

## A LOCAL EXISTENCE THEOREM FOR THE NAVIER-STOKES FLOW IN THE EXTERIOR TO A ROTATING OBSTACLE

TOSHIAKI HISHIDA    菱田 俊 明

Department of Applied Mathematics  
Faculty of Engineering  
Niigata University  
Niigata 950-2102 Japan  
(e-mail: hishida@eng.niigata-u.ac.jp)

ABSTRACT. Let us consider the three dimensional Navier-Stokes initial value problem in the exterior to a rotating obstacle. It is proved that a unique solution exists locally in time when the initial data  $a$  possess some regularity in the space  $L^2$  (similarly to the assumption given by Fujita and Kato [4]) and satisfy  $(\omega \times x) \cdot \nabla a \in H^{-1}$ , where  $\omega$  stands for the angular velocity of the rotating obstacle. An essential step for the proof is to deduce a certain smoothing property together with estimates near  $t = 0$  of the semigroup (it is not an analytic one) generated by the operator  $\mathcal{L}u = -P[\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u]$ , where  $P$  denotes the projection associated with the Helmholtz decomposition.

It is one of important problems in fluid mechanics to study the Navier-Stokes flow past a rotating obstacle. In order to understand the rotation effect mathematically, we will limit ourselves to a problem under the following simple situation; the angular velocity is constant and the translation is absent. In this article we discuss the locally in time existence of a unique solution to such a problem.

Let  $\mathcal{O} \subset \mathbb{R}^3$  be a compact, isolated rigid obstacle which is bounded by a smooth surface  $\Gamma$ , and  $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$  the exterior domain occupied by a viscous incompressible fluid. Assume that the obstacle  $\mathcal{O}$  is rotating about the  $x_3$ -axis with angular velocity  $\omega = (0, 0, 1)^T$ . Here and hereafter, super- $T$  denotes the transpose and all vectors are column ones;  $x = (x_1, x_2, x_3)^T$ ,  $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$  and so on. Set

$$\Omega(t) = \{y = O(t)x; x \in \Omega\}, \quad \Gamma(t) = \{y = O(t)x; x \in \Gamma\},$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

which actually vary as time  $t$  goes on (this is the situation under consideration) unless  $\mathcal{O}$  is axisymmetric, where

$$O(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now consider the fluid motion around  $\mathcal{O}$ , which is governed by the initial boundary value problem for the Navier-Stokes equation

$$(NS.1) \quad \left\{ \begin{array}{ll} \partial_t w + w \cdot \nabla_y w = \Delta_y w - \nabla_y q, & y \in \Omega(t), t > 0, \\ \nabla_y \cdot w = 0, & y \in \Omega(t), t \geq 0, \\ w = \omega \times y, & y \in \Gamma(t), t > 0, \\ w \rightarrow 0, & |y| \rightarrow \infty, t > 0, \\ w(y, 0) = a(y), & y \in \Omega, \end{array} \right.$$

where  $w = (w_1(y, t), w_2(y, t), w_3(y, t))$  and  $q = q(y, t)$  denote, respectively, unknown velocity and pressure of the fluid. The boundary condition on  $\Gamma(t)$  is the non-slip one since  $dy/dt = \dot{O}(t)O(t)^T y = \omega \times y$ , where  $\dot{O}(t) = (d/dt)O(t)$ . It is natural to reduce (NS.1) to the problem in the fixed domain  $\Omega$  by using the coordinate system  $x = O(t)^T y$  attached to the rotating obstacle. There are two ways to make the change of the fluid velocity. The one is

$$u(x, t) = O(t)^T w(y, t),$$

and the other is

$$v(x, t) = O(t)^T [w(y, t) - \omega \times y] = u(x, t) - \omega \times x.$$

We also make the change of the pressure by

$$p(x, t) = q(y, t).$$

Then we have

$$\begin{aligned}
\partial_t w &= O(t) \left[ \partial_t u + \left( \dot{O}(t)^T O(t) x \right) \cdot \nabla_x u + O(t)^T \dot{O}(t) u \right] \\
&= O(t) [\partial_t u - (\omega \times x) \cdot \nabla_x u + \omega \times u] \\
&= O(t) [\partial_t v - (\omega \times x) \cdot \nabla_x v + \omega \times v], \\
\Delta_y w &= O(t) \Delta_x u = O(t) \Delta_x v, \\
\nabla_y q &= O(t) \nabla_x p, \\
\nabla_y \cdot w &= \nabla_x \cdot u = \nabla_x \cdot v,
\end{aligned}$$

