

Minuscule Heaps Over Simply-Laced, Star-shaped Dynkin Diagrams

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1 Introduction

The aim of this paper is to classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams.

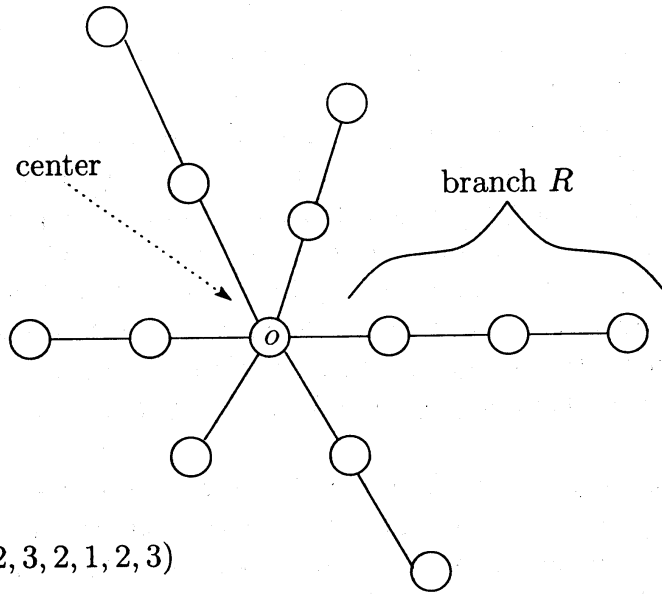
A simply-laced, star-shaped Dynkin diagram Γ is a simple graph (without loops or multiple edges) like the one in Figure 1. It has a node o , and several branches $R^1, R^2, \dots, R^l$ emanating from o . We call o the **center** of Γ , and the number of nodes on R^i (not including o) the **length** of the branch R^i . If $l \geq 3$, then o is uniquely determined by Γ . We mainly deal with such cases. If the length of R^i is l_i , then we say that Γ is of type $S(l_1, l_2, \dots, l_r)$.

Γ is an example of a Dynkin diagram, namely an encoding of a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$, associated to which is a Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ (see [1]). The set I indexing the rows and columns of A is the node set of Γ , which we denote by $N(\Gamma)$. $\mathfrak{g}(A)$ is a generalization of a finite dimensional semi-simple Lie algebra, say over $\mathbb{C}$, and defined by a certain presentation determined by A . All simple finite-dimensional cases (types $A_n (n \geq 1)$, $D_n (n \geq 4)$ and $E_n (n = 6, 7, 8)$) are included in our class.

Minuscule heaps arose in connection with the λ -minuscule elements of the Weyl group W of $\mathfrak{g}$. According to R. Proctor [6] and J. Stembridge [9] the notion of λ -minuscule elements of W was defined by D. Peterson in his unpublished work in the 1980's. Let λ be an integral weight for $\mathfrak{g}$. An element w of W is called λ -**minuscule** if it has a reduced decomposition $s_{i_1} s_{i_2} \dots s_{i_p}$ such that

$$s_{i_k}(s_{i_{k+1}} \dots s_{i_p} \lambda) = s_{i_{k+1}} \dots s_{i_p} \lambda - \alpha_{i_k} \text{ for all } 1 \leq k \leq p,$$

and is called minuscule if w is λ -minuscule for some integral weight λ . Here α_{i_k} is the simple root corresponding to s_{i_k} . It is known that a minuscule element is fully commutative, namely any reduced decomposition can be converted into any other by exchanging adjacent commuting generators several times (see [6, §15], [7, Theorem A] and [8, Theorem 2.2], or [9, Proposition



Type $S(2, 3, 2, 1, 2, 3)$

Figure 1: A star-shaped Dynkin diagram

2.1]). To a fully commutative element w , one can associate a Γ -labeled poset called its **heap**. A Γ -labeled poset is a triple $(P, \leq, \phi)$ in which $(P, \leq)$ is a poset and $\phi : P \rightarrow N(\Gamma)$ is any map (called the **labeling map**). A linear extension of a Γ -labeled poset naturally gives a word in the generators of W . The heap of a fully commutative element w is a Γ -labeled poset whose linear extensions give all reduced decompositions of w . A **minuscule heap** is the heap of a minuscule element of W . Stembridge obtained the following structural conditions for a finite Γ -labeled poset to be a minuscule heap ([8, Proposition 3.1]).

- (H1) If $p \rightarrow q$ in P , then $\phi(p)$ and $\phi(q)$ are either equal or adjacent in Γ . Moreover, if $p, q \in P$ are incomparable, then $\phi(p)$ and $\phi(q)$ are not equal, and not adjacent in Γ .
- (H2) If $p, q \in P, p < q, \phi(p) = \phi(q) = v$ and no element in $[p, q]$ except p, q are labeled v , then exactly two elements in $[p, q]$ have labels adjacent to v . (This is a simplified version accommodated to the simply-laced cases only.)

The interval appearing in (H2) is important in minuscule heaps, and will be called a v -interval. We start from this characterization, namely we define a **minuscule heap** over Γ to be a **finite** Γ -labeled poset $(P, \leq, \phi)$ satisfying (H1) and (H2). The isomorphism classes of minuscule heaps over Γ

corresponds bijectively with the minuscule element of W , where an isomorphism is defined to be a poset isomorphism commuting with the labeling maps. R. Proctor showed that, if Γ is simply-laced and λ is dominant, then the minuscule heap constructed from a λ -minuscule element is a d -complete poset, a notion defined by himself. d -complete posets enjoy nice properties such as the hook length formula and jeu de taquin, and are expected to be a nice class of posets that generalize Young diagrams. He introduced the operation of slant sum, and enumerated all 15 types of “slant-irreducible” d -complete posets, namely the ones irreducible with respect to the slant sum decomposition. Then J. Stembridge classified the slant-irreducible minuscule heaps over multiply-laced Dynkin diagrams Γ , where λ was still assumed to be dominant.

