

On a distance function between differentiable structures*

Yoshihiro SHIKATA *

1. Let M, N be smooth orientable manifolds with boundary and assume that the boundaries $\partial M, \partial N$ are diffeomorphic each other through a diffeomorphism f . Denote by $C(\partial M), C(\partial N)$ the collar neighbourhoods of $\partial M, \partial N$ respectively and let

$$\alpha : \partial M \times [0, 1) \rightarrow C(\partial M), \quad \beta : \partial N \times [0, 1) \rightarrow C(\partial N)$$

be the diffeomorphisms. Then the map which sends $\alpha(x, t)$ ($x \in \partial M, t \in [0, 1)$) into $\beta(F(x), 1-t)$, defines a diffeomorphism $F = F(f)$ between $C(\partial M), C(\partial N)$ and the identified space $M \bigcup_F N$ turns out to be a smooth manifold.

Lemma 1. Let M_i, N_i ($i = 1, 2$) be smooth manifolds with boundary and let f_1 be a diffeomorphism between ∂M_1 and ∂N_1 . If homeomorphisms $g_1: M_1 \rightarrow M_2$ and $g_2: N_1 \rightarrow N_2$ are diffeomorphic on some neighbourhoods of the closures of collar neighbourhoods $C(\partial M_1), C(\partial N_1)$, then there are collar neighbourhoods $C(\partial M_2), C(\partial N_2)$ and a diffeomorphism F_2 of $C(\partial M_2)$ onto $C(\partial N_2)$ so that $M_2 \bigcup_{F_2} N_2$ is homeomorphic to $M_1 \bigcup_{F_1} N_1$ by a homeomorphism $g_1 \bigcup g_2$ defined by

* During this work, the author is supported by SFB at Bonn University, also he wishes to express his hearty thanks to Prof. Klingenberg for his constant encouragement.

$$g_1 \cup g_2(x) = \begin{cases} g_1(x), & \text{if } x \in M_1 \\ g_2(x), & \text{if } x \in N_1 \end{cases}$$

Proposition 1 Let M_i, N_i, g_i ($i = 1, 2$), f , be as in

Lemma 1. Suppose moreover that with respect to

Riemannian metrics ρ_i, σ_i ($i = 1, 2$) on M_i, N_i respectively, the homeomorphism g_i ($i = 1, 2$) satisfy that

$$\rho_i(x, y)/k_i \leq \sigma_i(g_i(x), g_i(y)) \leq k_i \rho_i(x, y) \\ \text{for } x, y \in M_i,$$

then there exist Riemannian metrics τ_i on $M_i \cup_{F_i} N_i$ ($i = 1, 2$) such that

$$\tau_1(x, y)/\max(k_1, k_2) \leq \tau_2(g_1 \cup g_2(x), g_1 \cup g_2(y)) \\ \leq \max(k_1, k_2) \tau_1(x, y).$$

Proof Take a real valued smooth function φ such that

$$0 \leq \varphi(t) \leq 1, \quad \varphi(t) = 0 \text{ for } t \leq 0, \quad \varphi(t) = 1 \text{ for } t \geq 1,$$

$$0 \leq \varphi'(t) \quad \varphi'(t) = 0 \text{ for } t \leq 0 \text{ or } t \geq 1,$$

$$\varphi(1-t) = 1 - \varphi(t)$$

and let

$$\alpha_1: M_1 \times [0, 1) \rightarrow C(\partial M_1), \quad \beta_1: N_1 \times [0, 1) \rightarrow C(\partial N_1)$$

be diffeomorphisms onto the collar neighbourhoods. Then

$$\alpha_2 = g_1 \circ \alpha_1 ((g_1^{-1}|_{\partial M_2}), \text{id}), \quad \beta_2 = g_2 \circ \beta_1 ((g_2^{-1}|_{\partial N_2}), \text{id})$$

also are diffeomorphism of $\partial M_2 \times [0, 1)$, $\partial N_2 \times [0, 1)$

onto collar neighbourhoods $C(\partial M_2)$, $C(\partial N_2)$, respectively, moreover

