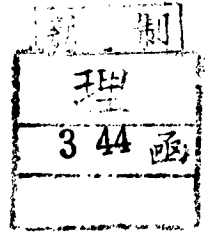




學位申請論文

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An Admissible Data Class of the Cauchy Problem
for Non-strictly Hyperbolic Operators

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§1. Introduction.

In this paper we shall study the Cauchy problem

$$(1.1) \quad \begin{aligned} P(x, t, D_x, D_t)u(x, t) &= f(x, t), \quad (x, t) \in \Omega \\ D_t^j u|_{t=0} &= u_j(x), \quad 0 \leq j \leq m-1 \end{aligned}$$

where $\Omega = \mathbb{R}^n \times [0, h]$, $h > 0$, $x = (x_1, \dots, x_n)$ and

$$(1.2) \quad P = D_t^m + \sum_{\substack{|v|+j \leq m \\ j \leq m-1}} a_{vj}(x, t) D_x^v D_t^j.$$

Here and hereafter we use the standard notations: $D_t = \frac{1}{i} \frac{\partial}{\partial t}$,

$D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, $v = (v_1, \dots, v_n)$ are multi-indices, $D_x^v = D_1^{v_1} \dots D_n^{v_n}$,

$|v| = v_1 + \dots + v_n$. We always assume P hyperbolic. Namely, letting

$\{\lambda_j(x, t, \xi)\}_{j=1, \dots, m}$ be the roots of the characteristic equation

$$(1.3) \quad P_m(x, t, \xi, \tau) = \tau^m + \sum_{|v|+j=m} a_{vj}(x, t) \xi^v \tau^j = 0$$

$(x, t) \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$, we assume that $\lambda_j(x, t, \xi)$ are real-valued.

Here we do not assume that $\{\lambda_j\}_{j=1, \dots, m}$ are distinct.

There are two approaches to this problem:

1. Find conditions on P for the Cauchy problem (1.1) to be C^∞ well-posed.
2. Given P , determine a class of data ($\subset C^\infty$) for which the Cauchy problem (1.1) has a unique solution in C^∞ .

In this paper we shall follow the second approach, introducing the so-called Gevrey class $\gamma_{loc}^{(\kappa)}$.

Definition 1.1 ($\gamma_{loc}^{(\kappa)}, \gamma_{loc}^{(\kappa)}; \kappa > 1$). $f(x) \in \gamma_{loc}^{(\kappa)}$ implies that for any compact set K there exist constants ρ and C such that

$$(1.4) \quad |D_x^\alpha f(x)| \leq C \rho^{-|\alpha|} (|\alpha|!)^\kappa, \quad x \in K, \quad \text{for any } \alpha.$$

$f(x) \in \gamma_{loc}^{(\kappa)}$ implies that (1.4) holds for any x .

It is convenient to denote the real-analytic class and the C^∞ -class by $\gamma_{loc}^{(1)}$ and $\gamma_{loc}^{(\infty)}$ respectively. Clearly, $\gamma_{loc}^{(1)} \subset \gamma_{loc}^{(\kappa)} \subset \gamma_{loc}^{(\kappa')} \subset \gamma_{loc}^{(\infty)}$, if $1 < \kappa < \kappa' < \infty$.

Let us say that $\gamma_{loc}^{(\kappa)}$ is an admissible data class (of the Cauchy problem (1.1)), if for any $u_j \in \gamma_{loc}^{(\kappa)}$ ($0 \leq j \leq m-1$) and any $f \in \gamma_{loc}^{(\kappa)}$ there exists a unique solution $u \in C^\infty$ of the Cauchy problem (1.1). Then it will be our problem to know whether $\gamma_{loc}^{(\kappa)}$ is an admissible data class of the problem (1.1) or not.

Remarking that if $\gamma_{loc}^{(\kappa)}$ is admissible and $\kappa' < \kappa$ then $\gamma_{loc}^{(\kappa')}$ is also admissible, we define $\kappa_0(P)$ by

(1.5) $\kappa_0(P) = \sup\{\kappa; \gamma_{loc}^{(\kappa)} \text{ is an admissible data class of the Cauchy problem (1.1)}\}.$

We call it the Gevrey index of P . Here we remark that if the coefficients of P are analytic or nearly analytic, then the set of κ 's in ~~the above~~ ^(1.5) is not empty, cf. [1]~[6].

It is an interesting problem to characterize $\kappa_0(P)$ in terms of the properties of the symbol of P . This is our theme in the present paper.

Up to now we already know the following results:

i) If the multiplicity of the characteristic roots is at most r (≥ 2) and the coefficients of P belong to some $\gamma^{(\kappa_1)}_{loc}$ ($1 < \kappa_1 < \frac{r}{r-1}$), then

$$(1.6) \quad \kappa_0(P) \geq \frac{r}{r-1}.$$

This theorem is due to Y. Ohya [1], J. Leray and Y. Ohya [2], R. Beals [3], V. Ja. Ivrii [4], J. M. Trepreau [5] and M.D. Bronshtein [6]. We should remark that if $u_j \in \gamma^{(\kappa_2)}_{loc}$ ($0 \leq j \leq m-1$) and $f \in \gamma^{(\kappa_2)}_{loc}$ ($\kappa_1 \leq \kappa_2 < \frac{r}{r-1}$), then the solution also belongs to the same $\gamma^{(\kappa_2)}_{loc}$.

ii) In the case of constant multiplicity, the characterization of $\kappa_0(P)$ was performed by H. Komatsu [7] and De Paris and Wagschal [8], cf. V. Ja. Ivrii [9] also. Concerning the case

where the principal part of the operator has constant coefficients, see S. Wakabayashi [10].

iii) V. Ja. Ivrii [11] gave some necessary conditions for $\gamma_{loc}^{(\kappa)}$ to be an admissible data class of (1.1).

In this article we will consider the operators with characteristics of variable multiplicity, assuming its multiplicity at most 2, and will give a new characterization of $\kappa_0(P)$, which is more precise than (1.6).

§2. Statement of results.

To state our results we use some notations.

Definition 2.1 ($\Gamma^{(\kappa)}$, $\Gamma^{k,(\kappa)}$; $\kappa \geq 1$, $k=0,1,\dots,\infty$). We say that $f(x)$ belongs to $\Gamma^{(\kappa)}$ if there exist constants ρ and C such that

$$(2.1) \quad \|D_x^\alpha f\| \leq C \rho^{-|\alpha|} |\alpha|!^\kappa, \quad \text{for any } \alpha.$$

We say also that $g(x,t)$ belongs to $\Gamma^{k,(\kappa)}$ if there exist constants ρ and C such that

$$(2.2) \quad \sup \|D_x^\alpha D_t^j g(\cdot, t)\| \leq C \rho^{-(|\alpha|+j)} (|\alpha|+j)!^\kappa, \quad 0 \leq t \leq h,$$

for any α, j ($j \leq k$). Here $\|\cdot\|$ denotes the L_x^2 -norm.

It follows easily from Sobolev's lemma that

$$(2.3) \quad \begin{aligned} \gamma^{(\kappa)} \cap C_0 &\subset \Gamma^{(\kappa)} \subset \gamma^{(\kappa)}, \\ \gamma^{(\kappa)}(\Omega) \cap C_0(\Omega) &\subset \Gamma^{\infty,(\kappa)} \subset \gamma^{(\kappa)}(\Omega). \end{aligned}$$

Definition 2.2 ($B_t^k[\overset{\circ}{S}^r(\kappa)]$). We say that $h(x,t,\xi)$ belongs to $B_t^k[\overset{\circ}{S}^r(\kappa)]$, if i): $h(x,t,\xi)$ is homogenous of degree r in ξ , and ii): there exists a constant ρ such that

$$(2.4) \quad |D_t^j D_x^\beta D_\xi^\alpha h(x,t,\xi)| \leq C_{j\alpha} \rho^{-|\beta|} |\beta|!^\kappa, \quad (x,t) \in \Omega, \quad |\xi|=1,$$

for any j ($\leq k$) and any α, β , where $C_{j\alpha}$ are constants independent of β .

We consider the Cauchy problem (1.1) under the following condition.