In this paper, we assume that Γ is simply-laced and star-shaped, but remove the assumption that λ is dominant. As an intermediary for classifying these minuscule heaps over such Γ , we introduce the notion of D -matrices (see §4). They represent the structure of the intervals $[b_o, t_o]$ of minuscule heaps, where b_o and t_o respectively are the smallest and largest elements labeled by o , the “central node” of Γ , respectively. We characterize the D -matrices for any fixed such Γ , and then give a complete description of the set of all minuscule heaps which share the structure of $[b_o, t_o]$ represented by each D -matrix. To describe these minuscule heaps, we introduce the notion of slant lattice over Γ (see §4). It plays the role of a “universal holder” to embed all minuscule heaps over Γ , and provides a “standard coordinate system” to compare them up to isomorphism. Our main results are Theorems 4.8 and 5.6.

The paper is organized as follows. §4, 5 form the main part of this paper, where we classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams. To reach there, we collect some basic facts in §2, and introduce the notion of the slant lattice in §3.

2 Preliminaries

First note that all poset appearing in this paper, including infinite ones, satisfy the following condition:

(*) If $p, q \in P$ and $p \leq q$, then there exists a finite sequence of elements of P , say $p_0, p_1, \dots, p_l$, such that $p_0 = p, p_l = q$ and p_i covers p_{i-1} for $1 \leq i \leq l$.

We call such a sequence $p_0, p_1, \dots, p_l$ a **saturated chain from p to q** .

Let $(P, \leq, \phi)$ be a Γ -labeled poset. For each $v \in N(\Gamma)$, we denote by P_v the set of all elements in P labeled v . For $\Gamma' \subset \Gamma$, we denote $\cup_{v \in N(\Gamma')} P_v$ by $P_{\Gamma'}$. It is each to see the following.

Proposition 2.1. *Let Γ be any Dynkin diagram. Let $(P, \leq, \phi)$ be a Γ -labeled poset satisfying (H1), and v a node of Γ . Then P_v is totally ordered.*

Now let $(P, \leq, \phi)$ be a minuscule heap over Γ . By the **support** of P we mean the image of ϕ , which is denoted by $\text{supp } P$. Minuscule heaps with acyclic support has additional nice properties. Following [8], we denote this condition by (H4), namely

(H4) $\text{supp } P$ is acyclic. ((H3) is used in [8] for another condition for dominant minuscule heaps.)

Note that (H4) is always satisfied if Γ is star-shaped (see §4), since such Γ are acyclic.

Proposition 2.2. *Let $(P, \leq, \phi)$ be a minuscule heap over Γ .*

(1) *If C is a convex subset of P , then $(C, \leq|_C, \phi|_C)$ is a minuscule heap over Γ , where $\leq|_C$ and $\phi|_C$ are the restrictions of the ordering $\leq$ and over Γ . In particular, all order ideals, order filters, intervals, open intervals, and connected components of P are minuscule heaps over Γ .*

(2) *The dual poset of P is a minuscule heap over Γ . Namely, $(P, \leq^*, \phi)$ is a minuscule heap over Γ .*

It is also easy to see the following.

Proposition 2.3. *Let $(P, \leq, \phi)$ be a Γ -labeled poset satisfying (H1). Then P is connected if and only if $\text{supp } P$ is connected.*

We say that two subdiagrams Γ_1 and Γ_2 of Γ are **strongly disjoint** if their node sets are disjoint and if no node of Γ_1 is adjacent to any node of Γ_2 in Γ .

Remark 2.4. Let $P_1, P_2, \dots, P_c$ be the connected components of P . Proposition 2.3 implies that the subdiagrams Γ_i of Γ with node sets $\phi(P_i)$, $i = 1, 2, \dots, c$, are connected and pairwise strongly disjoint. Hence $\Gamma_1, \Gamma_2, \dots, \Gamma_c$ are the connected components of $\text{supp } P$. This establishes a one-to-one correspondence between the connected components of P and those of $\text{supp } P$. A Γ -labeled poset is a minuscule heap over Γ if and only if its connected components are minuscule heaps over Γ and their supports are pairwise strongly disjoint.

Our aim is to classify the minuscule heaps P over simply-laced, star-shaped Γ up to isomorphism of Γ -labeled posets. By Remark 2.4, it is sufficient to study each connected component. At most one of the connected components contains o in its support, and the rest have supports of type A .

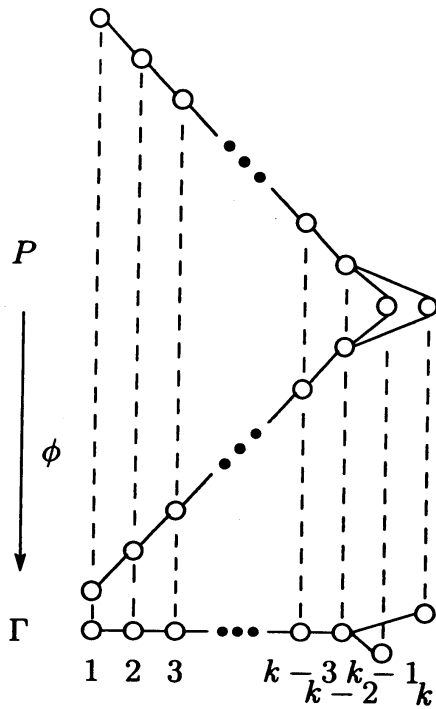


Figure 2: The double-tailed diamond $d_k(1)$

The type A minuscule heaps turn out to be all of the labeled posets described in [1, 1] (isomorphic to skew Young diagrams). So we concentrate on the case where $\text{supp } P$ is connected and $o \in \text{supp } P$.

