

Some New Properties of Lagrange Function

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1 Introduction and Preliminaries

Let us consider the following problem

$$\begin{aligned} \text{(MP)} \quad & \text{Minimize} \quad f(x) := (f_1(x), \dots, f_m(x)) \\ & \text{subject to} \quad g_t(x) \leq 0, \quad t \in T, \\ & \quad \quad \quad x \in C, \end{aligned}$$

where the functions $f_i : X \rightarrow \mathbb{R}, i \in M := \{1, \dots, m\}$ and $g_t : X \rightarrow \mathbb{R}, t \in T$ are locally Lipschitz on a Banach space X , T is an arbitrary (possibly infinite) index set, and C is a closed convex subset of X . We denote the feasible set by $F =: \{x \in C \mid g_t(x) \leq 0, t \in T\}$.

In this paper, due to Chankong-Haimes method, for $j \in M$ and $z \in C$, we associate to (MP) the following scalar problem,

$$\begin{aligned} \text{(P}_j(z)\text{)} \quad & \text{Minimize} \quad f_j(x) \\ & \text{subject to} \quad f_k(x) \leq f_k(z), k \in M^j := M \setminus \{j\}, \\ & \quad \quad \quad g_t(x) \leq 0, t \in T, \\ & \quad \quad \quad x \in C. \end{aligned}$$

Relationships between (MP) and (P_j(z)) are established and optimality conditions of the problems are given.

Let us denote by $\mathbb{R}^{(T)}$ a following linear space,

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For each $\lambda \in \mathbb{R}^{(T)}$, the supporting set corresponding to λ is $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$. It is a finite subset of T . We denote $\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}$. It is a nonnegative cone of $\mathbb{R}^{(T)}$. For $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t \in T} \subset Z$, Z being a real linear space, we understand that

$$\sum_{t \in T} \lambda_t z_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t z_t & \text{if } T(\lambda) \neq \emptyset, \\ 0 & \text{if } T(\lambda) = \emptyset. \end{cases}$$

Throughout this paper X is a Banach space, C is a nonempty closed convex subset of X , T is a compact topological space, $f : X \rightarrow \mathbb{R}$ is a locally Lipschitz function, and $g_t : X \rightarrow \mathbb{R}, t \in T$, are locally Lipschitz with respect to x uniformly in t , i.e.,

$$\forall x \in X, \exists U(x), \exists K > 0, |g_t(u) - g_t(v)| \leq K \|u - v\|, \forall u, v \in U(x), \forall t \in T.$$

The following concepts can be found in the Clarke's books [1, 2]. Let D be a nonempty closed convex subset of X . The normal cone to D at a point $z \in D$ coincides with the normal cone in the sense of convex analysis and given by $N_D(\cdot, z) := \{v \in X^* \mid v(x - z) \leq 0, \forall x \in D\}$.

Let $g : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$, is $g'(z; d) = \lim_{t \rightarrow 0^+} \frac{g(z+td) - g(z)}{t}$ if the limit exists. The Clarke generalized directional derivative of g at $z \in X$ in direction $d \in X$ is $g^c(z; d) := \limsup_{\substack{y \rightarrow z \\ t \rightarrow 0^+}} \frac{g(y+td) - g(y)}{t}$. The Clarke sub-

differential of g at $z \in X$, denoted by $\partial^c g(z)$, is defined by $\partial^c g(z) := \{v \in X^* \mid v(d) \leq g^c(z; d), \forall d \in X\}$.

A locally Lipschitz function g is said to be regular (in the sense of Clarke) at $z \in X$ if $g'(z; d)$ exists and

$$g^c(z; d) = g'(z; d), \forall d \in X.$$

Definition 1.1 Let C be a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function.

(i) The function f is said to be pseudoconvex at $x \in C$ if $f(y) < f(x) \Rightarrow u(y - x) < 0, u \in \partial^c f(x), y \in C$, equivalently, $u(y - x) \geq 0 \Rightarrow f(y) \geq f(x), u \in \partial^c f(x), y \in C$. The function f is said to be pseudoconvex on C if it is pseudoconvex at every $x \in C$.

(ii) The function f is said to be quasiconvex at $x \in C$ if $f(y) \leq f(x) \Rightarrow u(y - x) \leq 0, u \in \partial^c f(x), y \in C$, equivalently, $u(y - x) > 0 \Rightarrow f(y) > f(x), u \in \partial^c f(x), y \in C$. The function f is said to be quasiconvex on C if it is quasiconvex at every $x \in C$.

We need the following lemmas.

Lemma 1.1 Let $f : X \rightarrow \mathbb{R}$ be a locally Lipschitz pseudoconvex function. If there exists $u \in \partial^c f(x)$ such that $u(y - x) \geq 0, y, x \in X$, then $f(y) \geq f(x)$.

Lemma 1.2 Let $f : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. If f is pseudoconvex then f is quasiconvex.