and

$$\begin{aligned}
w \cdot \nabla_y w &= O(t) [u \cdot \nabla_x u] \\
&= O(t) [v \cdot \nabla_x v + (\omega \times x) \cdot \nabla_x v + \omega \times v + \omega \times (\omega \times x)].
\end{aligned}$$

The problem (NS.1) is thus reduced to the following (NS.2) and (NS.3) for  $\{v, p\}$  and  $\{u, p\}$ , respectively. The former is the problem with not only the Coriolis force  $2 \omega \times v$  but also the growing boundary condition at space infinity:

$$(\text{NS.2}) \left\{ \begin{array}{ll} \partial_t v + v \cdot \nabla_x v = \Delta_x v - 2 \omega \times v - \omega \times (\omega \times x) - \nabla_x p, & x \in \Omega, t > 0, \\ \nabla_x \cdot v = 0, & x \in \Omega, t \geq 0, \\ v = 0, & x \in \Gamma, t > 0, \\ v + \omega \times x \rightarrow 0, & |x| \rightarrow \infty, t > 0, \\ v(x, 0) = a(x) - \omega \times x, & x \in \Omega. \end{array} \right.$$

The latter is the problem with the convection term having the coefficient  $\omega \times x$  which is understood as the rigid motion rotating about the  $x_3$ -axis:

$$\begin{aligned}
 \text{(NS.3)} \quad & \left\{ \begin{array}{ll} \partial_t u + u \cdot \nabla_x u = \Delta_x u + (\omega \times x) \cdot \nabla_x u - \omega \times u - \nabla_x p, & x \in \Omega, \, t > 0, \\ \nabla_x \cdot u = 0, & x \in \Omega, \, t \geq 0, \\ u = \omega \times x, & x \in \Gamma, \, t > 0, \\ u \rightarrow 0, & |x| \rightarrow \infty, \, t > 0, \\ u(x, 0) = a(x), & x \in \Omega. \end{array} \right.
 \end{aligned}$$

Up to now the mathematical theory for the existence and uniqueness of solutions to the problem (NS.1) has been little developed. In his Habilitationsschrift [2] Borchers first attacked this problem, including the case where the angular velocity depends on time  $t$ . He dealt with the problem (NS.2) and proved the existence of weak solutions of class

$$v + \omega \times x (= u) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \forall T > 0,$$

with the energy inequality provided that  $a \in L^2(\Omega)$  satisfies

$$(1) \quad \nabla \cdot a = 0 \quad \text{in } \Omega, \quad \nu \cdot (a - \omega \times x) = 0 \quad \text{on } \Gamma,$$

where  $\nu$  is the unit exterior normal vector to  $\Gamma$ . We donot know the uniqueness of weak solutions and this feature is the same as the standard Navier-Stokes theory. Later on, in [3] Chen and Miyakawa have treated (NS.3) for  $\Omega = \mathbb{R}^3$ , that is, the Cauchy problem. They have discussed the existence of weak solutions with the so-called strong energy inequality and some decay properties of the constructed solutions.

The purpose of the present article is to prove that there exists a unique local solution to the problem (NS.3) whenever the initial data  $a \in L^2(\Omega)$  satisfying (1) possess some regularity and fulfill  $(\omega \times x) \cdot \nabla a \in H^{-1}(\Omega)$ .

