

On Some Elliptic Equations with Nonlocal Nonlinear Terms

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§.1 Introduction and Main Results.

Consider the following problem ; Find u such that

$$(E)_\gamma \quad \begin{cases} -\Delta u = (V * |u|^2) u + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega (\subset \mathbb{R}^n)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\lambda \in \mathbb{R}^1$, $V(x) = 1/|x|^\gamma$ ($0 < \gamma < n$) and

$$(V * |u|^2)(x) = \int_{\Omega} V(x-y) |u|^2(y) dy.$$

In this paper we want to show that the solvability of the problem $(E)_4$ ($n > 4$) is similar to that of the following Yamabe type equation ;

$$(Y) \quad \begin{cases} -\Delta u = u^{(n+2)/(n-2)} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

One of our motivations for studying the problem $(E)_4$ is the following question.

Question : Is there nonlinear elliptic boundary value problem which has strong dependence on the geometry (or the topology) of the domain Ω ? Is there nonlinear elliptic boundary value problem which

characterizes the geometry (or the topology) of the domain ?

For example, Dirichlet boundary value problem for the minimal surface equation has strong dependence on convexity of the domain. This fact is well known (see [GT]). For the problem (Y) with $\lambda = 0$ some results are known (see [P],[BC]).

(a) If the domain Ω is star-shaped, (Y) has no solution,

(b) If Ω is non-contractible and $n = 3$, (Y) has at least one solution.

From these facts, there is a following conjecture:

"There is a solution of (Y) with $\lambda = 0$."

$\Longleftrightarrow$ " The domain Ω is non-contractible."

For the problem $(E)_4$ we can prove the next statements.

< Effect of the domain (for $(E)_4$) .>

PROPOSITION.1. - Let Ω be an annulus domain in R^n ($n > 4$) (i.e. $\Omega = \{ x \in R^n ; 0 < a < |x| < b \}$, $a, b > 0$ are constants.) and $\lambda < \lambda_1$, where λ_1 is a first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then there is a radially symmetric solution of $(E)_4$.

PROPOSITION.2. - Let Ω be a star-shaped domain in R^n ($n > 5$) and $\lambda \leq 0$. Then there is no solution of $(E)_4$.

PROPOSITION.1 can be proved by direct variational method and PROPOSITION.2 is obtained by using Pohozaev's type identity.

REMARK.1. - (a) The condition $\lambda < \lambda_1$ is necessary for the solvability of $(E)_4$ (or (Y)).

(b) Under the conditions $0 < \lambda < n$ and $\lambda \leq 4$, any weak solution of $(E)_4$ is a classical solution.

We do not know whether non-contractibility of the domain implies the existence of solutions for $(E)_4$ or not. In attacking this question, we encounter the following questions.

Question.A. - Is the solution u of $(E)'$ unique ?

$$(E)' \quad \begin{aligned} & - \Delta u = (1/|x|^4 * |u|^2) u \quad \text{in } R^n, \\ & u > 0 \quad (u \in H^1(R^n)). \end{aligned}$$

REMARK.2. - Existence of solution for $(E)'$ is proved in this paper.

Question.B. - Is any solution of $(E)'$ radially symmetric ?

To the question B we give next partial result.

PROPOSITION.3. - Let u be a solution of $(E)'$ and $u(x) = O(1/|x|^{n-2})$ as $|x| \rightarrow \infty$. Then u is radially symmetric.

REMARK.3. - For $0 < \gamma < n$, $\gamma \leq 4$ symmetry properties of the solution for problem $(E)_\gamma$ can be proved for a symmetric domain by using a slight modification of the argument in Gidas, Ni and Nirenberg [GNN 1]. In particular, we obtain the next theorem.

THEOREM.1. - Let u be a solution of $(E)_\gamma$ ($0 < \gamma < n$, $\gamma \leq 4$) and $\Omega = B_R(o)$ ($= \{x \in R^n; |x| < R\}$). Then u is radially symmetric and $u_r(r) < 0$ for all $r \in (0, R]$.