Condition (C). The following (C-1), (C-2) and (C-3) are satisfied:

$$(C-1) \quad a_{vj}(x,t) \in \gamma^{(\kappa)}.$$

(C-2) The characteristic polynomial $P_m(x,t,\xi,\tau)$ can be decomposed as

$$(2.5) \quad P_m = \prod_{j=1}^{m-s} (\tau - \lambda_j(x,t,\xi)) \prod_{j=1}^s (\tau - \mu_j(x,t,\xi)), \quad s \leq 2m,$$

where λ_j, μ_j are real-valued and belong to $B_t^{m-1}[\overset{\circ}{S}^1(\kappa)]$. Furthermore there exists a positive constant δ such that if $j \neq k$ then

$$(2.6) \quad |\lambda_j - \lambda_k| \geq \delta, \quad |\mu_j - \mu_k| \geq \delta, \quad |\lambda_j - \mu_k| \geq \delta, \quad (x,t) \in \Omega, \quad |\xi|=1.$$

(C-3) For each $0 \leq j \leq s$, there exists $a_j(x,t,\xi) \in B_t^0[\overset{\circ}{S}^0(\kappa)]$ such that

$$(2.7) \quad \{\tau - \mu_j, \tau - \lambda_j\} = t^{-1}(\mu_j - \lambda_j)a_j,$$

where $\{ , \}$ denotes the Poisson's bracket.

Now, let $L_j(x, t, \xi)$ be the Levi's functions, namely

$$(2.8) \quad L_j(x, t, \xi) = P'_{m-1}(x, t, \xi, \frac{\mu_j + \lambda_j}{2}), \quad j=1, \dots, s$$

where $P'_{m-1} = \frac{1}{2} \partial_\tau D_t P_m + \frac{1}{2} \sum_{j=1}^n \partial_{\xi_j} D_{x_j} P_m - P_{m-1}$; $\partial_\tau = \frac{\partial}{\partial \tau}$, $\partial_{\xi_j} = \frac{\partial}{\partial \xi_j}$ and P_{m-1} is the homogenous part of degree $m-1$ in (ξ, τ) of P . We define the numbers $\{\rho_j, \sigma_j\}_{j=1, \dots, s}$ as follows.

$$(2.9) \quad \rho_j = \inf\{\rho \geq 1; \frac{t^{\rho} L_j}{\mu_j - \lambda_j} \in B_t^0[S^{m-2}(\kappa)]\},$$

where, if the set of ρ in the right side is empty, $\rho_j = \infty$.

$$(2.10) \quad \sigma_j = \sup\{\sigma \geq 0; t^{-\sigma} L_j \in B_t^0[S^{m-1}(\kappa)]\},$$

where, if the set of σ in the right side coincides with $R^+ = \{\sigma; 0 \leq \sigma < \infty\}$, then $\sigma_j = \infty$.

Let $\kappa^* = \kappa^*(P)$ be the number defined by

$$(2.11) \quad \kappa^* = \min_{1 \leq j \leq s} \frac{2\rho_j + \sigma_j}{\rho_j - 1},$$

where $\kappa^* = 2$ if $\rho_j = \infty$ for some j (even though $\sigma_j = \infty$), and $\kappa^* = \infty$ if $\rho_j = 1$ for every j . Then the following theorem holds.

Theorem. Under the condition (C), if $1 \leq \kappa < \kappa^*$, then the Cauchy problem (1.1) is $\Gamma^{(\kappa)}$ well-posed. That is to say, for any $u_j \in \Gamma^{(\kappa)}$ ($0 \leq j \leq m-1$) and any $f \in \Gamma^{k, (\kappa)}$, there exists a unique solu-

tion $u \in \Gamma^{m+k,(\kappa)}$ of the Cauchy problem (1.1). Here $k=0,1,\dots,\infty$.

Corollary. Suppose that κ is equal to some κ_1 ($1 < \kappa_1 < 2$) in the condition (C) and also in the definitions of ρ_j , σ_j and $\kappa^*(P)$. Then, under the condition (C),

$$(2.12) \quad \kappa_0(P) \geq \kappa^*(P) \geq 2.$$

Moreover, if $u_j \in \gamma_{loc}^{(\kappa_2)}(0 \leq j \leq m-1)$, $f \in \gamma_{loc}^{(\kappa_2)}$ and $\kappa_1 \leq \kappa_2 < \kappa^*(P)$, then the solution u of (1.1) also belongs to the same $\gamma_{loc}^{(\kappa_2)}$.

Proof of Corollary. Since the multiplicity of the characteristic roots is at most 2 and the coefficients of P belong to $\gamma^{(\kappa_1)}(1 < \kappa_1 < 2)$, we already know the existence of the domain of finite dependence, cf. [2], [4]. Therefore, using a partition of unity, one can easily verify that this corollary follows from the above theorem. (q.e.d)

We give here some remarks on Theorem and Corollary.

Remark 1. If $1 \leq \kappa < 2$, the consequence of Theorem remains true without assuming the condition (C-3), cf. [2].

Remark 2. Consider the case:

$$tL_j(\mu_j - \lambda_j)^{-1} \in B_t^0[\overset{\circ}{S}^{m-2}(\kappa)], \quad 1 \leq j \leq s,$$

where $\rho_j=1$ and consequently $\kappa^*(P)=\infty$. In this case, even if $\kappa=\kappa^*$ the consequence of Theorem remains true, namely the Cauchy problem (1.1) is $\Gamma^{(\infty)}$ well-posed. Here $\gamma^{(\infty)}=B$, $\Gamma^{(\infty)}=D_{L^2}^\infty$ and $B_t^k[\overset{\circ}{S}^r(\infty)]$

is the usual symbol class $B_t^k[\overset{\circ}{S}^r]$ of pseudo-differential operators. Cf. [12], [13] and [14].

Remark 3. Suppose that κ is equal to some κ_1 ($1 < \kappa_1 < 2$) in the conditions (C-1) and (C-2), and under these two conditions consider the case: for every j ($1 \leq j \leq s$) there exist $\theta_j > 0$ and $\delta_j(x, t, \xi) \in B_t^1[\overset{\circ}{S}^1(\kappa_1)]$ such that

$$(2.13) \quad \mu_j - \lambda_j = t^{\theta_j} \delta_j, \quad \inf |\delta_j(x, t, \xi)| > 0, \quad (x, t) \in \Omega, \quad |\xi| = 1.$$

In this case, the condition (C-3) is automatically satisfied. Hence it follows from Corollary that, whatever the lower order terms of P may be,

$$(2.14) \quad \kappa_0(P) \geq \min_{1 \leq j \leq s} 2\theta_j / (\theta_j - 1), \quad \theta_j = \max\{1, \theta_j\}.$$

Remark 4. Consider the following second order hyperbolic differential operator as an example.

$$(2.15) \quad P = D_t^2 - t^{2\ell} \sum_{j,k=1}^n a_{jk} D_j D_k - b_0 D_t - t^m \sum_{j=1}^n b_j D_j - c$$

where $\inf_{\substack{(x,t) \in \Omega \\ |\xi|=1}} \sum a_{jk}(x,t) \xi_j \xi_k > 0$, $\sum b_j(x,0) \xi_j \neq 0$, 2ℓ and m are non-

negative integers. If $\ell - 1 \leq m$, then the Cauchy problem for this operator is C^∞ well-posed (see the remark 2). Therefore we

consider the case of $\ell - 1 > m$, assuming that all of the coefficients belong to some $\gamma^{(\kappa_1)}_{-1}$ ($1 < \kappa_1 < 2$).

We see easily that the condition (C) is satisfied and that

$\kappa^*(P) = (2\ell - m)/(\ell - m - 1)$ for this operator. On the other hand, from Ivrii [11] we see also that if the coefficients are analytic and $\kappa > (2\ell - m)/(\ell - m - 1)$, then $\gamma_{\text{loc}}^{(\kappa)}$ is not an admissible data class. Thus, in this case $\kappa^*(P)$ coincides with $\kappa_0(P)$.

Our plan of the proof of Theorem is as follows. Modifying μ_j and λ_j near $\xi=0$, we may suppose that

$$(2.16) \quad \begin{aligned} &\mu_j(x, t, \xi), \lambda_j(x, t, \xi) \in B_t^{m-1}[S^1(\kappa)] \\ &|\mu_j - \mu_k| \geq \delta \langle \xi \rangle, |\lambda_j - \lambda_k| \geq \delta \langle \xi \rangle, |\mu_j - \lambda_k| \geq \delta \langle \xi \rangle, \text{ if } j \neq k, \end{aligned}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $B_t^{m-1}[S^1(\kappa)]$ is a symbol class of pseudo-differential operators which we introduce in the Appendix. Set

$$(2.17) \quad \begin{aligned} \Delta_i &= D_t - \mu_i(x, t, D_x), \quad i=1, \dots, m, \\ \Pi_i &= \Delta_i \Delta_{i-1} \dots \Delta_1, \quad i=1, \dots, m, \end{aligned}$$

where $\mu_{s+i} = \lambda_i$ ($i=1, \dots, m-s$) and $\mu_i(x, t, D_x)$ is the pseudo-differential operator with symbol $\mu_i(x, t, \xi)$.