To state our results precisely, we introduce notation. We use the same symbols for denoting the spaces of scalar and vector functions if there is no confusion. By  $C_0^\infty(\Omega)$  we denote the class of all  $C^\infty$  functions with compact supports in  $\Omega$ . Let  $H^s(\Omega)$  for  $s \geq 0$  be the usual  $L^2$  Sobolev spaces. If  $s$  is not an integer, then the space  $H^s(\Omega)$  is defined via the complex interpolation (see Lions and Magenes [11, Chapter 1]), that is,

$$H^s(\Omega) = [L^2(\Omega), H^m(\Omega)]_\theta, \quad s = \theta m, \quad m > 0 \text{ (integer)}, \quad 0 < \theta < 1.$$

The scalar product and the norm of  $L^2(\Omega) = H^0(\Omega)$  are respectively denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$ . The space  $H_0^s(\Omega)$ ,  $s > 0$ , is the completion of  $C_0^\infty(\Omega)$  in  $H^s(\Omega)$ , and  $H^{-s}(\Omega)$  stands for the dual space of  $H_0^s(\Omega)$ . Let  $C_{0,\sigma}^\infty(\Omega)$  be the class of all solenoidal (that is, divergence free) vector functions whose components are in  $C_0^\infty(\Omega)$ . By  $L_\sigma^2(\Omega)$  we denote the completion of  $C_{0,\sigma}^\infty(\Omega)$  in  $L^2(\Omega)$ . Then the space  $L^2(\Omega)$  of vector functions admits the following orthogonal decomposition, the Helmholtz decomposition (Temam [13, Chapter I]):

$$L^2(\Omega) = L_\sigma^2(\Omega) \oplus L_\pi^2(\Omega),$$

where

$$L_\pi^2(\Omega) = \{\nabla p \in L^2(\Omega); p \in L_{\text{loc}}^2(\overline{\Omega})\}.$$

Let  $P$  be the projection (the Fujita-Kato projection) from  $L^2(\Omega)$  onto  $L_\sigma^2(\Omega)$  associated with the decomposition above. Then the Stokes operator  $A : L_\sigma^2(\Omega) \rightarrow L_\sigma^2(\Omega)$  is defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega), \quad Au = -P\Delta u.$$

In view of (NS.3), the linear operator  $\mathcal{L} : L_\sigma^2(\Omega) \rightarrow L_\sigma^2(\Omega)$  we should consider is as follows:

$$\begin{cases} D(\mathcal{L}) = \{u \in D(A); (\omega \times x) \cdot \nabla u \in L^2(\Omega)\}, \\ \mathcal{L}u = Au - P[(\omega \times x) \cdot \nabla u - \omega \times u]. \end{cases}$$

It is proved that the operator  $\mathcal{L}$  is  $m$ -accretive, so that  $-\mathcal{L}$  generates a  $(C_0)$  semigroup  $\{e^{-t\mathcal{L}}; t \geq 0\}$  of contractions on  $L^2_\sigma(\Omega)$ . Furthermore, we have

$$(2) \quad \|u\|_{H^2(\Omega)} + \|P[(\omega \times x) \cdot \nabla u]\| \leq C \|(1 + \mathcal{L})u\|,$$

for all  $u \in D(\mathcal{L})$  (see [8]). On account of unboundedness of the coefficient of  $\mathcal{L}$ , the elliptic regularity estimate (2) is no longer trivial. It is thus not so easy to show the closedness of  $\mathcal{L}$  directly. But the accretivity implies that  $\mathcal{L}$  is closable. So, we prove that  $1 + \bar{\mathcal{L}}$  is surjective, where  $\bar{\mathcal{L}}$  is the closure of  $\mathcal{L}$ . For the proof, we solve the corresponding stationary problem by using the solutions in  $\mathbb{R}^3$  and in a bounded domain near the boundary  $\Gamma$  together with cut-off functions. For the recovery of the solenoidal condition in the localization, we make use of the result of Bogovskiĭ [1] on a continuous right-inverse of the divergence operator with zero boundary condition in bounded domains. At the next step, we show  $\bar{\mathcal{L}} = \mathcal{L}$  together with estimate (2). The fractional powers of  $\mathcal{L}$  are also well defined as closed operators in  $L^2_\sigma(\Omega)$ , and we see that  $D(\mathcal{L}^\alpha) \subset D(A^\alpha)$  with estimate

$$(3) \quad \|A^\alpha u\| \leq C_\alpha \|(1 + \mathcal{L})^\alpha u\|,$$

for all  $u \in D(\mathcal{L}^\alpha)$  and  $0 < \alpha \leq 1$ . Indeed, (3) for the case  $\alpha = 1$  is equivalent to (2), and the Heinz-Kato inequality for  $m$ -accretive operators (Tanabe [12, Chapter 2]) implies (3) for  $0 < \alpha < 1$ .