Let Γ and Γ' be Dynkin diagrams. We say that a Γ -labeled poset $(P, \leq, \phi)$ and a Γ' -labeled poset $(P', \leq', \phi')$ are **abstractly isomorphic** (or isomorphic if no confusion would arise) if there is a poset isomorphism $\alpha : P \rightarrow P'$ and an isomorphism of subdiagrams $\beta : \text{supp } P \rightarrow \text{supp } P'$ such that β maps the label of p to the label of $\alpha(p)$ for every $p \in P$. For an integer $k \geq 3$, we denote by $d_k(1)$, as was done by Proctor [6], the labeled poset illustrated in Figure 2. An interval $[p, q]$ abstractly isomorphic to $d_k(1)$ will be called a **double-tailed diamond**, with the special case where $k = 3$ being called a **diamond**.

Let Γ be any Dynkin diagram. The following proposition is due to Stembridge.

Proposition 2.5. [8, Proposition 3.3] *Let $(P, \leq, \phi)$ be a minuscule heap satisfying (H4). Let v be a node in $N(\Gamma)$ and let $[p, q]$ be a v -interval. Then $[p, q]$ is a double-tailed diamond. In particular, if q covers two distinct elements, then $[p, q]$ is a diamond.*

Remark 2.6. In [8], Stembridge calls $d_k(1)$ a subinterval of type D_k .

The following proposition is also due to Stembridge.

Proposition 2.7. [8, Corollary 3.4] *If a minuscule heap $(P, \leq, \phi)$ satisfies (H4), then P is a ranked poset, i.e. there exists a function $f : P \rightarrow \mathbb{Z}$, called a rank function, such that $f(q) = f(p) + 1$ for any covering pair $p \rightarrow q$, which we mean $p < q$ and $(p, q) = \emptyset$.*

3 The slant lattice

In this section we define the notion of the slant lattice over an acyclic Dynkin diagram Γ , and show that every minuscule heap over Γ can be “cover-embedded”, which we define below, into this poset. For the moment, we do not assume that Γ is acyclic. We only assume (H4).

Now we define the slant lattice. From this point, we assume that Γ itself is connected and acyclic, so that any minuscule heap over Γ satisfies (H4).

For $(u, i), (v, j) \in N(\Gamma) \times \mathbb{Z}$, we write $(u, i) \rightarrow (v, j)$ if and only if $j = i + 1$ and v, u are adjacent nodes of Γ . We write $\leq$ for the reflective and transitive closure of $\rightarrow$.

Lemma 3.1. *Suppose that Γ is connected and acyclic, and let $\rightarrow, \leq$ be the relations on $N(\Gamma) \times \mathbb{Z}$ defined above.*

(1) $\leq$ is a partial ordering in $N(\Gamma) \times \mathbb{Z}$.

Let (u, i) and (v, j) be elements of $N(\Gamma) \times \mathbb{Z}$.

(2) *If Γ contains at least 2 nodes, then we have $(u, i) \leq (v, j)$ if and only if $i \leq j$, $d(u, v) \leq j - i$, and $d(u, v) \equiv j - i \pmod{2}$. Here $d(u, v)$ denotes the distance between u and v in Γ , namely the smallest $l \in \mathbb{Z}_{\geq 0}$ such that there exists a sequence $u = u_0, u_1, \dots, u_l = v$ of nodes, or ∞ if no such l exists.*

(3) (u, i) is covered by (v, j) in $N(\Gamma) \times \mathbb{Z}$ if and only if $(u, i) \rightarrow (v, j)$.

If S is a subset of a poset P , we consider two orderings in S induced from P . One is just the restriction of the ordering P . The subset S equipped with this ordering will be simply called a **subposet** of P (in the ordinary sense if it is ambiguous). The other ordering, generally weaker than the one above, is obtained by first taking the covering relation in P , restricting it to S , and then taking its reflexive-and-transitive closure. It is straightforward to check that this is in fact a partial order. In this ordering, two elements $p, p' \in S$ are in order if and only if there is a (finite) saturated chain $p = p_0, p_1, \dots, p_l = p'$ of P consisting solely of elements of S . It can be checked that $p \in S$ is

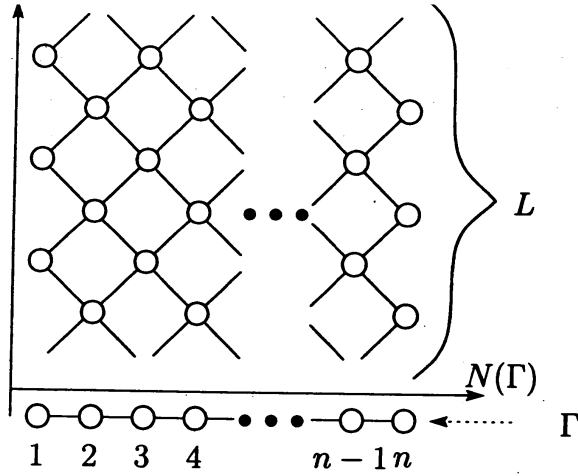


Figure 3: A slant lattice of type A

covered by $p' \in S$ in this ordering if and only if p is covered by p' in P . (This is not the case with the restriction of the ordering of P .) We call it the ordering **cover-induced** from P , and we call S together with this ordering a **cover-subposet** of P . Note that, for a general P , the ordering cover-induced on P itself may be strictly weaker than the original ordering, but our assumption (*) on P assures that this does not happen. Now suppose P and Q are posets. We say that a map $\phi : P \rightarrow Q$ is a **cover-embedding** if it gives a poset isomorphism of P with the cover-subposet $\phi(P)$ of Q , namely if p is covered by p' in P if and only if $\phi(p)$ is covered by $\phi(p')$ in Q .