Since, with no loss of generality, we may suppose the initial data to be null, we consider the Cauchy problem

$$(1.1') \quad \Pi_m u = (\Pi_m - P)u + f, \quad \text{null initial data.}$$

We construct its solution by the method of successive approximation. Let u^i be the solution of

$$(E)^i \quad \Pi_m u^i = f^{i-1}, \quad \text{null initial data,}$$

where $f^0 = f$, $f^{i-1} = (\Pi_m - P)u^{i-1}$ ($i \geq 2$). In §3 we will derive two types

of energy estimates of the solution of

$$(E) \quad \Pi_m u = f, \quad \text{null initial data.}$$

In §4, combining these estimates by the Hölder's inequality (Lemma 4.1), we will obtain the successive estimates of u^i and show the convergence of $\sum_{i=1}^{\infty} u^i$ under the assumption $\kappa < \kappa^*(P)$, which will give the desired solution of (1.1').

In the Appendix we will introduce a class of pseudo-differential operators with Gevrey coefficients and give some results on its calculus, which we use throughout this paper.

§3. Estimates of the solution of (E).

In this section we are going to obtain two types of estimates of the solution u of (E). The reasoning done below owes much to Mizohata-Ohya [15], Ohya [1] and Nishitani [13].

3.1. Reduction to a system, I. Set

$$(3.1) \quad \begin{aligned} u_0 &= u, \quad u_i = \Lambda_i u_{i-1}, \quad i=1, \dots, m-1, \\ U &= {}^t(u_0, u_1, \dots, u_{m-1}), \quad F = {}^t(0, \dots, 0, f), \\ E &= \begin{pmatrix} \Lambda^{m-2} & & & \\ & \ddots & & \\ & & \Lambda^{m-s-1} & \\ & & & \Lambda^{m-s-1} \\ & & & & 1 \end{pmatrix} \end{aligned}$$

where $\Lambda u = (2\pi)^{-n} \int e^{ix\xi} \langle \xi \rangle \hat{u}(\xi) d\xi$, $\hat{u}(\xi)$ is the Fourier transform of

$u(x)$. Then from (E) we have

$$(3.2) \quad D_t^E U = H E U + B^1 E U + F;$$

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & \Lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \Lambda \\ & & & \mu_s \\ \hline & & & & \lambda_1 & \Lambda & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \Lambda \\ & & & & & & & \lambda_{m-s} \end{pmatrix},$$

$$B^1 = \begin{pmatrix} B_{11}^1 & B_{12}^1 \\ 0 & B_{22}^1 \end{pmatrix} = \begin{pmatrix} b_{11}^1 & & & \\ & \ddots & & \\ & & b_{ss}^1 & \\ \hline & & & b_{s+1}^1 & & \\ & & & & \ddots & \\ & & & & & b_m^1 \end{pmatrix}, \quad b_j^1 \in B_t^0[S^0(\kappa)].$$

$H_1(x, t, \xi)$ is diagonalized by

$$(3.3) \quad N_1(x, t, \xi) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & n_{ij} & \\ 0 & & & 1 \end{pmatrix}; \quad n_{ij} = \langle \xi \rangle^{j-i} \prod_{k=i+1}^j (\mu_i - \mu_k)^{-1} \in B_t^0[S^0(\kappa)].$$

$N_1(x, t, D_x)$ has its left inverse (in the operator sense) of the form:

$$(3.4) \quad N_1^{-1}(x, t, D_x) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & r_{ij} & \\ 0 & & & 1 \end{pmatrix},$$

where $r_{ij}(x, t, D_x)$ is determined by the recurrence formula:

$$(3.5) \quad r_{ij} = -n_{ij} - r_{i,i+1} n_{i+1,j} - \dots - r_{i,j-1} n_{j-1,j} \in B_t^0[S^0(\kappa)],$$

and therefore $r_{ij}(x, t, \xi)$ is represented as

$$(3.5') \quad r_{ij}(x, t, \xi) = \langle \xi \rangle^{j-i} \prod_{k=i}^{j-1} (\mu_j - \mu_k)^{-1} + r'_{ij}, \quad r'_{ij} \in B_t^0[S^{-1}(\kappa)].$$

The diagonalizer N_2 of H_2 and its left inverse N_2^{-1} are obtained in the same way.

Hence, setting

$$(3.6) \quad N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \quad N^{-1} = \begin{pmatrix} N_1^{-1} & 0 \\ 0 & N_2^{-1} \end{pmatrix},$$

from (3.2) we have

$$(3.7) \quad D_t \text{NEU} = D \text{NEU} + B^2 \text{NEU} + N F;$$

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & & & \\ & \ddots & & \\ & & \mu_s & \\ & & & \lambda_1 & \\ & & & & \ddots \\ & & & & & \lambda_{m-s} \end{pmatrix}, \quad B^2 = \begin{pmatrix} B_{11}^2 & B_{12}^2 \\ 0 & B_{22}^2 \end{pmatrix},$$

where $B^2 = \{NH - DN + D_t N + NB^1\} N^{-1}$, which is, consequently B_{11}^2 and B_{22}^2 are also, trigonometric matrices whose components belong to $B_t^0[S^0(\kappa)]$.

It will be important in the next paragraph 3.2 to reduce the order of the off-diagonal components of B_{11}^2 to -1. For this purpose we diagonalize the matrix

$$\begin{pmatrix} \mu_1 & & & \\ & \ddots & & \\ & & b_{ij} & \\ 0 & & & \mu_s \end{pmatrix} (x, t, \xi); \quad b_{ij} \in B_t^0[S^0(\kappa)] \text{ are off-diagonal components of } B_{11}^2.$$

This matrix is diagonalized by

$$N_1' = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & n_{ij}' & \\ 0 & & & 1 \end{pmatrix}; \quad n_{ij}'(x, t, \xi) \in B_t^0[S^{-1}(\kappa)],$$

where n_{ij}' ($i < j$) is determined recurrently by

$$n_{ij}' = (b_{ij} + n_{i,i+1}' b_{i+1,j}' + \dots + n_{i,j-1}' b_{j-1,j}') (\mu_i - \mu_j)^{-1}$$

The left inverse (in the operator sense) of $N_1'(x, t, D_x)$ is

$$N_1'^{-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & r_{ij}' & \\ 0 & & & 1 \end{pmatrix}, \quad r_{ij}'(x, t, D_x) \in B_t^0[S^{-1}(\kappa)],$$

where r_{ij}' is determined by the same formula as (3.5).

Hence, if we set

$$(3.8) \quad N' = \begin{pmatrix} N_1' & 0 \\ 0 & I \end{pmatrix}, \quad N'^{-1} = \begin{pmatrix} N_1'^{-1} & 0 \\ 0 & I \end{pmatrix}, \quad V = N' N E U, \quad G = N F,$$

then (3.7) is reduced to

$$(E') \quad D_t V = D V + B V + G;$$

$$B = \{N'D - D N' + D_t N' + N' B^2\} N'^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

where B is a trigonometric matrix, the off-diagonal components of B_{11} belong to $B_t^0[S^{-1}(\kappa)]$ and the other components of B belong to $B_t^0[S^0(\kappa)]$.

3.2. Reduction to a system, II. Set

$$(3.9) \quad \delta_i = \mu_i - \lambda_i, \quad i = 1, \dots, s,$$

$$Q_1 = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_s \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}.$$

Then it follows from (E') that

$$(3.10) \quad D_t QV = DQV + B^3 V + G;$$

$$B^3 = \begin{pmatrix} B_{11}^3 & B_{12}^3 \\ 0 & B_{22}^3 \end{pmatrix} = \begin{pmatrix} [Q_1, D_1] + D_t Q_1 + Q_1 B_{11} & Q_1 B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

B^3 is trigonometric. The diagonal components of B_{11}^3 are

$$[\delta_i, \mu_i] + D_t \delta_i + \delta_i b_{ii}$$

$$= [D_t - \mu_i, D_t - \lambda_i] + \delta_i b_{ii}.$$

Hence, owing to the condition (C-3), there are $b_i, b_i' \in B_t^0[S^0(\kappa)]$ such that

$$(3.11) \quad (i,i) \text{ component of } B_{11}^3 = t^{-1} b_i \delta_i + t^{-1} b_i'.$$

Since the off-diagonal components of B_{11} belong to $B_t^0[S^{-1}(\kappa)]$, the off-diagonal components of B_{11}^3 belong to $B_t^0[S^0(\kappa)]$.