Our method to solve (NS.3) is to make use of the semigroup  $e^{-t\mathcal{L}}$  together with the fractional powers of  $A$  and  $\mathcal{L}$ . Although this approach itself is, in principle, standard (see Fujita and Kato [4], Giga and Miyakawa [6]), the semigroup  $e^{-t\mathcal{L}}$  is not

a usual one. The essential difficulty is the growth at space infinity of the coefficient  $\omega \times x$  of the operator  $\mathcal{L}$ , so that the convection term  $(\omega \times x) \cdot \nabla$  is not a perturbation of the Stokes operator  $A$ . In fact, the associated semigroup for the Cauchy problem in  $\mathbb{R}^3$  is explicitly given by

$$(4) \quad [U(t)f](x) = O(t)^T [e^{t\Delta}f](O(t)x), \quad x \in \mathbb{R}^3, t > 0,$$

where

$$[e^{t\Delta}f](x) = (4\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} f(y) dy,$$

and it is proved that the semigroup  $U(t)$  is never analytic on  $L^2_\sigma(\mathbb{R}^3)$  (see [9]). This is a different feature caused by the convection term  $(\omega \times x) \cdot \nabla$ . Thus, we cannot expect that  $e^{-t\mathcal{L}}$  is analytic. However, it has the remarkable smoothing effect. The following theorem asserts that  $e^{-t\mathcal{L}}f$  is in  $D(A)$  for all  $t > 0$  whenever  $f$  is slightly smooth, and that  $e^{-t\mathcal{L}}f$  is in  $D(\mathcal{L})$  for all  $t > 0$  under the additional assumption  $(\omega \times x) \cdot \nabla f \in H^{-\infty}(\Omega) \equiv \bigcup_{s \geq 0} H^{-s}(\Omega)$ .

**Theorem 1.** (i) Suppose that  $f \in D(A^\delta)$  for some  $0 < \delta \leq 1/2$ . Then  $e^{-t\mathcal{L}}f \in D(A)$  for all  $t > 0$ . Furthermore, there is a constant  $C = C(\delta) > 0$  such that

$$(5) \quad \|Ae^{-t\mathcal{L}}f\| \leq C t^{-1+\delta} \|f\|_{D(A^\delta)},$$

for all  $0 < t \leq 1$ .

(ii) Suppose that  $f \in D(A^\delta)$  for some  $0 < \delta < 1$ , and that  $(\omega \times x) \cdot \nabla f \in H^{-s}(\Omega)$  for some  $s \geq 0$ . Then  $e^{-t\mathcal{L}}f \in D(\mathcal{L})$  for all  $t > 0$  and

$$\mathcal{L}e^{-t\mathcal{L}}f \in C(0, \infty; L^2_\sigma(\Omega)), \quad e^{-t\mathcal{L}}f \in C^1(0, \infty; L^2_\sigma(\Omega)),$$

with

$$\frac{d}{dt}e^{-t\mathcal{L}}f + \mathcal{L}e^{-t\mathcal{L}}f = 0, \quad t > 0,$$

in  $L^2_\sigma(\Omega)$ . Furthermore, there are constants  $C = C(\delta) > 0$  and  $C' = C'(s) > 0$  such that

$$(6) \quad \begin{aligned} \|\mathcal{L}e^{-t\mathcal{L}}f\| &\leq C (t \wedge 1)^{-1+\delta} \|f\|_{D(A^\delta)} \\ &\quad + C' (t \wedge 1)^{-s/2} \left\{ \|(\omega \times x) \cdot \nabla f\|_{H^{-s}(\Omega)} + \|f\| \right\}, \end{aligned}$$

for all  $t > 0$ , where  $t \wedge 1 = \min\{t, 1\}$ .