For a minuscule heap P over Γ , there is a unique rank function f on P up to an additive constant for each connected component. Naturally f induces the following injection ν from P to $N(\Gamma) \times \mathbb{Z}$,

$$\nu : p \mapsto (\phi(p), f(p)).$$

We regard $N(\Gamma) \times \mathbb{Z}$ as a Γ -labeled poset by defining the label of each element (v, i) to be v .

Proposition 3.2. *Assume that Γ is acyclic. Let $(P, \leq, \phi)$ be a connected minuscule heap over Γ , let f be a rank function on P , and let ν be the map defined above. Then ν is a cover-embedding that commutes with the labeling maps.*

From now on, assume that Γ is connected. If we fix an element p of P , we can choose a rank function f such that $f(p) = 0$. We define a **slant lattice** L over Γ

$$\text{by } L = \{(q, u) \in N(\Gamma) \times \mathbb{Z} \mid f(q) - d(v, u) \equiv 0 \pmod{2}\}$$

(see Figure 3). If Γ contains at least two nodes, then L coincides with the connected component of the poset $N(\Gamma) \times \mathbb{Z}$ containing $(\phi(p), 0)$. Our definition of L depends on the choice of (p, v) , but it is unique up to a shift along the $\mathbb{Z}$ axis. Namely, Suppose we have another slant lattice L' constructed from another element $p' \in P$ and a rank function f' . If $f'(p) \equiv 0 \pmod{2}$, then we have $L' = L$. If $f'(p) \equiv 1 \pmod{2}$, then we have $L' = \{(v, i) \in N(\Gamma) \times \mathbb{Z} \mid (v, i-1) \in L\}$. If P is not connected, then we may choose f so as to embed $\text{Im } \nu \subset L$.

The ordering in L induced from $N(\Gamma) \times \mathbb{Z}$ in the usual sense coincides with the ordering cover-induced from $N(\Gamma) \times \mathbb{Z}$. The following is clear.

Corollary 3.3. *Let Γ be a connected acyclic Dynkin diagram, and let $(P, \leq \phi)$ be a connected minuscule heap over Γ . Let f be as above, and let L be the slant lattice over Γ defined by f . Then the corresponding ν is a cover-embedding of P into L .*

Let Q be a Γ -labeled poset such that there exist a cover-embedding $\nu : Q \rightarrow L$ and $\#Q < \infty$. Let f be the restriction of the second projection $\nu(Q) \subset N(\Gamma) \times \mathbb{Z} \rightarrow \mathbb{Z}$. For each $v \in N(\Gamma)$, we can set t_v (resp. b_v) to be the unique maximal (resp. minimal) element of Q_v since L_v is totally ordered. We say that Q_v is **full** if $f(t_v) - f(b_v) = 2r$, where $r + 1$ is the number of elements of Q_v .

Proposition 3.4. *Let $(P, \leq, \phi)$ be a minuscule heap satisfying $(H4)$, and let v be a node of Γ . P_v is full if and only if all v -intervals are diamonds.*

4 The star-shaped case: cores and D -matrices

Let Γ be a star-shaped Dynkin diagram, and let R be a branch of Γ . We denote the nodes of R by $R_1, R_2, \dots, R_l$ in the increasing order of the distances from o . We denote by $\bar{R}$ the subdiagram with node set $N(R) \cup \{o\}$, and we sometimes denote o by R_0 .

Let P be a minuscule heap over Γ with connected support containing o . By Proposition 2.1, P_o has a unique maximal (resp. minimal) element t_o (resp. b_o). Since $[b_o, t_o]$ is convex, it is also a minuscule heap. We call $[b_o, t_o]$ the **core** of P , and we say that P is **unadorned** if $P = [b_o, t_o]$ (see Fig 4). We proceed in two steps. In this section we classify the **unadorned** minuscule heaps over Γ by associating then with what we call D -matrices. In §5, we determine what adornments can be added to the core.

We can determine the possibilities of the R_i -intervals as follow.

