightarrow I, \quad L_a^2 : E(p) \rightarrow \{a + 2k \mid k \in \mathbb{Z}\},$$

called the h -labeling of type A_n associated to $a \in \mathbb{C}$, are defined as in Figure 1. For example, $z_a^p = z_{2,a} z_{2,a+2} z_{3,a+4} z_{3,a+6}$ for p in Figure 1. We define

$$(4.1) \quad z_a^p := \prod_{s \in E(p)} z_{L_a^1(s), L_a^2(s)}.$$

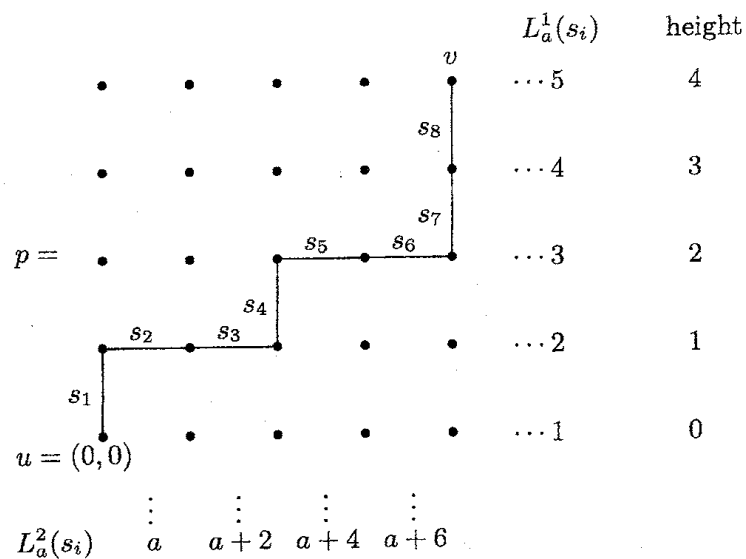


FIGURE 1. An example of a path p and its h -labeling.

Then we have

$$(4.2) \quad h_{r,a+2r-2}(z) = \sum_{(0,0) \xrightarrow{P} (r,n)} z_a^P.$$

For any skew diagrams λ/μ , let $l = d(\lambda)$, and let $\mathbf{u}_\mu = (u_1, \dots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \dots, v_l)$ be l -tuples of fixed initial and final points defined as $u_i = (\mu_i + 1 - i, 0)$ and $v_i = (\lambda_i + 1 - i, n)$. Let $\mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ be the set of l -tuples of paths $\mathbf{p} = (p_1, \dots, p_l)$ such that $u_i \xrightarrow{p_i} v_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_l$. We define the weight $z_a^{\mathbf{p}}$ and the signature $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ and $a \in \mathbb{C}$ by

$$z_a^{\mathbf{p}} = \prod_{i=1}^l z_a^{p_i}, \quad (-1)^{\mathbf{p}} = \text{sgn } \pi.$$

Then, the determinant (3.8) can be written as

$$(4.3) \quad \chi_{\lambda/\mu, a} = \sum_{\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

by (4.2). Applying the method of [10], we have

Proposition 4.1. *For any skew diagrams λ/μ ,*

$$(4.4) \quad \chi_{\lambda/\mu, a} = \sum_{\mathbf{p} \in P(A_n; \mu, \lambda)} z_a^{\mathbf{p}},$$

where $P(A_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \dots, p_l) \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any intersecting pairs of paths (p_i, p_j) .

The idea of its proof is to consider a weight-preserving, sign-changing involution ι on all $\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which possess an intersecting pair of paths. It immediately follows that the contributions of all such $\mathbf{p}$ to the right hand side of (4.3) are canceled with each other.

4.2. Tableaux description. We define the weight z_a^T for any tableau T of shape λ/μ as

$$z_a^T := \prod_{(i,j) \in \lambda/\mu} z_{T(i,j), a+2(j-i)},$$

where $T(i, j)$ is the entry of T at (i, j) , namely, the entry at the i th row and the j th column.

A tableau T which satisfies the rules **(V)** and **(H)** of A_n in (3.6) and (3.7) is called an A_n -tableau. Namely, an A_n -tableau is nothing but a semistandard tableau. We write the set of all the A_n -tableaux of shape λ/μ by $\text{Tab}(A_n, \lambda/\mu)$.