(iii) Let  $0 < \delta < 1/2$ . Then

$$\lim_{t \rightarrow 0} t^{1-\delta} \|Ae^{-t\mathcal{L}}f\| = 0,$$

for all  $f \in D(A^\delta)$ . For the same  $\delta$  as above, let  $0 \leq s < 2(1 - \delta)$ . Then

$$\lim_{t \rightarrow 0} t^{1-\delta} \|\mathcal{L}e^{-t\mathcal{L}}f\| = 0,$$

for all  $f \in D(A^\delta)$  satisfying  $(\omega \times x) \cdot \nabla f \in H^{-s}(\Omega)$ .

In Theorem 1 the case  $\delta = 0$  (namely,  $f \in L^2_\sigma(\Omega)$ ) is excluded on account of a technical difficulty caused by the solenoidal constraint. Indeed, in [7, Theorem 4] sharper results including  $\delta = 0$  have been established for the realization of a model operator  $\Delta + (\omega \times x) \cdot \nabla$  with the homogeneous Dirichlet boundary condition in  $L^2(\Omega)$ . But estimates (5) and (6) near  $t = 0$  together with the fractional powers of  $A$  and  $\mathcal{L}$  are very useful for the proof of local existence of a unique solution to (NS.3). The strategy for the proof of Theorem 1 is as follows. We first derive the similar smoothing effect to Theorem 1 for the semigroup  $U(t)$  given by (4). We next employ the method based on a refinement of the cut-off procedure developed in the proof of Theorem 4 of [7] combined with the result of Bogovskii [1] mentioned above. For the details, see [9].

We now fix  $\zeta \in C^\infty(\mathbb{R}^3)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta = 1$  near  $\Gamma$  and  $\zeta = 0$  for large  $|x|$ , and put

$$(7) \quad b(x) = -\frac{1}{2} \nabla \times \{\zeta(x)|x|^2\omega\}.$$

Then  $\nabla \cdot b = 0$  in  $\Omega$ ,  $b = \omega \times x$  on  $\Gamma$  and  $b = 0$  for large  $|x|$ . We set

$$\tilde{u}(x, t) = u(x, t) - b(x),$$

in (NS.3) and apply the projection  $P$  to the equation of motion to obtain the integral equation

$$(NS.4) \quad \tilde{u}(t) = e^{-t\mathcal{L}}[a - b] - \int_0^t e^{-(t-s)\mathcal{L}} P[\tilde{u} \cdot \nabla \tilde{u} + B\tilde{u}](s) ds, \quad t \geq 0,$$

in  $L_\sigma^2(\Omega)$ , where

$$B\tilde{u} = \tilde{u} \cdot \nabla b + b \cdot \nabla \tilde{u} + F[b],$$

$$F[b] = \Delta b + (\omega \times x) \cdot \nabla b - \omega \times b - b \cdot \nabla b.$$

The main theorem then reads as follows.

**Theorem 2.** Suppose that  $a - b \in D(\mathcal{L}^\gamma)$  for some  $1/4 < \gamma < 1/2$  and that  $(\omega \times x) \cdot \nabla a \in H^{-s}(\Omega)$  for some  $1 \leq s < 2(1 - \gamma)$ . Then there exist  $T > 0$  and a unique solution  $\tilde{u}$  to (NS.4) on the interval  $[0, T]$ , which is of class

$$\tilde{u} \in C([0, T]; L_\sigma^2(\Omega)),$$

and possesses the regularity  $\tilde{u}(t) \in D(A)$ ,  $0 < t \leq T$ , with the properties:

$$(8) \quad \lim_{t \rightarrow 0} \|\tilde{u}(t) - (a - b)\|_{D(A^\gamma)} = \lim_{t \rightarrow 0} \|u(t) - a\|_{D(A^\gamma)} = 0,$$