~~and~~ the identification map F_2 obtained from α_2 , β_2 , and $(g_2|_{\partial N_1}) \circ f_1 \circ (g_1^{-1}|_{\partial M_2})$ satisfies that

$$g_2 \circ F_1 = F_2 \circ g_1 \quad \text{on } C(\partial M_1).$$

Define quadratic forms $\tilde{\tau}_i$ on $M_i \cup_{F_i} N_i$ ($i = 1, 2$) by

$$(\tilde{\tau}_i)_x = \begin{cases} (\tilde{\rho}_i)_x & , x \in M_i - C(\partial M_i), \\ \varphi(t(x))(\tilde{\rho}_i)_x + (1-\varphi(t(x)))(F_i^* \tilde{\sigma}_i)_x, & x \in C(\partial M_i), \\ (\tilde{\sigma}_i)_x & , x \in N_i - C(\partial N_i). \end{cases}$$

where $t(x)$ denotes the t -coordinate of x in the collar neighbourhood! (and $\sim$ indicates the quadratic form of the metric) Then it is easy to see that the well

defined quadratic forms $\tilde{\tau}_i$ ($i = 1, 2$) give Riemannian metrics τ_i on $M_i \cup_{F_i} N_i$. Since

$$\begin{aligned} \rho_1(x, y)/k_1 &\leq \rho_2(g_1(x), g_1(y)) \leq k_1 \rho_1(x, y) \\ \sigma_1(F_1(x), F_1(y))/k_2 &\leq \sigma_2(\cancel{g_2^{-1}(x)}, g_2 F_1(x), g_2 F_1(y)) \\ &\leq k_2 \sigma_1(F_1(x), F_1(y)) \end{aligned}$$

it holds that

$$\tilde{\rho}_1/k_1 \prec g_1^* \tilde{\rho}_2 \prec k_1 \tilde{\rho}_1,$$

$$F_1^* \tilde{\sigma}_1/k_2 \prec g_1^*(F_2^* \tilde{\sigma}_2) = (g_2 F_1)^* \tilde{\sigma}_2 \prec k_2 F_1^* \tilde{\sigma}_1.$$

Therefore the metrics τ_i satisfy that

$$\tilde{\tau}_1/\max(k_1, k_2) \prec g_1^* \tilde{\tau}_2 \prec \max(k_1, k_2) \tilde{\tau}_1$$

on $C(\partial M_i)$, thus from the construction of $g_1 \cup g_2$ we may conclude that

$$\begin{aligned} \tau_1(x, y)/\max(k_1, k_2) &\leq \tau_2((g_1 \cup g_2)(x), (g_1 \cup g_2)(y)) \\ &\leq \max(k_1, k_2) \tau_1(x, y). \end{aligned}$$

Let M_i ($i = 1, 2$) be smooth manifolds with metrics ρ_i ($i = 1, 2$) and f be a map of M_1 into M_2 , then we define

$\ell(f: \mathcal{P}_1, \mathcal{P}_2)$ by

$$\ell(f: \mathcal{P}_1, \mathcal{P}_2) = \inf \left\{ k \geq 1 / \mathcal{P}_1(x, y) / k \leq \mathcal{P}_2(f(x), f(y)) \right. \\ \left. \leq k \mathcal{P}_1(x, y), \quad \text{for any } x, y \in M \right\}$$

Definition Let $\Sigma_i (i = 1, 2)$ be differential structures on a combinatorial manifold X represented by smooth manifolds $M_i (i = 1, 2)$ with Riemannian metrics $\mathcal{P}_i (i = 1, 2)$. The distance $d(\Sigma_1, \Sigma_2)$ between the differential structures is defined to be

$$d(\Sigma_1, \Sigma_2) = \log \left(\inf \ell(f: \mathcal{P}_1, \mathcal{P}_2) \right),$$

where the infimum is taken over all the piecewise linear equivalences f of M_1 onto M_2 and all the Riemannian metrics $\mathcal{P}_1, \mathcal{P}_2$. It is known ([S]) that d is actually a distance function.

Theorem 1 Let $\Sigma_{i,j} (i = 1, 2, j = 1, 2)$ be differential structures on combinatorial manifolds X_i , $(i=1,2)$ respectively, then it holds that

$$d(\Sigma_{1,1} \# \Sigma_{1,2}, \Sigma_{2,1} \# \Sigma_{2,2}) \leq \max(d(\Sigma_{1,1}, \Sigma_{2,1}), d(\Sigma_{1,2}, \Sigma_{2,2}))$$

where $\Sigma_{i,1} \# \Sigma_{i,2}$ denotes the differential structure obtained by the connected sum.