For any $\mathbf{p} = (p_1, \dots, p_l) \in P(A_n; \mu, \lambda)$, we associate a tableau $T(\mathbf{p})$ of shape λ/μ such that the i th row of $T(\mathbf{p})$ is given by $\{L_a^1(s) \mid s \in E(p_i)\}$ listed in the increasing order. See Figure 2 for an example. Clearly, $T(\mathbf{p})$ satisfies the horizontal rule because of the h -labeling rule of $\mathbf{p}$, and $T(\mathbf{p})$ satisfies the vertical rule since $\mathbf{p} \in P(A_n; \mu, \lambda)$ do not have any intersecting pairs of paths. Therefore, we obtain a map

$$(4.5) \quad T : P(A_n; \mu, \lambda) \ni \mathbf{p} \mapsto T(\mathbf{p}) \in \text{Tab}(A_n, \lambda/\mu)$$

for any skew diagrams λ/μ . In fact,

Proposition 4.2. *The map T (4.5) is a weight-preserving bijection.*

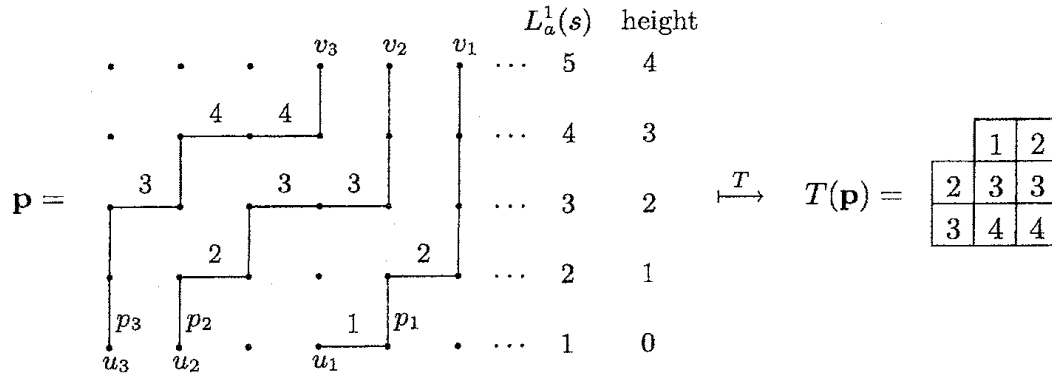


FIGURE 2. An example of p and the tableau $T(p)$ for $(\lambda, \mu) = ((3^3), (1))$.

By Proposition 4.1 and 4.2, we reproduce the result of [5].

Theorem 4.3 ([5]). *If λ/μ is a skew diagram, then*

$$\chi_{\lambda/\mu, a} = \sum_{T \in \text{Tab}(A_n, \lambda/\mu)} z_a^T.$$

5. TABLEAUX DESCRIPTION OF TYPE C_n

In this section, we consider the case that $\mathfrak{g}$ is of type C_n .

5.1. Paths description. In view of the definition of the generating function of $H_a(z, X)$ in (3.2), we define an h -path and its h -labeling as follows:

Definition 5.1. Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. An h -path of type C_n is a path $u \xrightarrow{p} v$ such that the initial point u is at height $-n$ and the final point v is at height n , and the number of the eastward steps at height 0 is even. Let $P(C_n)$ denote the set of all the h -paths of type C_n .

For any $p \in P(C_n)$, the h -labeling (L_a^1, L_a^2) of type C_n associated to $a \in \mathbb{C}$ is defined as in Figure 3.