Therefore, (3.10) is written as

$$(3.10) \quad \begin{aligned} D_t QV &= \begin{pmatrix} D_1 & Q_1 \circ B_{12} \\ 0 & D_2 \end{pmatrix} QV + t^{-1} \begin{pmatrix} B'_1 & 0 \\ 0 & 0 \end{pmatrix} QV \\ &+ \begin{pmatrix} t^{-1} B'_{11} & B'_{12} \\ 0 & B_{22} \end{pmatrix} V + G, \end{aligned}$$

where $B'_{12} = Q_1 B_{12} - Q_1 \circ B_{12}$ and every component of B'_1 , B'_{11} , B'_{12} and B_{22} belongs to $B_t^0[\mathcal{B}^0(\kappa)]$.

The symbol of $\begin{pmatrix} D_1 & Q_1 \circ B_{12} \\ 0 & D_2 \end{pmatrix}$ is diagonalized by

$$(3.12) \quad \tilde{N}(x, t, \xi) = \begin{pmatrix} I & \tilde{N}_{12} \\ 0 & I \end{pmatrix}; \quad \tilde{n}_{ii} = b_{ii}, \quad \tilde{n}_{ij} = \delta_{ij} b_{ij} (\mu_i - \lambda_j)^{-1},$$

where $\tilde{n}_{ij}$ and b_{ij} denote (i, j) component of $\tilde{N}_{12}$ and B_{12} , respectively. The inverse operator of $\tilde{N}(x, t, D_x)$ is $\tilde{N}^{-1} = \begin{pmatrix} I & -\tilde{N}_{12} \\ 0 & I \end{pmatrix}$.

Hence, if we set

$$(3.13) \quad W = \tilde{N} QV, \quad \tilde{B}' = \tilde{N} \begin{pmatrix} B'_{11} & t B'_{12} \\ 0 & t B_{22} \end{pmatrix}, \quad \tilde{G} = t^{-1} \tilde{B}' V + \tilde{N} G,$$

then we have

$$(E) \quad D_t W = DW + t^{-1} \tilde{B} W + \tilde{G},$$

where $B \in B_t^0[S^0(\kappa)]$.

3.3. Energy estimate, I. Our aim in this paragraph is to obtain an energy estimate of the solution of

$$(E') \quad D_t V = DV + BV + G, \quad V|_{t=0} = 0.$$

It is easy to see that

$$(3.14) \quad \frac{d}{dt} \|\Lambda^r V\| \leq K_0 \|\Lambda^r V\| + \|[\Lambda^r, D]V\| + \|\Lambda^r BV\| + \|\Lambda^r G\|,$$

for any $r \geq 0$, where $K_0 = \|D - D^*\|_{\mathcal{L}(L^2, L^2)}$. Here D belongs to $B_t^0[S^1(\kappa)]$

and B to $B_t^0[S^0(\kappa)]$.

The following lemma holds.

Lemma 3.1. There exist constants ρ and K such that

$$(3.15) \quad \begin{aligned} \|[\Lambda^r, D]V\| &\leq \sum_{j=1}^{r^*} \binom{r^*}{j} (2\rho)^{-j+1} (j-1)!^K K \|\Lambda^{r+1-j} V\| \\ &\quad + (2\rho)^{-r} r!^K K \|V\|, \end{aligned}$$

for any $r > 0$, and

$$(3.16) \quad \|\Lambda^r BV\| \leq \sum_{j=0}^{r^*-1} \binom{r^*}{j} (2\rho)^{-j} j!^K K \|\Lambda^{r-j} V\| + (2\rho)^{-r} r!^K K \|V\|,$$

for any $r \geq 0$, where $r! = \Gamma(r+1)$ and r^* denotes the smallest integer larger than or equal to r .

The proof of this lemma will be given in the Appendix.

Using this lemma, we have

$$(3.17) \quad \begin{aligned} \frac{d}{dt} \|\Lambda^r V\| \leq (K_0 + r^* K + K) \|\Lambda^r V\| + \sum_{j=2}^{r^*} \binom{r^*+1}{j} (2\rho)^{-j+1} (j-1)!^K K \|\Lambda^{r+1-j} V\| \\ + 2(2\rho)^{-r} r!^K K \|V\| + \|\Lambda^r G\|, \end{aligned}$$

for any $r \geq 0$, where, in case of $r=0$, the 3-rd term in the right side does not appear.

Now, we assume that

$$(3.18) \quad \|\Lambda^r G(\cdot, t)\| \leq t^s \rho^{-(r+p)} (r+p)!^K A K_r(t), \quad \text{for any } r \geq 0,$$

where $s (\geq 0)$, $p (\geq 1)$ and $A (\geq 0)$ are parameters and

$K_r(t) = \exp\{K_0 + 2(r^*+1)K\}t$. For simplicity, we use the abbreviation:

$$(3.19) \quad w_r(s, p; t) = t^s \rho^{-(r+p)} (r+p)!^K K_r(t).$$

Then the following proposition holds.

Proposition 3.1. Under the assumption (3.18), the solution V of (E') satisfies the inequality:

$$(3.20) \quad \|\Lambda^r V(\cdot, t)\| \leq \frac{C}{s+1} A w_r(s+1, p; t), \quad r \geq 0,$$

where C is a certain constant independent of s , p , A and r .

Proof. When $r=0$, the 2-nd and 3-rd terms in the right side of (3.17) do not appear. Therefore

$$\|V\| \leq \int_0^t e^{(K_0+K)(t-u)} A w_0(s,p;u) du \leq \frac{A}{s+1} w_0(s+1,p;t).$$

Next, supposing that (3.20) is valid for any $0 \leq r \leq n$, we show that (3.20) is valid also for $n < r \leq n+1$. Here n is an arbitrary non-negative integer. It follows from (3.17) that

$$\begin{aligned} \|A^r V\| &\leq \int_0^t e^{(K_0+(r^*+1)K)(t-u)} \left\{ \sum_{j=2}^{r^*} \binom{r^*+1}{j} (2\rho)^{-j+1} (j-1)!^{\kappa_K} \right. \\ &\quad \times \frac{C}{s+1} A w_{r-j+1}(s+1,p,u) + 2(2\rho)^{-r} r!^{\kappa_K} \frac{A}{s+1} w_0(s+1,p;u) \\ &\quad \left. + w_r(s,p;u) \right\} du \\ &\leq \frac{A}{s+1} w_r(s+1,p;t) \left\{ \sum_{j=2}^{r^*} \binom{r^*+1}{j} \binom{r+p}{j-1}^{-\kappa} \frac{2^{-j+1} \kappa C}{(r^*+1)K} \right. \\ &\quad \left. + \binom{r+p}{r}^{-\kappa} \frac{2^{-r} 2K}{(r^*+1)K} + 1 \right\}. \end{aligned}$$

Since $\kappa \geq 1$ and $p \geq 1$,

$$\{\dots\dots\} \leq C \sum_{j=2}^{r^*} \frac{r^*+1}{j} \frac{2^{-j+1}}{r^*+1} + 2 \leq \frac{C}{2} + 2.$$

Therefore it is sufficient if we take $C=4$.

(q.e.d.)

By the way, the first s components of G in (E') are null. Hence, putting $V = {}^t(v_1, \dots, v_m)$, $V_1 = {}^t(v_1, \dots, v_s)$ and $V_2 = {}^t(v_{s+1}, \dots, v_m)$, we have

$$(3.21) \quad D_t V_1 = D_1 V_1 + B_{11} V_1 + B_{12} V_2.$$

The following lemma will often be used hereafter.

Lemma 3.2. If $B \in B_t^0[s^0(\kappa)]$ satisfies (3.16) in the Lemma 3.1 and V satisfies the inequality:

$$\|\Lambda^r V(\cdot, t)\| \leq A w_r(s, p; t), \quad \text{for any } r \geq 0,$$

then it holds that

$$\|\Lambda^r B V(\cdot, t)\| \leq C A w_r(s, p; t), \quad \text{for any } r \geq 0,$$

where C is a certain constant independent of s , p , A and r .