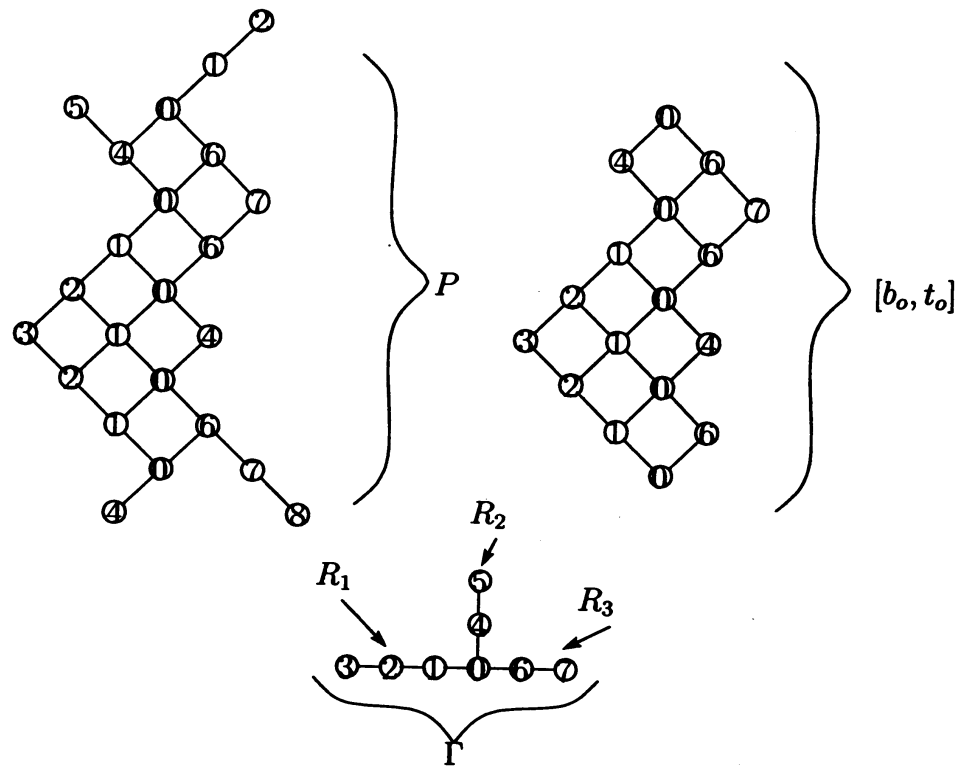


Figure 4: A minuscule heap and its core

Proposition 4.1. *Let $(P, \leq, \phi)$ be a minuscule heap over Γ . Let R be a branch of Γ , and let l be its length. Then*

- *Any o -interval in P is a diamond, namely P_o is full.*
- *If $1 \leq h < l$, then any R_h -interval in P is either a diamond or isomorphic to $d_{h+3}(3)$.*
- *Any R_l -interval in P is isomorphic to $d_{l+3}(3)$.*

(In particular, if Γ is of type A then P_v is full for each $v \in N(\Gamma)$.)

From now on, choose a rank function on f with $f(b_o) = 0$ and choose a slant lattice L which contains $(o, 0)$, namely which contains $\text{Im } \nu$. We may identify P with $\text{Im } \nu$.

Now fix a branch R and determine the shape of $[b_o, t_o] \cap P_{\bar{R}}$. We distinguish between two kinds of o -intervals, namely the ones containing an element labeled R_1 (which we call R -diamonds) and the rest (non- R -diamonds).

Let Γ' be the Dynkin diagram of type A_n with node set $\{1, 2, \dots, n\}$ and L' be a slant lattice over Γ' containing $(1, 1)$. We define a subset Q of L' by

$$Q := \{(v, q) | 1 \leq v \leq n, v \leq q \leq 2n - v\}.$$

We regard Q as a cover-subposet of L , and call a Γ' -labeled poset isomorphic to Q a **wing** over Γ' (see Figure 5) of **width** n .

Proposition 4.2. (1) *In the above notation, $[o_k, o_{k+s}] \cap P_{\bar{R}}$ is a wing over $\bar{R}$.*

(2) *$[b_o, t_o] \cap P_{\bar{R}}$ is contained in the union of all wings over $\bar{R}$ in P .*

(3) *Two adjacent R -diamond blocks are separated by exactly one non- R -diamond.*

(4) *$[b_o, t_o] \cap P_{\bar{R}}$ is contained in the union of all wings over $\bar{R}$. If two R -diamonds in P are separated by non- R -diamonds only, then the number of such non- R -diamonds must be one.*

Let $b_0 = o_0, o_1, \dots, o_c = t_o$ be the elements of P_o in the increasing order. Then $[o_0, o_1], [o_1, o_2], \dots, [o_{c-1}, o_c]$ give all o -intervals of P . We call a sequence of o -intervals $[o_k, o_{k+1}], [o_{k+1}, o_{k+2}], \dots, [o_{k+s-1}, o_{k+s}]$ an **R -diamond block** if $[o_k, o_{k+1}], [o_{k+1}, o_{k+2}], \dots, [o_{k+s-1}, o_{k+s}]$ are R -diamonds and $[o_{k-1}, o_k], [o_{k+s}, o_{k+s+1}]$ are non- R -diamonds (or $k = 0$ or $k + s = c$). We call s the **length** of this R -diamond block.

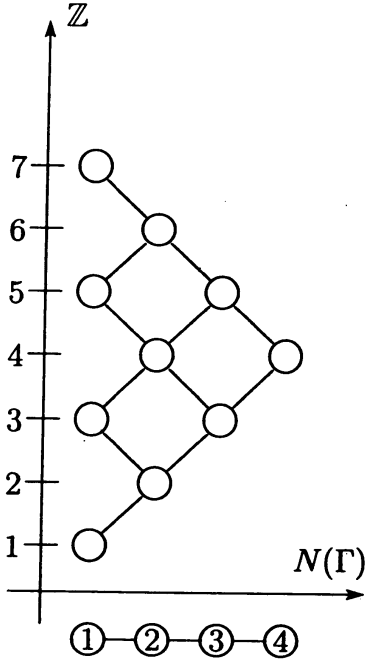


Figure 5: A wing over the Dynkin diagram of type A_4

Proposition 4.3. *Let $a_1, \dots, a_r$ be the lengths of the R -diamond blocks in $[b_o, t_o]$ arranged from bottom to top. Then the sequence $a_1, \dots, a_r$ is unimodal: i.e. we have $a_1 \leq a_2 \leq \dots \leq a_i \geq \dots \geq a_{r-1} \geq a_r$, for some $1 \leq i \leq r$.*