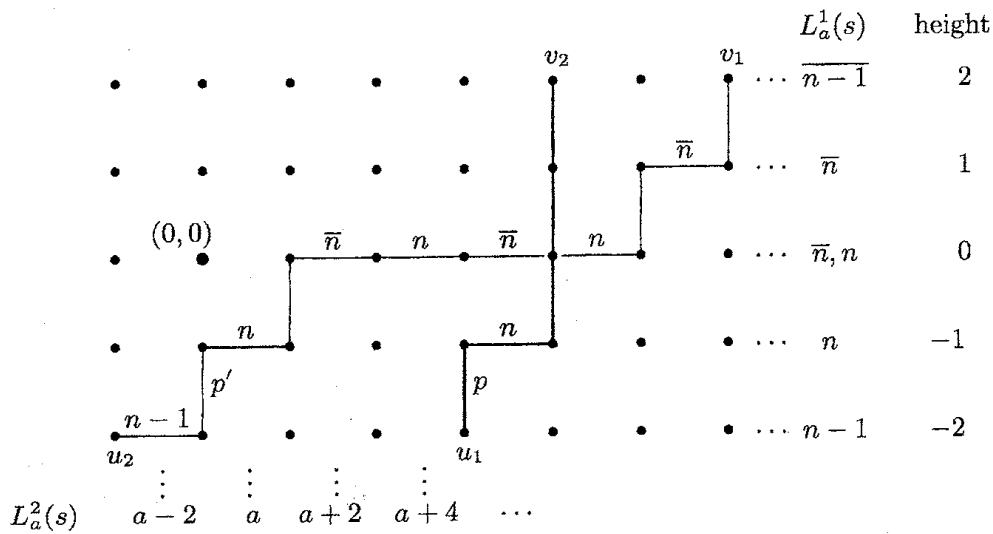


FIGURE 3. An example of h -paths of type C_n and their h -labelings.

By (3.2), we have

$$(5.1) \quad h_{r,a+2r-2}(z) = \sum_{(0,-n) \xrightarrow{P} (r,n)} z_a^P,$$

where z_a^P is defined as in (4.1) by the h -labeling of type C_n .

For any skew diagram λ/μ , let $l = d(\lambda)$, and let $\mathbf{u}_\mu = (u_1, \dots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \dots, v_l)$ be l -tuples of initial and final points defined as $u_i = (\mu_i + 1 - i, -n)$ and $v_i = (\lambda_i + 1 - i, n)$. As in the case of type A_n , we set $\mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ be the set of l -tuples of paths $\mathbf{p} = (p_1, \dots, p_l)$ such that $u_i \xrightarrow{p_i} v_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_l$. and define $z_a^{\mathbf{p}}$ and $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ and $a \in \mathbb{C}$ by the h -labeling of type C_n . Then, the determinant (3.8) can be written as

$$\chi_{\lambda/\mu, a} = \sum_{\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

by (5.1).

The first difference from the A_n case is that, the involution ι is not defined on all $\mathbf{p} = (p_1, \dots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which possess an intersecting pair of paths (p_i, p_j) , because of the definition of the h -paths of type C_n (Definition 5.1). To define the involution for the C_n case, we give the definition of the specially (resp. ordinarily) intersecting pair of paths (see [15]). Applying the method of [10], all $\mathbf{p} = (p_1, \dots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ with an ordinarily intersecting pair (p_i, p_j) are canceled, and we have

Proposition 5.2. *For any skew diagrams λ/μ ,*

$$(5.2) \quad \chi_{\lambda/\mu, a} = \sum_{\mathbf{p} \in P(C_n; \mu, \lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

where $P(C_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \dots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which does not have any ordinarily intersecting pair of paths (p_i, p_j) .

The second difference from A_n is that, the sum (5.2) is alternating. However, we conjecture that the sum (5.2) turns out to be a positive sum for any skew diagram λ/μ . To obtain a positive sum, we use the relations

$$(5.3) \quad z_{i,a} z_{\bar{i}, a-2n+2i-4} = z_{i-1, a} z_{\bar{i}-1, a-2n+2i-4}, \quad i = 1, \dots, n; \quad a \in \mathbb{C}.$$

The third difference is that there exist some $\mathbf{p} \in P(C_n; \mu, \lambda)$ that have a “transposed” pair of paths (p_i, p_j) . (Roughly speaking, if the initial points $\mathbf{u} = (u_1, u_2)$ and final points $\mathbf{v} = (v_1, v_2)$ are in order and $u_1 \xrightarrow{p_1} v_2, u_2 \xrightarrow{p_2} v_1$, then (p_1, p_2) is called a transposed pair of paths. For example, (p, p') in Figure 3 is transposed.) For such $\mathbf{p}$, we cannot define a tableau of shape λ/μ as same as $T(\mathbf{p})$ in (4.5), and therefore, the map T on $P(C_n; \mu, \lambda)$ to a certain set of tableaux of shape λ/μ cannot be defined as same as (4.5) of A_n .