Proof. It follows from (3.16) that

$$\begin{aligned} \|\Lambda^r B V\| &\leq \sum_{j=0}^{r^*-1} \binom{r^*}{j} (2\rho)^{-j} j!^\kappa K A w_{r-j}(s, p; t) + (2\rho)^{-r} r!^\kappa K A w_0(s, p; t) \\ &\leq K A w_r(s, p; t) \left\{ \sum_{j=0}^{r^*-1} \binom{r^*}{j} \binom{r+p}{j}^{-\kappa} 2^{-j+2} \binom{r+p}{r}^{-\kappa} \right\} \\ &\leq 3 K A w_r(s, p; t). \end{aligned}$$

Therefore it is sufficient if we take $C=3K$. (q.e.d.)

Now, let us be back to (3.21). Since $B_{12} \in B_t^0[S^0(\kappa)]$ and V_2 satisfies (3.20), using the above lemma, we see that

$$\|\Lambda^r B_{12} V_2\| \leq \frac{C}{s+1} A w_r(s+1, p; t), \quad \text{for any } r \geq 0.$$

Therefore, by the same way as the proof of Proposition 3.1, one can prove the following proposition.

Proposition 3.2. If G satisfies (3.18), then V_1 satisfies

$$(3.22) \quad \|\Lambda^r V_1(\cdot, t)\| \leq \frac{CA}{(s+2)(s+1)} w_r(s+2, p; t), \quad \text{for any } r \geq 0,$$

where C is a certain constant independent of s , p , A and r .

3.4. Energy estimate, II. Our aim in this paragraph is to obtain an energy estimate of QV , assuming that G in (E') satisfies (3.18).

It follows from $(\tilde{E})$ that

$$\frac{d}{dt} \|\Lambda^r W\| \leq K_0 \|\Lambda^r W\| + \|[\Lambda^r, D]W\| + t^{-1} \|\Lambda^r \tilde{B}W\| + \|\Lambda^r \tilde{G}\|.$$

Therefore, using the Lemma 3.1, we have

$$(3.23) \quad \begin{aligned} \|\Lambda^r W\| \leq & \int_0^t e^{(K_0 + r^* K)(t-u)} \left(\frac{t}{u}\right)^K \left\{ \sum_{j=2}^r \binom{r^*}{j} (2\rho)^{-j+1} (j-1)! K \times \right. \\ & \left. \times \|\Lambda^{r-j+1} W\| + (2\rho)^{-r} r! K \|W\| + \frac{1}{u} \sum_{j=1}^r \binom{r^*}{j} (2\rho)^{-j} j! K \times \right. \end{aligned}$$

$$\times \|\Lambda^{r-j} W\| + \frac{1}{u} (2\rho)^{-r} r!^K K \|W\| + \|\Lambda^{r\sim} \tilde{G}\| \} du,$$

where, in case of $r=0$, the ~~2-nd~~ and ~~4~~-th terms in the right side do not appear.

By the way, since $\tilde{G} = t^{-1} \tilde{B}' V + \tilde{N} G$; $\tilde{B}', \tilde{N} \in B_t^0[S^0(\kappa)]$, using the Proposition 3.1 and the Lemma 3.2 we see that

$$(3.24) \quad \|\Lambda^{r\sim} \tilde{G}(\cdot, t)\| \leq C A w_r(s, p; t), \quad r \geq 0.$$

Let us denote CA in the above inequality by $\tilde{A}$. Then, if s is large we can show that

$$(3.25) \quad \|\Lambda^r W\| < \frac{C\tilde{A}}{s+1} w_r(s+1, p; t), \quad r \geq 0.$$

In fact, when $r=0$, it follows from (3.23) and (3.24) that, if $s \geq 2K$,

$$\begin{aligned} \|W\| &\leq \int_0^t e^{K_0(t-u)} \left(\frac{t}{u}\right)^K \tilde{A} w_0(s, p; u) du \\ &\leq \frac{\tilde{A}}{s-K+1} w_0(s+1, p; t) \leq \frac{2\tilde{A}}{s+1} w_0(s+1, p; t). \end{aligned}$$

Next, letting n be an arbitrary non-negative integer, we assume that (3.25) holds for any $0 \leq r \leq n$. Then it follows from (3.23) and (3.24) that for $n < r \leq n+1$

$$\|\Lambda^r W\| \leq \int_0^t e^{(K_0 + r^* K)(t-u)} \left(\frac{t}{u}\right)^K \left\{ \sum_{j=2}^r \binom{r^*}{j} (2\rho)^{-j+1} (j-1)!^K K \frac{C\tilde{A}}{s+1} \times \right.$$

$$\begin{aligned}
& \times w_{r-j+1}(s+1, p; u) + (2\rho)^{-r} r!^{\kappa} K_{s+1}^{\frac{2\tilde{A}}{s+1}} w_0(s+1, p; u) \\
& + \frac{1}{u} \sum_{j=1}^{r^*-1} \binom{r^*}{j} (2\rho)^{-j} j!^{\kappa} K_{s+1}^{\frac{C\tilde{A}}{s+1}} w_{r-j}(s+1, p; u) \\
& + \frac{1}{u} (2\rho)^{-r} r!^{\kappa} K_{s+1}^{\frac{2\tilde{A}}{s+1}} w_0(s+1, p; u) + \tilde{A} w_r(s, p; u) \} du \\
& \leq \frac{\tilde{A}}{s+1} w_r(s+1, p; t) \left\{ \sum_{j=2}^{r^*} \binom{r^*}{j} \binom{r+p}{j-1}^{-\kappa} \frac{CK2^{-j+1}}{(r^*+2)K} + \binom{r+p}{r}^{-\kappa} \frac{2K2^{-r}}{(r^*+2)K} \right. \\
& \quad \left. + \sum_{j=1}^{r^*-1} \binom{r^*}{j} \binom{r+p}{j}^{-\kappa} \frac{CK2^{-j}}{s-K+1} + \binom{r+p}{r}^{-\kappa} \frac{2K2^{-r}}{s-K+1} + \frac{s+1}{s-K+1} \right\}.
\end{aligned}$$

Since $\kappa \geq 1$ and $p \geq 1$, if $s \geq 4K$ then

$$\begin{aligned}
\{ \dots \} & \leq \sum_{j=2}^{r^*} \frac{r^*-j+1}{j} \times \frac{C2^{-j+1}}{r^*+2} + 1 + \sum_{j=1}^{r^*-1} \frac{CK2^{-j}}{s-K+1} + \frac{2K+s+1}{s-K+1} \\
& \leq \frac{5}{6}C + 3.
\end{aligned}$$

Therefore, if we take $C=18$, (3.25) is valid also for $n < r \leq n+1$.

Thus (3.25) has been verified.

Now that $QV = \tilde{N}^{-1}W$; $\tilde{N}^{-1} \in B_t^0[S^0(\kappa)]$, using the Lemma 3.2 and (3.25) we have the proposition:

Proposition 3.3. If G satisfies (3.18) and $s \geq 4K$, then the solution V of (E') satisfies the inequality:

$$(3.26) \quad \|\Lambda^r QV\| \leq \frac{CA}{s+1} w_r(s+1, p; t), \quad \text{for any } r \geq 0,$$

where C is a certain constant independent of s , p , A and r .

§4. Convergence of Σu^i .

According to the way in the preceding section, we set

$$(4.1) \quad \begin{aligned} u_0^i &= u^i, \quad u_k^i = \Delta_k u_{k-1}^{i-} \quad (k=1, \dots, m-1), \\ U^i &= {}^t(u_0^i, \dots, u_{m-1}^i), \quad F^i = {}^t(0, \dots, 0, f^i), \\ V^i &= N'NEU^i, \quad G^i = NF^i. \end{aligned}$$

Then $(E)^i$ is reduced to

$$(E')^i \quad D_t V^i = DV^i + BV^i + G^{i-1}, \quad V^i|_{t=0} = 0, \quad i=1, 2, \dots$$

4.1. Reformulation of G^i , $i \geq 1$. Taking $\{1, \Pi_1, \dots, \Pi_{m-1}\}$ as a basis, we can represent $\Pi_m - P$ in the form:

$$(4.2) \quad \Pi_m - P = c_0 \Pi_{m-1} + c_1 \Pi_{m-2} + \dots + c_{m-1}.$$

Therefore, remarking that $f^i = (\Pi_m - P)u^i$, we have

$$(4.3) \quad F^i = \begin{pmatrix} 0 \\ c_{m-1} & \dots & c_0 \end{pmatrix} U^i = \begin{pmatrix} 0 \\ \tilde{c}_{m-1} & \dots & \tilde{c}_0 \end{pmatrix} N^{-1} N'^{-1} V^i,$$

where $\tilde{c}_{m-j} = c_{m-j} \Lambda^{-(m-j-1)} \in B_t^0[s^1(\kappa)]$ for $j=1, \dots, s$, and

$\tilde{c}_{m-j} = c_{m-j} \Lambda^{-(m-j)} \in B_t^0[S^0(\kappa)]$ for $j=s+1, \dots, m$.