$$(9) \quad \lim_{t \rightarrow 0} t^{\alpha-\gamma} \|\tilde{u}(t)\|_{D(A^\alpha)} = 0, \quad \gamma < \alpha \leq 1,$$

$$(10) \quad \|\tilde{u}(t)\|_{D(A^\alpha)} \leq C_\alpha K_0 t^{-\alpha+\gamma}, \quad 0 < t \leq T, \quad \gamma \leq \alpha \leq 1,$$

where

$$K_0 = \|a - b\|_{D(\mathcal{L}^\gamma)} + \|(\omega \times x) \cdot \nabla a\|_{H^{-s}(\Omega)} + \| |x|b \| + \|F[b]\|_{H^1(\Omega)}.$$

The proof is given in [9]. We conclude this article with some comments on Theorem 2.

*Remark.* (i) In view of (7), the assumption  $a - b \in D(\mathcal{L}^\gamma) \subset D(A^\gamma)$  (see (3)) with  $\gamma > 1/4$  implies that  $a = \omega \times x$  on  $\Gamma$  (cf. Fujiwara [5]).

(ii) The critical case  $\gamma = 1/4$  is the well known exponent of Fujita and Kato [4]. If Theorem 1 for  $\delta = 0$  were deduced, then we could show Theorem 2 for the case  $\gamma = 1/4$ .

(iii) Under the assumption  $(\omega \times x) \cdot \nabla a \in H^{-2(1-\gamma)}(\Omega)$ , it is also possible to construct a unique solution. But the behavior (9) of such a solution is not clear.

(iv) The solution obtained in Theorem 2 is the so-called mild solution. Since we find the solution  $\tilde{u}(t)$  with values in  $D(A)$  and it does not belong to  $D(\mathcal{L})$  in general, it seems to be difficult to derive the differentiability of  $\tilde{u}$  with respect to time  $t$ .

(v) Theorem 2 holds true with  $\omega = (0, 0, 1)^T$  replaced by  $\omega = (0, 0, \omega_0)^T$  for every  $\omega_0 \in \mathbb{R}$ . The existence interval  $T = T(|\omega_0|) > 0$  is then monotonically decreasing with respect to  $|\omega_0|$ .

(vi) When the obstacle  $\mathcal{O}$  is not rotating, that is  $\omega = 0$ , the problem (NS.3) possesses a unique local strong solution for  $a \in L_\sigma^3(\Omega) \supset D(A^{1/4})$ , where  $L_\sigma^3(\Omega)$  denotes the completion of  $C_{0,\sigma}^\infty(\Omega)$  in  $L^3(\Omega)$ . If  $\|a\|_{L^3(\Omega)}$  is sufficiently small, then the solution is extended globally in time. This is the result of Iwashita [10].

## REFERENCES

1. Bogovskiĭ, M. E., *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. **20** (1979), 1094–1098.
2. Borchers, W., *Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes Gleichungen inkompressibler viskoser Flüssigkeiten*, Habilitationsschrift, University of Paderborn, 1992.
3. Chen, Z. M., Miyakawa, T., *Decay properties of weak solutions to a perturbed Navier-Stokes system in  $\mathbb{R}^n$* , Adv. Math. Sci. Appl. (to appear).
4. Fujita, H., Kato, T., *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.
5. Fujiwara, D., *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad. **43** (1967), 82–86.
6. Giga, Y., Miyakawa, T., *Solutions in  $L_r$  of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), 267–281.
7. Hishida, T., *Heat equations in the exterior to a rotating obstacle*, preprint.
8. Hishida, T., *The Stokes semigroup with rotation effect in exterior domains*, preprint.
9. Hishida, T., *An existence theorem for the Navier-Stokes flow in the exterior to a rotating obstacle*, preprint.
10. Iwashita, H.,  *$L_q - L_r$  estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in  $L_q$  spaces*, Math. Ann. **285** (1989), 265–288.
11. Lions, J. L., Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, Berlin, 1972.
12. Tanabe, H., *Equations of Evolution*, Pitman, London, 1979.
13. Temam, R., *Navier-Stokes Equations*, 3rd (revised) ed., North-Holland, Amsterdam, 1984.