Proof Represent $\Sigma_{i,j}$ by smooth manifolds $M_{i,j}$, and for

$\varepsilon > 0$ take piecewise diffeomorphisms g_i of $M_{i,1}$ into $M_{i,2}$ and Riemannian metrics $\mathcal{P}_{i,j}$ on $M_{i,j}$ so that

$$\log \ell(g_i; \mathcal{P}_{i,1}, \mathcal{P}_{i,2}) \leq d(\Sigma_{i,1}, \Sigma_{i,2}) + \varepsilon$$

Assume that g_i are diffeomorphic on neighbourhoods of points $p_i \in M_{i,1}$, then after cutting out small imbedded disks around p_i , $M'_{i,j}$ and g_i turns out to satisfy the assumption of Proposition 1 with $k_i = \ell(g_i; \mathcal{P}_{i,1}, \mathcal{P}_{i,2})$.

Since identified manifolds $M'_{1,j} \cup M'_{2,j}$ represent the connected sum $\Sigma_{1,j} \# \Sigma_{2,j}$, we have that

$d(\Sigma_{1,1} \# \Sigma_{2,1}, \Sigma_{1,2} \# \Sigma_{2,2}) \leq \max(\log k_1, \log k_2)$
finishing the proof.

Corollary 1 Let Γ_k be the group of k -dimensional homotopy spheres, then it holds that

$d(\Sigma_1 + \Sigma_3, \Sigma_2 + \Sigma_3) = d(\Sigma_1, \Sigma_2)$
for any $\Sigma_i \in \Gamma_k$ ($i = 1, 2, 3$).

Corollary 2 The subset $\Gamma_k(a)$ of Γ_k given by

$$\Gamma_k(a) = \{ \Sigma \in \Gamma_k / d(S^k, \Sigma) \leq a \}$$

turns out to be a subgroup of Γ_k , where S^k denotes the standard k -sphere.

Corollary 3 Let M_i ($i = 1, 2$) be k -dimensional manifolds such that $M_2 \approx M_1 \# \Sigma$ (diffeomorphic) with $\Sigma \in \Gamma_k(a)$, then

$$d(M_1, M_2) \leq a.$$

Corollary 4 Let $\text{Diff } S^{k-1}$ denote the set of orientation preserving diffeomorphisms onto itself and let π denote the projection of $\text{Diff } S^{k-1}$ onto Γ_k . Take the usual metric $||$ on S^{k-1} induced from that of $R^k \supset S^{k-1}$, then it holds that

$$d(S^k, \pi(f)) \leq \log \ell(f; ||, ||).$$

Proof Extend f radially to a homeomorphism g of disk D^k onto itself which bounds the sphere S^{k-1} and apply Lemma 1 to disks D^k , g , id and f :

$$\begin{array}{ccccc} D^k & \supset & \partial D^k & \xrightarrow{f} & \partial D^k \subset D^k \\ \downarrow g & & & & \downarrow \text{id} \\ D^k & & & & D^k \end{array}$$

to obtain a homeomorphism $g \cup \text{id}$ and a diffeomorphism F_2 of ∂D^k onto itself which can be chosen to be identity. Since it is obvious that

$$\ell(f; ||, ||) = \ell(g; ||, ||),$$

Proposition 1 yields that

$$d(S^{k-1} \cup_{F_2} S^{k-1}, \pi(f)) \leq \log \ell(f; ||, ||).$$

2. The partial converse to Corollary 3 holds as in the following:

Proposition 2 Let f be a homeomorphism between k -dimensional manifolds M_i , ($i = 1, 2$) with Riemannian metrics ρ_i ($i = 1, 2$) and assume that f is diffeomorphic except finite number of points $P_1, \dots, P_m \in M_1$ then $M_2 \approx M_1 \# \Sigma(\text{diffeomorphic})$ with $\Sigma \in \Gamma_k(\log \ell(f; \rho_1, \rho_2))$.