Now we give the definition of efficient solution of (MP).

Definition 1.2 A point $z \in F$ is said to be an efficient solution of (MP) if there exists no other $x \in F_M$ such that

$$f_i(x) \leq f_i(z), \text{ for all } i \in M$$

and

$$f_{i_0}(x) < f_{i_0}(z), \text{ for some } i_0 \in M.$$

The criteria of Chankong-Haimes method applied for (MP) is as follows.

Lemma 1.3 A feasible point z of (MP) is an efficient solution if and only if it is a solution of $(P_j(z))$ for each $j \in M$.

2 Optimality Conditions

First of all, let us consider the following scalar optimization problem in order to recall some concepts of solution for a single objective optimization problem.

$$\begin{aligned} \text{(P)} \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad g_t(x) \leq 0, t \in T, \\ & \quad \quad \quad x \in C \end{aligned}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz function and functions $g_t, t \in T$ and C are as above. Also, the feasible set of (P) is denoted by $F_P := \{x \in C | g_t(x) \leq 0, t \in T\}$.

Let $x \in \mathbb{R}^n$. We need the following condition,

$$(\mathcal{A}) : \exists d \in T_C(x) : g_t^\circ(x; d) < 0, \text{ for all } t \in I(x) := \{t \in T | g_t(x) = 0\}.$$

According to Theorems 4.1 and 4.2 presented in [10] (where the problem (P) is defined on a Banach space), we derive the following theorems for the case of the involved functions are defined on $\mathbb{R}^n$ and the index set T is compact. The proofs can be omitted.

Theorem 2.1 *Let z be an optimal solution for (P). Assume that the condition $(\mathcal{A})$ holds for z . Then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial f(z) + \sum_{t \in T} \lambda_t \partial g_t(z) + N_C(z), \quad g_t(z) = 0 \text{ for all } t \in T(\lambda). \quad (2.1)$$

We now establish optimality conditions for $(P_j(z))$ and (MP). The following condition is associated to the problem $(P_j(z))$, and the feasible set of $(P_j(z))$ is denoted by $F_j(z)$.

Let $x \in \mathbb{R}^n$, $I(x) = \{t \in T \mid g_t(x) = 0\}$, $H_j(x) = \{k \in M^j \mid f_k(x) = f_k(z)\}$, and $\bar{T}(x) = I(x) \cup H_j(x)$.

$$(\mathcal{A}_j) : \exists d \in T_C(x) : \begin{cases} g_t^\circ(x; d) < 0, \text{ for all } t \in I(x), \\ f_k^\circ(x; d) < 0, \text{ for all } k \in H_j(x). \end{cases}$$

Lemma 2.1 *Let z be an optimal solution of $P_j(z)$, assume that the condition $(\mathcal{A}_j)$ holds for z , then there exist $\mu_k \geq 0, k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z), g_t(z) = 0, \forall t \in T(\lambda). \quad (2.2)$$

Lemma 2.2 *Let $z \in F_j(z)$. Assume that the function f_j is pseudoconvex, the functions $f_k, k \in M^j$ and $g_t, t \in T$ are quasiconvex. If there exist $\mu_k \geq 0, k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that (2.2) holds. Then z is an optimal solution for $(P_j(z))$.*

Theorem 2.2 (Necessary Condition) *If z is an efficient solution, then there exist $\tau > 0$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that the following condition holds*

$$0 \in \sum_{i \in M} \tau_i \partial^c f_i(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N(C, z), \quad \lambda_t g_t(z) = 0, \quad \forall t \in T. \quad (2.3)$$

Theorem 2.3 (Sufficient Condition) *Let $z \in F$, assume that $\tau^T f$ is pseudoconvex at z and $\lambda_t g_t, t \in T$ are quasiconvex. If there exist $\tau > 0$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that (2.3) holds, then z is an efficient solution of (MP).*

3 Mixed Duality

The dual problem of (MP) in a mixed type of Wolfe type and in Mond-Weir type is formulated by

$$\begin{aligned}
(\text{MD}) \quad & \text{Maximize} \quad f(y) + \sum_{t \in T} \lambda_t g_t(y) e \\
& \text{subject to} \quad 0 \in \sum_{i \in M} \tau_i \partial^c f_i(y) + \sum_{t \in T} (\lambda_t + \mu_t) \partial^c g_t(y) + N(C, y), \\
& \quad \mu_t g_t(y) \geq 0, t \in T, \\
& \quad \tau^T e = 1, \tau > 0, \tau \in \mathbb{R}^m, e = (1, \dots, 1) \in \mathbb{R}^m, \\
& \quad (y, \tau, \lambda, \mu) \in C \times \mathbb{R}^m \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)}.
\end{aligned}$$

Let us denote by G the feasible set of (MD). The optimal values of the problems (MP) and (MD) are denoted by $V(\text{MP})$ and $V(\text{MD})$, respectively.