Let $\overset{\circ}{c}_{m-j}(x, t, \xi)$ be the principal symbol of c_{m-j} . Then, taking (3.4)~(3.6) and (3.8) into account, we see that the principal symbol of the (m, j) component of the matrix operating to V^i in (4.3) is

$$\overset{\circ}{c}_{m-1} \langle \xi \rangle^{j-1} \prod_{k=1}^{j-1} (\mu_j - \mu_k)^{-1} + \overset{\circ}{c}_{m-2} \langle \xi \rangle^{j-2} \prod_{k=2}^{j-1} (\mu_j - \mu_k)^{-1} + \dots + \overset{\circ}{c}_{m-j}.$$

Therefore, if we set

$$\ell_j = \{ \overset{\circ}{c}_{m-1} + \overset{\circ}{c}_{m-2} (\mu_j - \mu_1) + \dots + \overset{\circ}{c}_{m-j} \prod_{k=1}^{j-1} (\mu_j - \mu_k) \} \langle \xi \rangle^{-m+2}, \quad 1 \leq j \leq s, \quad (4.4)$$

$$L = \begin{pmatrix} \ell_1 & & & \\ & \ddots & & \\ & & \ell_{s_0} & \\ & & & \ddots \\ & & & & 0 \end{pmatrix},$$

then we can represent G^i as follows:

$$(4.5) \quad G^i = BLV^i + B'V^i; \quad B, B' \in B_t^0[S^0(\kappa)].$$

ℓ_j above is closely related with the Levi's function $L_j(x, t, \xi)$ defined in (2.8). Let $\sigma_{m-1}(\Pi_m^{-P})$ denote the principal symbol of Π_m^{-P} . Then,

$$\sigma_{m-1}(\Pi_m^{-P}) = \overset{\circ}{c}_{0 \prod_{j=1}^{m-1}} (\tau - \mu_j) + \overset{\circ}{c}_{1 \prod_{j=1}^{m-2}} (\tau - \mu_j) + \dots + \overset{\circ}{c}_{m-1}.$$

Therefore we see that

$$(4.6) \quad \sigma_{m-1}(\Pi_m - P)|_{\tau=\mu_j} = \ell_j \langle \xi \rangle^{-m+2}, \quad j=1, \dots, s.$$

On the other hand, because

$$\begin{aligned} \Pi_m - P_m &= \Delta_m \Delta_{m-1} \dots \Delta_1 - \Delta_m \circ \Delta_{m-1} \circ \dots \circ \Delta_1 \\ &= \sum_{i=2}^m \Delta_m \dots \Delta_{i+1} \{ \Delta_i (\Delta_{i-1} \circ \dots \circ \Delta_1) - \Delta_i \circ \Delta_{i-1} \circ \dots \circ \Delta_1 \}, \end{aligned}$$

we see that $\sigma_{m-1}(\Pi_m - P_m)$ is equal to

$$\begin{aligned} & \sum_{j>i} \{ \partial_{\tau}(\tau-\mu_j) D_t(\tau-\mu_i) + \sum_{k=1}^n \partial_{\xi_k}(\tau-\mu_j) D_{x_k}(\tau-\mu_i) \} \prod_{h \neq i, j} (\tau-\mu_h) \\ &= \frac{1}{2} \partial_{\tau} D_t P_m + \frac{1}{2} \sum_{k=1}^n \partial_{\xi_k} D_{x_k} P_m + \frac{1}{2} \sum_{j>i} \{ \tau-\mu_j, \tau-\mu_i \} \prod_{h \neq i, j} (\tau-\mu_h) \\ & \quad + \frac{1}{2} \sum_i \partial_{\xi_k} D_{x_k} \mu_i \prod_{h \neq i} (\tau-\mu_h). \end{aligned}$$

Hence, using the condition (C-3), we can represent ℓ_j as follows:

$$(4.7) \quad \ell_j = (L_j + m_j) \langle \xi \rangle^{-m+2}; \quad m_j = t^{-1}(\mu_j - \lambda_j) q_j, \quad q_j \in B_t^0[S^{m-2}(\kappa)]$$

4.2. Successive estimation. We assume that f in (1.1') satisfies

$$(4.8) \quad \|\Lambda^r f(\cdot, t)\| \leq A \rho^{-r} r!^{\kappa}, \quad 0 \leq t \leq h, \quad \text{for any } r \geq 0.$$

Then, by the Lemma 3.2 we see that

$$(4.9) \quad \|\Lambda^r G^0\| \leq C_1 A_{w_r}(0,1;t), \quad r \geq 0.$$

The proposition 3.1 being applicable, it follows that

$$(4.10) \quad \|\Lambda^r V^1\| \leq C_2 C_1 A_{w_r}(1,1;t), \quad r \geq 0.$$

From (4.3) G^1 is represented as $G^1 = MV^1$; $M \in B_t^0[S^1(\kappa)]$. Remark ing that $\Lambda^r G^1 = \Lambda^{r+1} \Lambda^{-1} MV^1$; $\Lambda^{-1} M \in B_t^0[S^0(\kappa)]$, by the lemma 3.2 we se that

$$(4.11) \quad \|\Lambda^r G^1\| \leq C_3 C_2 C_1 \overset{A}{w}_{r+1}(1,1;t) \leq C_3 C_2 C_1 \overset{A}{w}_r(1,2;t), \quad r \geq 0.$$

By repeating the above arguments it is easily shown that

$$(4.12) \quad \begin{aligned} \|\Lambda^r V^i\| &\leq \frac{1}{i!} C_3^{i-1} C_2^i C_1 A_{w_r}(i,i;t), \quad r \geq 0 \\ \|\Lambda^r G^i\| &\leq \frac{1}{i!} C_3^i C_2^i C_1 A_{w_r}(i,i+1;t), \quad r \geq 0. \end{aligned}$$

Letting n be an integer larger than $4K$, we repeat the above successive estimating up to $i=n$. For $i > n$, using not only the Proposition 3.1 but also the Propositions 3.2 and 3.3, ~~we will be~~ ^{we are} able to give estimates of V^i and G^i ~~sharper~~ ^{with sharper form} than the above.

From (4.12) we have

$$(4.13) \quad \|\Lambda^r G^n\| \leq \tilde{A}_{w_r}(n,n+1;t), \quad r \geq 0,$$

where $\tilde{A} = C_3^n C_2^n C_1^n A/n!$. Therefore by the propositions 3.1, 3.2 and 3.3 we see that

$$\begin{aligned} \|\Lambda^r V^{n+1}\| &\leq \frac{C_4 \tilde{A}}{n+1} w_r(n+1, n+1; t), \\ (4.14) \quad \|\Lambda^r V_1^{n+1}\| &\leq \frac{C_5 \tilde{A}}{(n+1)^2} w_r(n+2, n+1; t), \\ \|\Lambda^r QV^{n+1}\| &\leq \frac{C_6 \tilde{A}}{n+1} w_r(n+1, n+1; t), \quad \text{for any } r \geq 0. \end{aligned}$$

We estimate $\Lambda^r \tilde{L}_j v_j^{n+1}$ and $\Lambda^r \tilde{m}_j v_j^{n+1}$ ($1 \leq j \leq s$), taking (4.5) and (4.7) into account. Here $\tilde{L}_j = L_j \Lambda^{r+m+2}$, $\tilde{m}_j = m_j \Lambda^{-m+2}$ and $V^{n+1} = (v_1^{n+1}, \dots, v_m^{n+1})$. We are assuming $\kappa < \kappa^*$. Therefore, from the definitions of ρ_j , σ_j and κ^* , there are $\rho_j' \geq \rho_j$, $\sigma_j' \leq \sigma_j$, $a_j \in B_t^0[S^0(\kappa)]$ and $b_j \in B_t^0[S^1(\kappa)]$ such that

$$\begin{aligned} \tilde{L}_j &= t^{-\rho_j'} a_j \cdot \delta_j = t^{-\rho_j'} \{a_j \delta_j + a_j'\}, \quad a_j' \in B_t^0[S^0(\kappa)], \\ (4.15) \quad \tilde{L}_j &= t^{\sigma_j'} b_j, \\ \kappa < \kappa'^* &= \min_{1 \leq j \leq s, \rho_j' - 1} \frac{2\rho_j' + \sigma_j'}{\rho_j' - 1} \leq \kappa^*. \end{aligned}$$

In the case of $\rho_j = \infty$, the first expression of $\tilde{L}_j$ is meaningless and will not be used actually.