5.2. Tableaux description. To formulate the tableaux description, we define a tableau $T(\mathbf{p})$ for any $\mathbf{p} = (p_1, \dots, p_l) \in P(C_n; \mu, \lambda)$ such that any pair of paths (p_i, p_j) is neither ordinarily intersecting nor transposed. Then we have a weight-preserving bijection

$$(5.4) \quad T : \tilde{P}(C_n; \mu, \lambda) \rightarrow \widetilde{\text{Tab}}(C_n, \lambda/\mu) := \{T(\mathbf{p}) | \mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)\}$$

as in (4.5), where $\tilde{P}(C_n; \mu, \lambda)$ is the set of all $\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any transposed pair of paths. The set $\widetilde{\text{Tab}}(C_n, \lambda/\mu)$ can also be described by the set of all tableaux T of shape λ/μ which satisfy the following “horizontal” and “vertical” rules as in the A_n case, which reduce to the rules (H) and (V) for C_n in (3.7) and (3.6), when λ/μ is a one-row diagram for (H) and one-column diagram for (V).

(H) Each $(i, j) \in \lambda/\mu$ satisfies both of the following conditions:

- $T(i, j) \preceq T(i, j+1)$ or $(T(i, j), T(i, j+1)) = (\bar{n}, n)$.
- $(T(i, j-1), T(i, j), T(i, j+1)) \neq (\bar{n}, \bar{n}, n), (\bar{n}, n, n)$.

(V) Each $(i, j) \in \lambda/\mu$ satisfies at least one of the following conditions:

- $T(i, j) \prec T(i+1, j)$.
- $T(i, j) = T(i+1, j) = n$, $(i+1, j-1) \in \lambda/\mu$ and $T(i+1, j-1) = \bar{n}$.
- $T(i, j) = T(i+1, j) = \bar{n}$, $(i, j+1) \in \lambda/\mu$ and $T(i, j+1) = n$.

We expect that the alternating sum (5.2) can be translated into a positive sum by a certain set of tableaux $\text{Tab}(C_n, \lambda/\mu) \subset \widetilde{\text{Tab}}(C_n, \lambda/\mu)$ as

$$(5.5) \quad \chi_{\lambda/\mu, a} = \sum_{T \in \text{Tab}(C_n, \lambda/\mu)} z_a^T.$$

Thus, the tableaux in $\text{Tab}(C_n, \lambda/\mu)$ are described by a horizontal rule and the vertical rule with an “extra” rule which selects them out of $\widetilde{\text{Tab}}(C_n, \lambda/\mu)$.

The idea in [15] to obtain from (5.2) is as follows: We introduce some weight-preserving, sign-inverting injections which “resolves” a transposed pair of paths, which is well-defined if $d(\lambda) \leq n$. The relation in (5.3) plays a crucial role for the weight-preserving property of these injections. Then, we can show that the contributions for (5.2) from all $\mathbf{p}$ with a transposed pair of paths almost cancel with each other.

In the following subsection, we give the tableaux description for λ/μ of $d(\lambda) = 2$. In this case, the set $\tilde{P}(C_n; \mu, \lambda)$ is exactly the set of all $\mathbf{p} = (p_1, p_2) \in P(C_n; \mu, \lambda)$ such that (p_1, p_2) is not transposed. Moreover, we do not have to consider any cancellations between all the transposed pairs $\mathbf{p} = (p_1, p_2)$. The contributions for all the transposed pairs $\mathbf{p} = (p_1, p_2)$ are all negative, which turn into the extra rule.

5.3. Skew diagrams of two rows. Let $\text{Tab}(C_n, \lambda/\mu)$ be the set of all the HV -tableaux T with the following extra condition:

(E-2R) If T contains a subtableau (excluding a and b)

$$(5.6) \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline n & n & \cdots & n \\ \hline \bar{n} & \bar{n} & \cdots & \bar{n} \\ \hline \end{array}}^k \begin{array}{c} a \\ b \end{array} \end{array}$$

where k is an odd number, then at least one of the following conditions holds:

- (1) Let (i_1, j_1) be the position of the top-right corner of the subtableau (5.6). Then $(i_1, j_1 + 1) \in \lambda/\mu$ and $a := T(i_1, j_1 + 1) = n$.
- (2) Let (i_2, j_2) be the position of the bottom-left corner of the subtableau (5.6). Then $(i_2, j_2 - 1) \in \lambda/\mu$ and $b := T(i_2, j_2 - 1) = \bar{n}$.