Theorem 3.1 (Weak Duality) *Let x and (y, τ, λ, μ) be the feasible solution of (MP) and (MD), respectively. Assume that $(\tau^T f + \sum_{t \in T} (\lambda_t + \mu_t) g_t)$ is pseudoconvex, $f_i, i \in M$ and $g_t, t \in T$ are regular on C . Then the following cannot hold:*

$$f(x) < f(y) + \sum_{t \in T} \lambda_t g_t(y) e.$$

Theorem 3.2 (Strong Duality) *Suppose that y is an efficient solution for (MP) and weak duality theorem (Theorem 3.1) holds, then there exist $\lambda, \mu \in \mathbb{R}_+^{(T)}$ and $\tau_i, i \in M$ such that (y, τ, λ, μ) is an efficient solution for (MD).*

4 Properties of Lagrange Function

The Lagrange function associated to (MP) is formulated by

$$L(x, \lambda) = \begin{cases} f(x) + \sum_{t \in T} \lambda_t g_t(x) e, & (x, \lambda) \in C \times \mathbb{R}_+^{(T)} \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $\lambda \in \mathbb{R}_+^{(T)}$, the function $L(\cdot, \lambda)$ is locally Lipschitz on X .

From now on, we suppose that the function $L(\cdot, \lambda)$ is pseudoconvex on X for every $\lambda \in \mathbb{R}_+^{(T)}$, and $f, g_t, t \in T$, are regular on X .

Theorem 4.1 *If z is an efficient solution of (MP) and there exists $j \in M$ such that the condition $(\mathcal{A}_j)$ holds for z , then $(z, \bar{\tau}, \bar{\lambda}, 0)$ and $(z, \bar{\tau}, 0, \bar{\lambda})$ are solutions of (MD).*

Theorem 4.2 *Suppose that $(y^*, \tau^*, \lambda^*, \mu^*)$ is a weakly efficient solution of (MD).*

i) It holds $L(y, \lambda^ + \mu^*) = V(\text{MD})$ for all $y \in G_1 := \{y \in C \mid (y, \tau^*, \lambda^*, \mu^*) \in G\}$ and $\mu_t^* g_t(y^*) = 0$ for all $t \in T$.*

ii) Furthermore, if $V(\text{MD}) = V(\text{MP})$ then $L(y, \lambda^ + \mu^*) = V(\text{MD})$ for all $y \in \text{Sol}(\text{MP})$ and $(\lambda_t^* + \mu_t^*) g_t(y) = 0$ for all $t \in T$.*

References

- [1] F.H. Clarke, Optimization and Nonsmooth Analysis, Willey-Interscience, New York (1983).
- [2] F.H. Clarke, Ledyaev Yu.S., Stern J.S., and Wolenski P.R., Nonsmooth Analysis and Control Theory, Springer-Verlag, Berlin (1998).
- [3] N. Dinh, V. Jeyakumar and G.M. Lee, Lagrange multiplier characterizations of solution sets of constrained pseudolinear optimization problems, Optim. 55 (2006) 241-250.
- [4] N. Dinh, M.A. Goberna, M.A. Lopez and T.Q. Son, New Farkas-type constraint qualifications in convex infinite programming, ESAIM Control Optim. Calc. Var. 13 (2007) 580-597.

- [5] N. Dinh, B.S. Mordukhovich and T.T.A. Nghia, Subdifferentials of value functions and optimality conditions for some classes of DC and bilevel infinite and semi-infinite programs, *Math. Program.* 123 (2009) 101-138.
- [6] V. Jeyakumar, G.M. Lee, and N. Dinh, Lagrange Multiplier Conditions Characterizing the Optimal Solution Sets of Cone-Constrained Convex Programs, *J. Optim. Theory Appl.* 123 (2004) 83-103.
- [7] V. Jeyakumar, G.M. Lee and N. Dinh, Characterizations of solution sets of convex vector minimization problems, *European J. Oper. Res.* 174 (2006) 1380-1395.
- [8] V. Chankong, Y.Y. Haimes (1983), *Multiobjective Decision Making: Theory and Methodology*, Amsterdam: North-Holland.
- [9] D.S. Kim and T.Q. Son, Characterizations of Solution Sets of a class of Nonconvex Semi-Infinite Programming Problems, *J. Nonlinear Convex Anal.* 12 (2011) 429-440.
- [10] T.Q. Son, J.J. Strodiot, V.H. Nguyen, ε -Optimality and ε -Lagrangian duality for a nonconvex programming problem with an infinite number of constraints, *J. Optim. Theory Appl.* 141 (2009) 389-409.
- [11] T.Q. Son, D.S. Kim and N.N. Tam, Weak stability and strong duality of a class of nonconvex infinite programs via augmented Lagrangian, *J. Global Optim.*, 53 (2012) 165-184.