Next we consider the effect of lower order perturbations to the problem $(E)_4$. For $0 < \lambda < \lambda_1$, solvability of the problem $(E)_4$ does not depend on the shape of the domain.

< Effect of lower order perturbations >

THEOREM.2. - Let $0 < \lambda < \lambda_1$, $n > 4$ and Ω be an arbitrary bounded domain. Then there is a solution of $(E)_4$.

Next we consider the following lower order perturbation problem:

$$-\Delta u = (1/|x|^{4*} |u|^2) u + \lambda (1/|x|^{\gamma*} |u|^2) u \quad \text{in } \Omega,$$

$$(P) \quad \begin{aligned} u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $0 < \gamma < 4$ and $0 < \lambda$.

THEOREM.3. - Let $\lambda > 0$. Assume that $1 < \gamma < 4$ if $n = 5$ and $0 < \gamma < 4$ if $n \geq 6$. Then there is a solution of (P).

REMARK.4. - For the equation (E)₄ we can obtain analogous results to some interesting results for Yamabe type equation (Y) (e.g. [CFP], [CFS] etc.).

REMARK.5. - Nonlinear term $(V * |u|^2) u$ appears, for example, in Hartree type equation and Choquard type equation (see e.g. [Lio 1], [Lio 3], [GV], [HT]).

In the next section we explain the idea and some technical tools in proving THEOREM.2 and THEOREM.3.

§.2 Sketch of proof of THEOREM.2 and THEOREM.3.

Notations: For $0 < \lambda < \lambda_1$, define $\sigma(\lambda)$ as follows:

$$\sigma(\lambda) = \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ u \neq 0}} \frac{\left(\|\nabla u\|_{L^2(\Omega)}^2 - \lambda \|u\|_{L^2(\Omega)}^2 \right)^2}{J(u)},$$

$$\text{where } J(u) = \iint_{\Omega \times \Omega} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^4} dx dy.$$

$$\text{Define } \Sigma = \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ u \neq 0}} \frac{\|\nabla u\|_{L^2(\Omega)}^4}{J_0(u)},$$

where $J_0(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^4} dx dy$.

Note that we can get $\sigma(0) \geq \Sigma$ easily, but later we prove $\sigma(0) = \Sigma$. One of the key steps of our argument is the following.

PROPOSITION.4. - There is a minimizer for Σ and the minimizer is (in some sense) uniquely given by the following function

$$U(x) = \frac{1}{(1 + |x|^2)^{(n-2)/2}}. \quad (\text{For any minimizer } u \text{ there are constants } c, d \text{ and } y \in \mathbb{R}^n \text{ such that } u(x) = c U((x - y)/d).)$$

This is a consequence of E.H.Lieb's theorem [L] and the well known fact for Sobolev's inequality. Note that the inequality :

$$J(u) \leq C \|\nabla u\|_{L^2(\Omega)}^4 \quad \text{for any } u \in H^1(\mathbb{R}^n),$$

where C is a constant, is obtained by using Hardy-Littlewood-Sobolev's inequality and Sobolev's inequality. The following Brezis and Lieb's type lemma plays an important role throughout our arguments.

LEMMA.1. - Let $u \in H_0^1(\Omega)$ and $v_n \longrightarrow 0$ weakly in $H_0^1(\Omega)$ ($n \longrightarrow \infty$).

Then there is a limit of $[J(u + v_n) - J(v_n)]$ and

$$\lim_{n \longrightarrow \infty} [J(u + v_n) - J(v_n)] = J(u) \text{ holds.}$$

< Sketch of proof of THEOREM.2 >

Our Strategy is on the same line to that of Brezis and Nirenberg [BN]. Using LEMMA.1, we can prove the next proposition.

PROPOSITION.5. - If $\sigma(\lambda) < \sigma(0)$, then $\sigma(\lambda)$ is attained.

Next we shall prove ,

PROPOSITION.6. - Under the assumption of THEOREM.2, the inequality

$\sigma(\lambda) < \sigma(0)$ holds.