Then

Theorem 5.3. For any skew diagrams λ/μ with $d(\lambda) = 2$ and $n \geq 2$, the equality (5.5) holds.

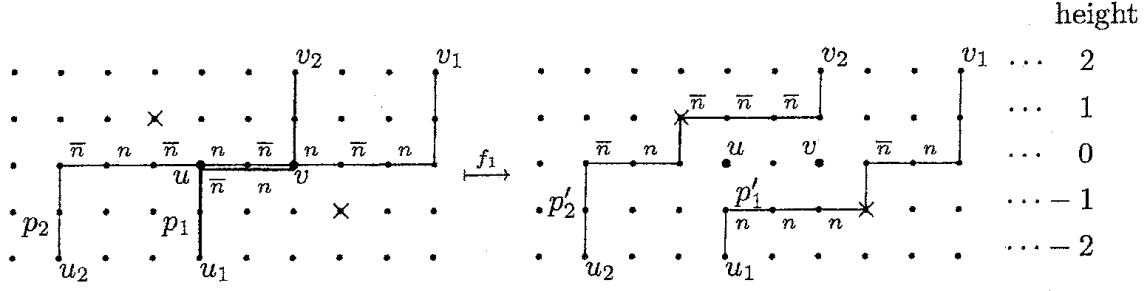


FIGURE 4. An example of $\mathbf{p} = (p_1, p_2) \in P_1(C_n; \mu, \lambda)$ and the map f_1

Proof. Let $P_1(C_n; \mu, \lambda)$ (resp. $P_0(C_n; \mu, \lambda)$) be the set of all $\mathbf{p} = (p_1, p_2) \in P(C_n; \mu, \lambda)$ such that (p_1, p_2) is transposed (resp. (p_1, p_2) is not transposed). In this case, we have $P_0(C_n; \mu, \lambda) = \tilde{P}(C_n; \mu, \lambda)$.

Define a weight-preserving, sign-inverting injection

$$f_1 : P_1(C_n; \mu, \lambda) \rightarrow P_0(C_n; \mu, \lambda)$$

as follows (see Figure 4 for example): Let u (resp. v) be the leftmost (resp. rightmost) intersecting point of p_1 and p_2 at height 0. We assume that $u - (0, 1)$ and $v + (0, 1)$ is on p_1 while $u - (1, 0)$ and $v + (1, 0)$ is on p_2 . Set $f_1(p_1, p_2) = (p'_1, p'_2)$ by

$$\begin{aligned} p'_1 : u_1 &\xrightarrow{p_1} u + (0, -1) \longrightarrow v + (1, -1) \longrightarrow v + (1, 0) \xrightarrow{p_2} v_1, \\ p'_2 : u_2 &\xrightarrow{p_2} u + (-1, 0) \longrightarrow u + (-1, 1) \longrightarrow v + (0, 1) \xrightarrow{p_1} v_2. \end{aligned}$$

Roughly speaking, f_1 resolves the transposed pair (p_1, p_2) . From (5.2) and (5.4), we have

$$\chi_{\lambda/\mu, a} = \sum_{T \in \widetilde{\text{Tab}}(C_n, \lambda/\mu)} z_a^T - \sum_{\mathbf{p} \in \text{Im } f_1} z_a^{T(\mathbf{p})}.$$

The set $\{T(\mathbf{p}) \mid \mathbf{p} \in \text{Im } f_1\}$ consists of all the tableaux in $\widetilde{\text{Tab}}(C_n, \lambda/\mu)$ prohibited by the extra rule (E-2R). $\square$