Let Γ be the Dynkin diagram of type $S(l_1, \dots, l_r)$ and let $R^1, \dots, R^r$ be the branches of Γ of length $l_1, \dots, l_r$ respectively. We call an $r \times m$ integer matrix $B = (b_{i,j})$, where m is any nonnegative integer, a D -matrix for Γ if it satisfies the following conditions:

- (1) $b_{i,j} = 0$ or 1 for all i and j .
- (2) For each j , we have $\sum_{i=1}^r b_{i,j} = 2$.
- (3) For each i , the i th row has the form

$$(\overbrace{0, \dots, 0}^{c_i}, \overbrace{1, \dots, 1}^{a_{i,1}}, 0, \overbrace{1, \dots, 1}^{a_{i,2}}, 0, \dots, 0, \overbrace{1, \dots, 1}^{a_{i,s_i-1}}, 0, \overbrace{1, \dots, 1}^{a_{i,s_i}}, \overbrace{0, \dots, 0}^{d_i}) \quad (\#)$$

for some $s_i \in \mathbb{Z}_{\geq 0}$, $a_{i,1}, a_{i,2}, \dots, a_{i,s_i} \in [1, l_i]_{\mathbb{Z}}$ and $c_i, d_i \in \mathbb{Z}_{\geq 0}$, and the sequence $a_{i,1}, a_{i,2}, \dots, a_{i,s_i}$ is unimodal. If $s_i = 0$, then this means that all entries in row i are 0.

We include an empty matrix as a special case where $m = 0$. What we saw above and the shape of o -intervals lead us to the following.

Lemma 4.4. Let P, b_o, t_o, f as above. Define an $r \times (f(t_o)/2)$ -matrix $B = (b_{i,j})$ by

$$b_{i,j} = \begin{cases} 1 & \text{if } [o_{j-1}, o_j] \text{ is an } R^i\text{-diamond} \\ 0 & \text{otherwise,} \end{cases}$$

where o_j is the element of P_o with rank $2j$. Then B is a D -matrix for Γ . We call this B the D -matrix of P .

Example 4.5. The D -matrix B for Γ constructed from P of Fig. 4 is

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Conversely, we can construct an unadorned minuscule heap for each D -matrix for Γ as follows. Recall that we have fixed a slant lattice L over Γ containing $(o, 0)$.

Lemma 4.6. Let B be a D -matrix for Γ with m columns. Put

$$Q = \{(o, 0), (o, 2), \dots, (o, 2m)\} \\ \cup \bigcup_{i=1}^r \{(R_h^i, j) \in L \mid 1 \leq h \leq l_i, b_{i,k} = 1 \text{ for all } k \in [j-h, j+h]\}.$$

Let $\leq$ denote the ordering in Q cover-induced from L , and let $\phi : Q \rightarrow N(\Gamma)$ denote the restriction of the first projection $N(\Gamma) \times \mathbb{Z} \rightarrow N(\Gamma)$. Then $(Q, \leq, \phi)$ is a unadorned minuscule heap over Γ .

Example 4.7. Let us construct the minuscule heap Q from B in Example 4.5. By the definition of Q , we have $Q_0 = \{(0, 0), (0, 2), (0, 4), (0, 6), (0, 8)\}$. P_{R_1} consists of wing of width 3. P_{R_2} consists of 2 wings of width 1. P_{R_3} consists of 2 wings, and each widths are 2 and 1 from bottom (see Fig. 6). In fact, Q is isomorphic to the core of P .

Let $\mathcal{H}_0$ denote the set of isomorphic classes of unadorned minuscule heaps over Γ . We can summarize the results of this section as follows. This is the first part of our main result.

Theorem 4.8. There is a one-to-one correspondence between $\mathcal{H}_0$ and the D -matrices for Γ .

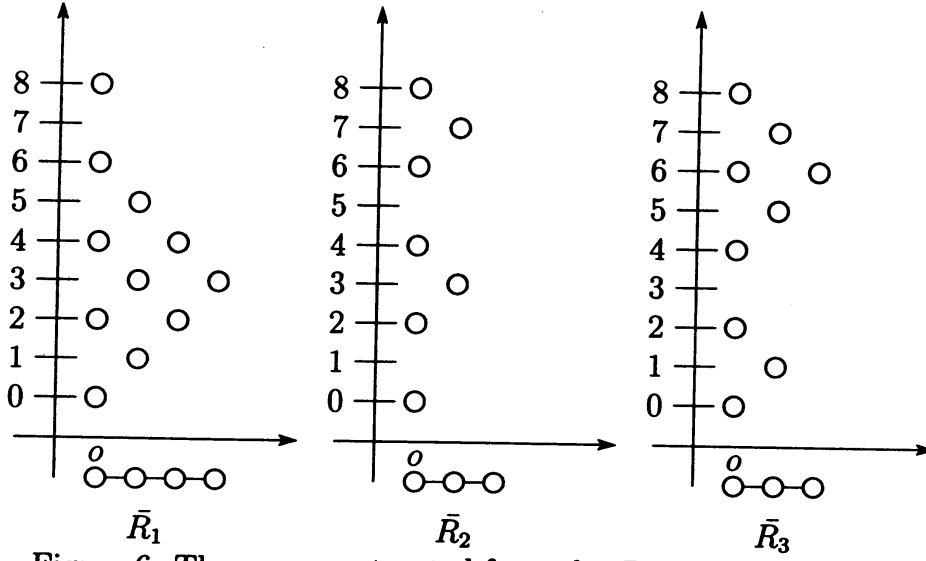


Figure 6: The core constructed from the D -matrix B as in Example 4.5

5 The star-shaped case: adornments

In this section, we determine what adornments can be added to the cores.