Hence we get the following inequalities from (4.14) using the Lemma 3.2.

$$\begin{aligned}
(4.16) \quad & \|\Lambda^r L_j v_j^{n+1}\| \leq \frac{C_7 \tilde{A}}{n+1} w_r(n+1-\rho_j^-, n+1; t), \quad r \geq 0, \\
& \|\Lambda^r L_j v_j^{n+1}\| \leq \frac{C_8 \tilde{A}}{(n+1)^2} w_{r+1}(n+2+\sigma_j^-, n+1; t), \quad r \geq -1.
\end{aligned}$$

To combine these two inequalities we prepare the following lemma, which is ~~the reformed one~~ ^{a variant} of the Hölder's inequality.

Lemma 4.1. If $p \geq 0$, $q \geq 0$ and $p+q=1$, then

$$\|\Lambda^r u\| \leq \|\Lambda^{r-p} u\|^q \|\Lambda^{r+q} u\|^p.$$

Proof. By the Hölder's inequality,

$$\begin{aligned}
& \|\Lambda^{r-p} u\|^{2q} \|\Lambda^{r+q} u\|^{2p} \\
&= \left[\int \{ \langle \xi \rangle^{r-p} |\hat{u}(\xi)| \}^{2q \frac{1}{q}} d\xi \right]^q \left[\int \{ \langle \xi \rangle^{r+q} |\hat{u}(\xi)| \}^{2p \frac{1}{p}} d\xi \right]^p \\
&\geq \int \langle \xi \rangle^{2r} |\hat{u}(\xi)|^2 d\xi = \|\Lambda^r u\|^2, \quad d\xi = (2\pi)^{-n} d\xi. \quad (\text{q.e.d})
\end{aligned}$$

Combining the two inequalities in (4.16) by this lemma, we have

$$(4.17) \quad \|\Lambda^r L_j v_j^{n+1}\| \leq \frac{C_9 \tilde{A}}{(n+1)^{1+q}} w_{r+q}(n+v_j, n+1; t), \quad r \geq 0,$$

where $v_j = 1 + (1 + \sigma_j^-)q - \rho_j^- p$. In the same way one can verify that

$$(4.18) \quad \|\Lambda^r \tilde{m}_j v_j^{n+1}\| \leq \frac{C_{10} \tilde{A}}{(n+1)^{1+q}} w_{r+q}(n+2q, n+1; t), \quad r \geq 0.$$

Therefore, owing to (4.5) and (4.7) we have

$$\|\Lambda^r G^{n+1}\| \leq \frac{C_{11} \tilde{A}}{(n+1)^{1+q}} w_r(n+v, n+1+q; t) + \frac{C_{12} \tilde{A}}{n+1} w_r(n+1, n+1; t)$$

where $v = \min\{1, 2q, v_1, \dots, v_s\}$. Remark that

$$(n+1)^q (r+n+1)! < (r+1+n)^q (r+1+n)! \sim (r+1+n+q)!.$$

Then, finally we have

$$(4.19) \quad \|\Lambda^r G^{n+1}\| \leq \frac{C_{13} \tilde{A}}{(n+1)^{1+q}} w_r(n+v, n+1+q; t), \quad r \geq 0.$$

Here q is regarded as a parameter.

If $v \geq 0$ we can repeat the above arguments. Therefore the following proposition is verified by induction.

Proposition 4.1. If $v \geq 0$, then it holds that

$$(4.20) \quad \|\Lambda^{r_V} V^{n+i+1}\| \leq \frac{C_5 C_{13}^i \tilde{A}}{(n+iv+1) \prod_{j=0}^{i-1} (n+jv+1)^{1+q}} w_r(n+iv+1, n+1+iq; t),$$

$$(4.21) \quad \|\Lambda^{r_{V_1}} V_1^{n+i+1}\| \leq \frac{C_6 C_{13}^i \tilde{A}}{(n+iv+1)^2 \prod_{j=0}^{i-1} (n+jv+1)^{1+q}} w_r(n+iv+2, n+1+iq; t),$$

$$(4.22) \quad \|\Lambda^{r_{QV}} QV^{n+i+1}\| \leq \frac{C_7 C_{13}^i \tilde{A}}{(n+iv+1) \prod_{j=0}^{i-1} (n+jv+1)^{1+q}} w_r(n+iv+1, n+1+iq; t),$$

$$(4.23) \quad \|\Lambda^{r_G} G^{n+i+1}\| \leq \frac{C_{13}^{i+1} \tilde{A}}{\prod_{j=0}^i (n+jv+1)^{1+q}} w_r(n+iv+v, n+1+iq+q; t),$$

$r \geq 0, i = 0, 1, 2, \dots$

4.3. Convergence of Σu^i . Remark that $EU^i = N^{-1}N'^{-1}V^i$; $N^{-1}, N'^{-1} \in B_t^0[S^0(\kappa)]$. Then from the Proposition 4.1 we have

$$(4.24) \quad \|\Lambda^r EV^{n+i+1}\| \leq \frac{C_5' C_{13}^1 A}{(n+iv+1) \prod_{j=0}^{i-1} (n+jv+1)^{1+q}} w_r(n+iv+1, n+1+iq; t).$$

By the way, we can represent $D_t^{j-1} \Lambda^{m-j} u$ as follows:

$$\begin{aligned} D_t^{j-1} \Lambda^{m-j} u &= \sum_{k=0}^{j-1} d_{j,j-k} \Delta_k \dots \Delta_1 u \\ &= \sum_{k=0}^{s-1} \tilde{d}_{j,j-k} \Lambda^{m-2-k} u_k + \sum_{k=s}^{j-1} \tilde{d}_{j,j-k} \Lambda^{m-1-k} u_k, \end{aligned}$$

where $d_{j,j-k} \in B_t^0[S^{m-1-k}(\kappa)]$, $\tilde{d}_{j,j-k} = d_{j,j-k} \Lambda^{-m+2+k} \in B_t^0[S^1(\kappa)]$ when $k=0, 1, \dots, s-1$, and $\tilde{d}_{j,j-k} = d_{j,j-k} \Lambda^{-m+1+k} \in B_t^0[S^0(\kappa)]$ when $k=s, \dots, j-1$.

Therefore, if we define the norm $\|\Lambda^r u\|$ by

$$(4.25) \quad \|\Lambda^r u\|^2 = \sum_{j=1}^m \|D_t^{j-1} \Lambda^{m-j+r} u\|^2, \quad r \geq 0,$$

then it follows that

$$(4.26) \quad \|\Lambda^r u^{n+i+1}\| \leq \frac{C_5'' C_{13}^1 A}{(n+iv+1) \prod_{j=0}^{i-1} (n+jv+1)^{1+q}} w_{r+1}(n+iv+1, n+1+iq; t).$$

Now, we remark the followings:

- 1°. $(n+iv+1)^q (r+n+2+iq)! \leq 2^q (r+n+2+iq)^q (r+n+2+iq)! \sim 2^q (r+n+2+iq+q)!$.
- 2°. $\binom{r+n+2+iq+q}{iq+q} \leq 2^{r+n+2+iq+q}$.

$$3^0. \prod_{j=0}^i (n+jv+1)^{1+q} \geq v^{(1+q)i} (i+1)!^{1+q}.$$

Then, if $v > 0$, we see that

$$(4.27) \quad \|\Lambda^r u^{n+i+1}\| \leq C'' A t^{n+1} C^{-r} (r+n+2)!^\kappa (C t^v)^i \frac{(iq+q)!^\kappa}{(i+1)!^{1+q}},$$

where $C = C_{13} 2^{\kappa q} v^{-(1+q)} \rho^{-q}$, $C' = e^{2Kh} 2^\kappa \rho^{-1}$ and C'' is a constant independent of r and i .

Now that by Stirling's formula

$$\frac{(iq)!^\kappa}{i!^{1+q}} \sim \frac{\{\sqrt{2\pi iq} (iq/e)^{iq}\}^\kappa}{\{\sqrt{2\pi i} (i/e)^i\}^{1+q}},$$

we have the following proposition.