5.4. Conjecture on the implicit form of the extra rule. Let $P_1(C_n; \mu, \lambda)$ be the set of all $\mathbf{p} = (p_1, \dots, p_l) \in P(C_n; \mu, \lambda)$ ($l = d(\lambda)$) such that one pair of paths (p_i, p_j) is transposed. Let $P_1^{i, i+1}(C_n; \mu, \lambda)$ be the set of $\mathbf{p} \in P_1(C_n; \mu, \lambda)$ such that the pair (p_i, p_{i+1}) is transposed. Then we have $P_1(C_n; \mu, \lambda) = \sum_{i=1}^{l-1} P_1^{i, i+1}(C_n; \mu, \lambda)$. We conjecture that one can always resolve the transposed pair (p_i, p_{i+1}) (without producing any ordinarily intersections) and define a weight-preserving, sign-inverting injection $f_1^{i, i+1} : P_1^{i, i+1}(C_n; \mu, \lambda) \rightarrow P_0(C_n; \mu, \lambda)$. Furthermore, we conjecture that

Conjecture 5.4. For any skew diagram λ/μ of $d(\lambda/\mu) \leq n$,

$$\text{Tab}(C_n, \lambda/\mu) = \widetilde{\text{Tab}}(C_n, \lambda/\mu) \setminus \{T(\mathbf{p}) \mid \mathbf{p} \in \bigcup_{i=1}^{d(\lambda)-1} \text{Im } f_1^{i, i+1}\}.$$

In other words, for a tableau $T \in \widetilde{\text{Tab}}(C_n, \lambda/\mu)$, the extra rule (E), which gives the condition for T to be in $\text{Tab}(C_n, \lambda/\mu)$, is implicitly stated in terms of paths as follows:

(E) If $\mathbf{p} \in P(C_n; \mu, \lambda)$ corresponds to T , then $\mathbf{p}$ is not obtained from some $\mathbf{p}' \in P_1(C_n; \mu, \lambda)$ as the resolution of the transposed pair.

REFERENCES

- [1] T. Akasaka, M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 5, 839–867.
- [2] T. Arakawa, *private communication*.
- [3] T. Arakawa, *Drinfeld functor and finite-dimensional representations of Yangian*, Comm. Math. Phys. **205** (1999), no. 1, 1–18.
- [4] T. Arakawa, T. Suzuki, *Duality between $\mathfrak{sl}_n(C)$ and the degenerate affine Hecke algebra*, J. Algebra **209** (1998), no. 1, 288–304.
- [5] V. V. Bazhanov, N. Reshetikhin, *Restricted solid-on-solid models connected with simply laced algebras and conformal field theory*, J. Phys. A **23** (1990), no. 9, 1477–1492.
CK V. Chari, M. Kleber, *Symmetric functions and representations of quantum affine algebras*, Contemp. Math. **297** (2002), 27–45.
- [6] V. Chari, A. Pressley, *Quantum affine algebras and their representations*, Canadian Math. Soc. Conf. Proc. **16** (1995), 59–78.
- [7] V. Chari, A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1995.
- [8] E. Frenkel, N. Reshetikhin, *Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$ -algebras*, Comm. Math. Phys. **178** (1996) 237–264.
- [9] E. Frenkel, N. Reshetikhin, *The q -characters of representations of quantum affine algebras and deformations of $\mathcal{W}$ -algebras*, Contemp. Math. **248** (1999), 163–205.
- [10] I. Gessel, G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. in Math. **58** (1985), no. 3, 300–321.
KNS
- [11] A. Kuniba, Y. Ohta, J. Suzuki, *Quantum Jacobi-Trudi and Giambelli formulae for $U_q(B_r^{(1)})$ from the analytic Bethe ansatz*, J. Phys. A **28** (1995), no. 21, 6211–6226.
- [12] A. Kuniba, M. Okado, J. Suzuki, Y. Yamada, *Difference L operators related to q -characters*, J. Phys. A **35** (2002), no. 6, 1415–1435.
- [13] A. Kuniba, J. Suzuki, *Analytic Bethe ansatz for fundamental representations of Yangians*, Comm. Math. Phys. **173** (1995), no. 2, 225–264.
LR
- [14] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, Oxford, 1995.
- [15] W. Nakai, T. Nakanishi, *Paths, tableaux, and q -characters of quantum affine algebras: the C_n case*, in preparation.
- [16] B. E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Graduate Texts in Mathematics **203**, Springer-Verlag, New York, 2001.