We need some calculations for the proof of PROPOSITION.6. Let

$$Q_\lambda(v) = \frac{(\|\nabla v\|_{L^2(\Omega)}^2 - \lambda \|v\|_{L^2(\Omega)}^2)^2}{J(u)}. \quad \text{Note that there is a}$$

positive constant c_0 such that $U_0(x) = c_0 U(x)$ is a solution of

$$-\Delta U_0 = \left(\frac{1}{|x|^4} * |U_0|^2 \right) U_0 \quad \text{in } \mathbb{R}^n.$$

We may assume that $0 \in \Omega$. Let $\phi_\varepsilon(x) = \xi(x) U_{0,\varepsilon}(x)$, where $\varepsilon > 0$,

$$U_{0,\varepsilon}(x) = \frac{1}{\varepsilon^{(n-2)/2}} U_0\left(\frac{x}{\varepsilon}\right) \text{ and } \xi \in C_0^\infty(\Omega) \text{ is a function such}$$

that $0 \leq \xi \leq 1$, $\xi = 1$ on $B_\rho(0)$ and $\xi = 0$ on $^c B_{2\rho}(0)$. The constant ρ

> 0 should be taken sufficiently small in order to $\phi_\varepsilon \in H_0^1(\Omega)$.

LEMMA.2. - (1) $Q_\lambda(\phi_\varepsilon) \longrightarrow \Sigma$ as $\varepsilon \longrightarrow 0$.

(2) $J(\phi_\varepsilon) = J(U_0) + o(\varepsilon^{n-2})$ (as $\varepsilon \longrightarrow 0$).

(3) $\|\nabla \phi_\varepsilon\|_{L^2(\Omega)}^2 = \|\nabla U_0\|_{L^2(\mathbb{R}^n)}^2 + o(\varepsilon^{n-2})$ (as $\varepsilon \longrightarrow 0$).

(4) There are constants $C_1, C_2 > 0$ such that

$$C_1 \varepsilon^2 \leq \|\phi_\varepsilon\|_{L^2(\Omega)}^2 \leq C_2 \varepsilon^2.$$

(5) There are positive constant δ and $C \in \mathbb{R}^1$ such that

$$Q_\lambda(\phi_\varepsilon) = \Sigma (1 - \delta \varepsilon^2 + C \varepsilon^{n-2} + o(\varepsilon^2)) \text{ as } \varepsilon \longrightarrow 0.$$

(Note that $n > 4$ implies $\varepsilon^{n-2} = o(\varepsilon^2)$.)

Hence (5) implies $Q_\lambda(\phi_\varepsilon) < \Sigma$ for sufficiently small $\varepsilon > 0$.

Therefore we get $\sigma(\lambda) < \Sigma$. $\square$

< Sketch of proof of THEOREM.3 >

$$\begin{aligned} \text{Let } F(u) = & \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{4} \int_{\Omega} \left(\frac{1}{|x|^4} * |u|^2 \right) |u|^2 dx \\ & - \frac{\lambda}{4} \int_{\Omega} \left(\frac{1}{|x|^\gamma} * |u|^2 \right) |u|^2 dx. \end{aligned}$$

We claim that "F satisfies (PS)-condition for $(-\infty, \frac{1}{4}\Sigma)$ ".

PROPOSITION.6. - Assume $n \geq 5$, $0 < \gamma < 4$ and $\lambda > 0$. Then for any sequence $\{u_j\} \subset H^1_0(\Omega)$ satisfying

(a) $F(u_j) \longrightarrow c$, (b) $F'(u_j) \longrightarrow 0$ in $H^{-1}(\Omega)$ and

(c) $c < \frac{1}{4} \Sigma$, there is a subsequence which converges in $H^1_0(\Omega)$ strongly.

Using the function U_0 we can prove

PROPOSITION.7. - Under the assumptions in THEOREM.3, there is a

$v_0 \in H^1_0(\Omega)$ such that $\sup_{t \geq 0} F(t v_0) < \frac{1}{4} \Sigma$.

Using PROPOSITION.6 and PROPOSITION.7, THEOREM.3 can be proved by Mountain Pass Lemma. $\square$

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