Proof Imbed small k -disks D_i around P_i , then the images $f(D_i)$ turn out to be submanifolds in N . Apply Lemma 1 to manifolds D_i , $f(D_i)$, diffeomorphism $f|_{\partial D_i}$ and homeomorphism id, f^{-1}

$$\begin{array}{ccccc} D_i & \supset & \partial D_i & \xrightarrow{f|_{\partial D_i}} & \partial(f(D_i)) & \subset & f(D_i) \\ \downarrow \text{id} & & & & & & \downarrow f^{-1} \\ D_i & \supset & \partial D_i & \xrightarrow{\text{id}} & \partial D_i & \subset & D_i \end{array}$$

to obtain homotopy spheres $\Sigma_i = D_i \cup_{F_1} f(D_i)$ and a homeomorphism $\text{id} \cup f^{-1}$ between the homotopy sphere and the sphere

S_i . Because of Proposition 1 there are Riemannian metrics σ_1^i, σ_2^i on Σ_i, S_i , respectively, so that

$$\ell(\text{id} \cup f; \sigma_1^i, \sigma_2^i) \leq \ell(f; \rho_1, \rho_2).$$

Therefore we have that

$$\Sigma_i \in \Gamma_k^{(\log \ell)}(f; \rho_1, \rho_2).$$

On the other, since it is easy to see that

$$M_2 \approx M_1 \# \Sigma_1 \# \Sigma_2 \dots \# \Sigma_m,$$

This finishes the proof.

In general concerning the first obstruction of Munkres ([M]) to smoothing f , we obtain the following:

Proposition 3 Let M_i ($i = 1, 2$) be smoothly triangulated manifolds with Riemannian metrics ρ_i ($i = 1, 2$) and let L be a m -dimensional subcomplex of M_1 . If a homeomorphism f of M_1 onto M_2 is diffeomorphic mod. L , and if $\ell(f; \rho_1, \rho_2) < \ell_0 \doteq 1.32$ for the positive root ℓ_0 of $x^3 - x - 1 = 0$, then the first obstruction chain $\lambda(f)$ of Munkres to smoothing f lies in

$$\Gamma_{k-m}^{(\log \ell)} (\ell(f) (1 - (\ell^3(f) - \ell(f))^2)^{-1/4})$$

Proof Munkres obstruction is obtained as follows:

Take an m -simplex $\sigma \in L$ and take trivializations of normal bundles as coordinate systems around σ and $f(\sigma)$ so that the tubular neighbourhoods of σ , $f(\sigma)$ are diffeomorphic to $\sigma \times \mathbb{R}^{k-m}$, $f(\sigma) \times \mathbb{R}^{k-m}$, respectively, then if $\varepsilon > 0$ is sufficiently small, $\pi \cdot f \cdot i_p$ is a homeomorphism of the ε -disk D_ε around 0 into $\mathbb{R}^{k-m}$ for the inclusion $i_p: \mathbb{R}^{k-m} \rightarrow p \times \mathbb{R}^{k-m}$ and for the projection $\pi: f(\sigma) \times \mathbb{R}^{k-m} \rightarrow \mathbb{R}^{k-m}$ thus the obstruction $\lambda(f)(\sigma)$ is defined to be the homotopy sphere obtained by glueing the boundaries of D_ε and $\pi \cdot f \cdot i_p(D_\varepsilon)$ through $\pi \cdot f \cdot i_p$.

Hence it is sufficient for the proof of Proposition 3 to compute $\ell(\pi \cdot f \cdot i_p; \rho_1, \rho_2)$ (see Proposition 1) and because of the regularity of f at L ([M] p.526 (4)) the computation is reduced to the following Assertion;

Assertion Let g be a map between manifolds N_i ($i = 1, 2$) with Riemannian metrics ϕ_i ($i = 1, 2$) satisfying that

$$l(g; \phi_1, \phi_2) < \kappa < l_0$$

then if g is differentiable along any vector of an m dimensional vector space $V \subset T_p(N_1)$, the angle θ between the vector $\exp_2^{-1} \cdot g \cdot \exp_1(y)$, 0 and the plane $dg(V)$ is not too small, in fact θ satisfies that

$$\cos \theta < \kappa^3 - \kappa < 1,$$

for any y in orthogonal linear subspace W to V , provided $|y|$ is sufficiently small.