Let Γ be a simply-laced, star-shaped Dynkin diagram of type $S(l_1, l_2, \dots, l_r)$, and L be a slant lattice over Γ which contains $(o, 0)$. Let $(P, \preceq) \subset L$ be a connected minuscule heap over Γ with $o \in \text{supp } P$ and $f(b_o) = 0$, where f is the restriction of the second projection $N(\Gamma) \times \mathbb{Z} \rightarrow \mathbb{Z}$ and $\preceq$ is the ordering cover-induced from L .

Let R be a branch. We note that $P_{\bar{R}}$ is may not be a minuscule heap.

For every $p, q \in P$, a sequence $p = p'_0, p'_1, \dots, p'_l = q$ in P such that either $p'_{i-1} \rightarrow p'_i$ or $p'_i \rightarrow p'_{i-1}$ holds for each $i, 1 \leq i \leq l$ is called a **Hasse walk** from p to q . The following is a key lemma in the proof we omitted below.

Lemma 5.1. *Let p be an element of $P_{\bar{R}}$. Put $h = d(o, \phi(p))$, where $d(\cdot, \cdot)$ is the distance of two nodes as we have set in §4. Then there exists a unique Hasse walk $p_0, p_1, \dots, p_h$ in $P_{\bar{R}}$ such that*

- (1) $\phi(p_0) = o, \phi(p_i) = R_i (1 \leq i \leq h)$ and $p_h = p$.
- (2) If p_{j-1} is covered by p_j , then no element of $P_{R_{j-1}}$ covers p_j in P .

(These conditions say that, if we regard the sequence as a walk from p to p_0 , we keep moving closer to P_o incessantly, and we go up instead of down whenever possible.)

We call such a Hasse walk in Lemma 5.1 the **approach to p from above**. We call a Hasse walk which is the approach to p from above in the dual poset

the **approach to p from below**. In the sequel, we investigate the form of a connected component Q of the cover-subposet $P_{\bar{R}}$ (resp. P_R) of P (and hence of L). We simply call such a subset a connected component of $P_{\bar{R}}$.

By Lemma 5.1, we have $\phi(Q) = \{o, R_1, R_2, \dots, R_m\}$ for some $m \geq 0$. The following three Propositions determine the possible shapes of Q .

Proposition 5.2. *Let Q be a connected component of $P_{\bar{R}}$. If $\phi(Q)$ is of type A_{m+1} , then $\#Q_{R_m} = 1$.*

Proposition 5.3. *Let $(R_0, j_0), (R_1, j_1), \dots, (R_m, j_m)$ be the approach to the unique element of Q_{R_m} from above and let $(R_0, i_0), (R_1, i_1), \dots, (R_m, i_m)$ be the approach to the unique element of Q_{R_m} from below. Then we have*

$$Q = \bigcup_{0 \leq k \leq m} \{(R_k, h) \in L \mid i_k \leq h \leq j_k\}. \quad (1a)$$

We call the approach to the unique element of Q_{R_m} from above (resp. below) **the upper (resp. the lower) boundary of Q** .

Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers. We define $\mathcal{B}_{\gamma, \delta}^{\alpha, \beta}$ (see Figure 7) to be the set of all subsets N of L such that

$$N = \{(R_k, h) \mid i_k \leq h \leq j_k, 0 \leq k \leq \gamma\}$$

for some Hasse walks $(R_0, i_0), (R_1, i_1), \dots, (R_\gamma, i_\gamma)$ and $(R_0, j_0), (R_1, j_1), \dots, (R_\gamma, j_\gamma)$ in L such that

$$(2a) \quad j_0 - i_0 = 2\alpha,$$

$$(2b) \quad (R_0, i_0) \rightarrow (R_1, i_1) \rightarrow \dots \rightarrow (R_\beta, i_\beta),$$

$$(2c) \quad (R_\beta, i_\beta) \leftarrow (R_{\beta+1}, i_{\beta+1}) \text{ if } \beta \neq \gamma,$$

$$(2d) \quad (R_0, j_0) \leftarrow (R_1, j_1) \leftarrow \dots \leftarrow (R_\delta, j_\delta),$$

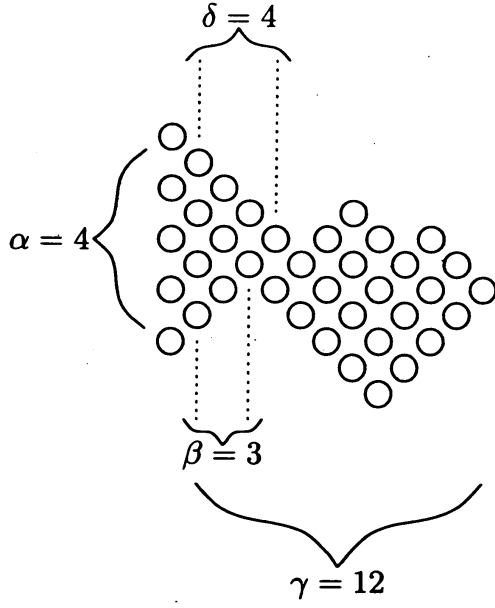
$$(2e) \quad (R_\delta, j_\delta) \rightarrow (R_{\delta+1}, j_{\delta+1}) \text{ if } \delta \neq \gamma, \text{ and}$$

$$(2f) \quad i_k \leq j_k \text{ for each } 0 \leq k \leq \gamma - 1, \text{ and } i_\gamma = j_\gamma.$$

Proposition 5.4. *Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers. Then $\mathcal{B}_{\gamma, \delta}^{\alpha, \beta} \neq \emptyset$ if and only if (3a)-(3c) hold.*

$$(3a) \quad \alpha, \beta, \delta \leq \gamma \leq l$$

$$(3b) \quad \beta \leq \alpha \text{ or } \delta \leq \alpha$$

Figure 7: An element of $\mathcal{B}_{12,4}^{4,3}$

(3c) If $\beta, \delta < \gamma$, then $\gamma \geq \alpha + 2$.