Proof of Assertion Taking an ε -disk D_ε of 0 in $T_p(N_1)$, we may assume that $\tilde{g} = \exp_2^{-1} \cdot g \cdot \exp_1$ also satisfies that

$$l(\tilde{g}; ||, ||) < \kappa < l_0.$$

Let $x \in V$ be such that $|x| = |y|$, then it holds that

$$\begin{aligned} 2 \langle f(x), f(y) \rangle &= |f(x)|^2 + |f(y)|^2 - |f(x) - f(y)|^2 \\ &< \kappa(|x|^2 + |y|^2) - |x - y|^2 / \kappa \\ &= 2|x|^2 (\kappa - 1/\kappa) \end{aligned}$$

also it holds that

$$2 \langle f(x), f(y) \rangle > 2|x|^2 (1/\kappa - \kappa),$$

therefore we have that

$$|\cos(\overrightarrow{f(x)}, \overrightarrow{f(y)})| < \kappa^3 - \kappa,$$

finishing the proof of Assertion.

Thus taking the regularity of f into consideration, ^(we) may conclude that by an application of Assertion to $g = f \cdot i_p$,

$$\kappa^{-1}(1 - (\kappa^3 - \kappa)^2)^{1/2} \leq \rho_2(\pi f i_p(x), \pi f i_p(y)) / \rho_1(x, y) \leq \kappa$$

On a small disk around 0, completing the proof of Proposition 3.

3. The method in 1, 2 applies to obtain a weak estimation of the pinching of a exotic sphere. Let M_1, M_2 be combinatorially equivalent compact manifolds, then according to the construction of Hirsch-Munkres (H), we may have a sequence of compact manifolds L_i ($i=1 \dots k$) such that

- i) L_i are combinatorially equivalent to M_1, M_2 .
- ii) $L_1 = M_1, L_k = M_2$ (diffeomorphic).
- iii) L_{i+1} is obtained by attaching of $\Sigma^j \times I^{n-j}$ to L_i through a certain attaching map. ($\Sigma^j \in \Gamma^j$).

Now suppose M_1, M_2 have different (integral) Pontrjagin class, then for some i, L_i, L_{i+1} have also different Pontrjagin classes. Since we know that manifolds having different Pontrjagin classes are of distance $\geq 1/2 \log 3/2$ (S_2), we have that

$$\begin{aligned}
 (1) \quad 1/2 \log 3/2 &\leq d(L_i, L_{i+1}) \\
 &\leq \max(d(L_i, L_i), d(S^j \times I^{n-j}, \Sigma^j \times I^{n-j})) \\
 &\leq d(S^j, \Sigma^j).
 \end{aligned}$$

Here the last inequality follows from an easily proved Lemma below:

Lemma 2 If M_i, N_i denote a pair of combinatorially equivalent compact manifolds ($i=1, 2$) then

$$d(M_1 \times M_2, N_1 \times N_2) \leq \max(d(M_1, N_1), d(M_2, N_2))$$

On the other as is improved by Karcher (unpublished, see also (S_3)) δ -pinched Riemannian manifold M_δ ($\delta \geq 9/16$) has distance $4(1-\sqrt{\delta})$ from the standard sphere S , therefore if the exotic sphere Σ^j in (1) is expressed as a δ -pinched manifold M_δ , δ must satisfy that

$$1/2 \log 3/2 \leq 4(1 - \sqrt{\delta}).$$

hence

$$\delta \leq 0.64,$$

thus we may conclude that a certain exotic sphere of dimension ≤ 16 which appears in the obstruction chain to smoothing a combinatorial equivalence can not be pinched by 0.64, because we know that there are compact 16 manifolds having different Pontrjagin classes.

References

- (M) J. Munkres On the smoothing of.....
Ann. of Math. '60 521-554
- (S) Y. Shikata On a distance function.....
Osaka J. Math. '66 293-301
- (S_2) Y. Shikata On Pontrjagin classes.....
J. of Math. Osaka City Univ. '63 73-86
- (S_3) Y. Shikata On the differentiable pinching
Osaka J. Math. '67 279-287