If $\beta < \gamma$ or $\delta < \gamma$, then $\gamma \geq \alpha + 1$.

Let B be a D -matrix for Γ . Let P be a minuscule heap over Γ , not necessarily unadorned, whose $[b_o, t_o]$ is represented by B . Let R be a branch of Γ and let $(\overbrace{0, \dots, 0}^{c_R}, \overbrace{1, \dots, 1}^{a_{R,1}}, \overbrace{0, 1, \dots, 1}^{a_{R,2}}, \dots, \overbrace{0, 1, \dots, 1}^{a_{R,e}}, \overbrace{0, \dots, 0}^{d_R})$ be the row of B corresponding to R . Put

$$\alpha_R = (\alpha_{R,1}, \alpha_{R,2}, \dots, \alpha_{R,h_R}) = (\overbrace{0, 0, \dots, 0}^{c_R}, a_{R,1}, a_{R,2}, \dots, a_{R,e}, \overbrace{0, 0, \dots, 0}^{d_R}).$$

Let $Q_R^1, Q_R^2, \dots, Q_R^{h_R}$ be the connected components of $P_{\bar{R}}$ from bottom to top. Then there are unique nonnegative integers $\beta_{R,i}, \gamma_{R,i}, \delta_{R,i}$ such that $Q_R^i \in \mathcal{B}_{\gamma_{R,i}, \delta_{R,i}}^{\alpha_{R,i}, \beta_{R,i}}$. Like α_R , we put $\beta_R = (\beta_{R,1}, \beta_{R,2}, \dots, \beta_{R,h_R})$, $\gamma_R = (\gamma_{R,1}, \gamma_{R,2}, \dots, \gamma_{R,h_R})$ and $\delta_R = (\delta_{R,1}, \delta_{R,2}, \dots, \delta_{R,h_R})$.

Example 5.5. Let us calculate $\alpha, \beta, \gamma, \delta$ corresponding to Fig. 4.

$$\alpha_{R_1} = (3, 0), \alpha_{R_2} = (0, 1, 1), \alpha_{R_3} = (1, 2)$$

$$\beta_{R_1} = (3, 2), \beta_{R_2} = (0, 1, 2), \beta_{R_3} = (1, 2)$$

$$\gamma_{R_1} = (3, 2), \gamma_{R_2} = (1, 1, 2), \gamma_{R_3} = (3, 2)$$

$$\delta_{R_1} = (3, 0), \delta_{R_2} = (1, 1, 1), \delta_{R_3} = (3, 2)$$

Conversely, we can construct minuscule heaps from a collection of such elements of $\mathcal{B}_{\gamma,\delta}^{\alpha,\beta}$ as follows. The following theorem gives a complete parameterization of the (isomorphism classes) of minuscule heaps having a fixed D -matrix B . This is the second part of our main result. We omit the arguments to check that the resulting subsets of L are actually minuscule heaps.

Theorem 5.6. [3] *Let B be a D -matrix for Γ , and let P denote the unadorned minuscule heap over Γ corresponding to B constructed in Lemma 4.6. For each branch R , define an integer sequence $\alpha_R = (\alpha_{R,i})_{i=1}^{h_R}$ from B as above. Let $\beta_R = (\beta_{R,i})_{i=1}^{h_R}$, $\gamma_R = (\gamma_{R,i})_{i=1}^{h_R}$, $\delta_R = (\delta_{R,i})_{i=1}^{h_R}$ be integer sequences satisfying the following conditions:*

- (1) *For each R and $1 \leq i \leq h_R$, the quadruple $\alpha_{R,i}, \beta_{R,i}, \gamma_{R,i}, \delta_{R,i}$ satisfy the conditions in Proposition 5.4.*
- (2) *For each R , the sequence γ_R is unimodal.*
- (3) *If $\delta_{R,i} < \gamma_{R,i}$, then $\beta_{R,i+1} = \gamma_{R,i+1} < \delta_{R,i}$ ($1 \leq i < h_R$).*
- (4) *If $\beta_{R,i} < \gamma_{R,i}$, then $\delta_{R,i-1} = \gamma_{R,i-1} < \beta_{R,i}$ ($1 < i \leq h_R$).*

For each R and $1 \leq i \leq h_R$, choose $Q^{R,i} \in \mathcal{B}_{\gamma_{R,i}, \delta_{R,i}}^{\alpha_{R,i}, \beta_{R,i}}$, and replace the i th wing over R (counted from the bottom) in P by $Q^{R,i}$. These $Q^{R,i}$ do not overlap with one another, and the resulting Γ -labeled poset P' is a minuscule heap over Γ with connected support containing o , having B as the D -matrix, the $Q^{R,i}$, $i = 1, 2, \dots, h_R$, being the connected components of P'_R for each R . Moreover, all minuscule heaps over Γ with connected support containing o are obtained in this manner.

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