Proposition 4.2. If $v > 0$ and $q\kappa < 1+q$, then

$$(4.28) \quad \sum_{i=1}^{\infty} \|\Lambda^r u^i\| \leq \text{const.} \rho'^{-r} r!^\kappa, \quad 0 \leq t \leq h.$$

Furthermore, even when $v > 0$ and $q\kappa = 1+q$, the above cosequence remains true for $0 \leq t \leq h_0$, if h_0 is taken sufficiently small.

If $\rho_j < \infty$ for every j , then the condition assumed above is equivalent to

$$(4.29) \quad 0 < q \leq 1, \quad q < \frac{1}{\kappa-1}, \quad q > \frac{\rho_j' - 1}{\rho_j' + \sigma_j' + 1}, \quad j=1, \dots, s.$$

On the other hand, we are assuming that $1 < \kappa < \kappa^*$. Hence there is a parameter q which satisfies (4.29); see (4.15) In the case of

$\rho_j = \infty$ for some j , it is sufficient if we take $q=1$.

Thus the existence of ^a~~the~~ solution of (1.1') has been proved.

By the latter half of the Proposition 4.2, we see also the local (in t) existence of ^a~~the~~ solution in the case: $\rho_j = \infty$ for some j and $\kappa = \kappa^*$ ($=2$).

The uniqueness of the solution will be proved as follows.
Let u be a solution of

$$(4.30) \quad \Pi_m u = (\Pi_m - P)u, \quad \text{null initial data.}$$

Suppose that

$$(4.31) \quad \|\Lambda^r u\| \leq \text{const.} \rho^{-r} r!^\kappa, \quad \text{for any } r \geq 0.$$

Then, by the same way as the proof of the existence theorem, we can show that

$$(4.32) \quad \|\Lambda^r u\| \leq \text{const.} C^{-r} r!^\kappa (Ct^v)^{i-1} (iq)!^\kappa / i!^{1+q},$$

$r \geq 0$, $i=1,2,\dots$, where C, C' are constants independent of r and i .

If $v > 0$ and $q\kappa < 1+q$, then the right hand term in (4.32) tends to zero as ⁱ~~i~~ to ∞ . As shown above, if $\kappa < \kappa^*$ then there is a parameter q which satisfies $v > 0$ and $q\kappa < 1+q$. Hence the uniqueness has been proved, too.

At the end, we give some remarks:

1). If $1 < \kappa < 2$, the consequence of the Theorem remains true

without assuming the condition (C-3), because we may take $q=1$ for the condition " $v>0$ and $q\kappa<1+q$ " to be satisfied.

2). One can prove what we stated in Remark 2 by taking $q=0$, because $v=0$ and by taking n large one may suppose that $\frac{c_{13}}{n+1}<1$ in (4.19). Cf. [16].

Appendix

We introduce a class of (Gevrey) pseudo-differential operators and give some results on its calculus which we used throughout the paper, referring to H. Kumano-go [17] and S. Steinberg [18].

Definition A.1. A function $h(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is called a symbol of class $S^m(\kappa, \rho)$, $-\infty < m < \infty$, $\kappa > 1$, $\rho > 0$, if it satisfies

$$(1) \quad |h_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_\alpha \rho^{-|\beta|} |\beta|!^\kappa \langle \xi \rangle^{m-|\alpha|}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for any α and β , where $h_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha D_x^\beta h(x, \xi)$ and C_α is a constant independent of β . We put $S^m(\kappa) = \bigcup_{\rho>0} S^m(\kappa, \rho)$.

We define the pseudo-differential operator corresponding to $h(x, \xi) \in S^m(\kappa, \rho)$ by

$$(2) \quad Hu = h(x, D_x)u = \int e^{ix\xi} h(x, \xi) u(\xi) d\xi,$$

and say that $H (=h(x, D_x))$ belongs to $S^m(\kappa, \rho)$. We call $\sigma(H)=h(x, \xi)$ the symbol of H . We set $S^m(\kappa) = \bigcup_{\rho>0} S^m(\kappa, \rho)$. Then the following theorem holds.

Theorem A.1. 1). If $H_i \in S^{m_i}(\kappa, \rho)$, $i=1,2$, then $H_1 H_2$ belongs to $S^{m_1+m_2}(\kappa, \rho')$, $\rho' < \rho$. Moreover, the symbol of $H_1 H_2$ has the asymptotic expansion: For any positive integer N ,

$$(3) \quad \sigma(H_1 H_2) \equiv \sum_{|\alpha| < N} \frac{1}{\alpha!} h_1^{(\alpha)}(x, \xi) h_2^{(\alpha)}(x, \xi), \quad \text{mod. } S^{m_1+m_2-N}(\kappa, \rho'),$$

where $h_i(x, \xi) = \sigma(H_i)$.

2). If $H \in S^m(\kappa, \rho)$, then $H^* \in S^m(\kappa, \rho')$, $\rho' < \rho$. Moreover, for any positive integer N ,

$$(4) \quad \sigma(H^*) \equiv \sum_{|\alpha| < N} \frac{1}{\alpha!} \bar{h}^{(\alpha)}(x, \xi), \quad \text{mod. } S^{m-N}(\kappa, \rho'),$$

where $h(x, \xi) = \sigma(H)$. Here H^* stands for the adjoint operator of H , namely

$$\int (Hu)(x) \overline{v(x)} dx = \int u(x) \overline{(H^*v)(x)} dx, \quad \text{for any } u, v \in S.$$

To prove the theorem, we introduce a class of pseudo-differential operators with double symbol.

Definition A.2. A function $h(x, \xi, x', \xi') \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ is called a double symbol of class $S^{m, m'}(\kappa, \rho)$, $-\infty < m, m' < \infty$, $\kappa > 1$, $\rho > 0$, if it satisfies

$$(5) \quad |h_{(\beta, \beta')}^{(\alpha, \alpha')}(x, \xi, x', \xi')| \leq C_{\alpha, \alpha'} \rho^{-|\beta + \beta'|} |\beta|!^\kappa |\beta'|!^\kappa \langle \xi \rangle^{m - |\alpha|} \langle \xi' \rangle^{m' - |\alpha|}$$

$(x, \xi, x', \xi') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, for any $\alpha, \beta, \alpha', \beta'$, where $h_{(\beta, \beta')}^{(\alpha, \alpha')} =$

$= \partial_{\xi}^{\alpha} \partial_{\xi'}^{\alpha'} D_x^{\beta} D_{x'}^{\beta'} h(x, \xi, x', \xi')$ and $C_{\alpha, \alpha'}$ is a constant independent of β and β' .

We define the operator corresponding to $h(x, \xi, x', \xi') \in S^{m, m'}(\kappa, \rho)$ by

$$(6) \quad \begin{aligned} Hu &= h(x, D_x, x', D_{x'})u \\ &= Os-\iiint e^{-i(y\xi + y'\xi')} h(x, \xi, x+y, \xi') u(x+y+y') dy dy' d\xi d\xi', \end{aligned}$$

$u \in B$, where $Os-\iiint$ denotes the oscillatory integral, cf[17].

We say that $H (=h(x, D_x, x', D_{x'}))$ belongs to $S^{m, m'}(\kappa, \rho)$.

The following three propositions are already known, see [17].

1. If $h(x, D_x, x', D_{x'}) \in S^{m, m'}(\kappa, \rho)$, then

$$h(x, D_x, x', D_{x'})u = \iiint e^{i(x-x')\xi + ix'\xi'} h(x, \xi, x', \xi') \hat{u}(\xi') d\xi' \cdot dx' \cdot d\xi,$$

$u \in S$, where the integration is supposed to be done in the described order.

2. Let $H_i \in S^{m_i}(\kappa, \rho)$, $\sigma(H_i) = h_i(x, \xi)$, $i=1, 2$, and set $h(x, \xi, x', \xi') = h_1(x, \xi) h_2(x', \xi')$. Then $h(x, \xi, x', \xi') \in S^{m_1, m_2}(\kappa, \rho)$ and

$$H_1 H_2 u = h(x, D_x, x', D_{x'})u, \quad u \in S.$$

3. Let $H \in S^m(\kappa, \rho)$, $\sigma(H) = h(x, \xi)$, and set $\bar{h}^R(\xi, x') = \overline{h(x', \xi)}$. Then $\bar{h}^R(\xi, x') \in S^{m, 0}(\kappa, \rho)$ and

$$H^*u = \bar{h}^R(D_x, x')u, \quad u \in S.$$

Therefore, Theorem A.1 follows from

Theorem A.2. Let $h(x, \xi, x', \xi') \in S^{m, m'}(\kappa, \rho)$, and set

$$h_L(x, \xi) = Os - \iint e^{-iy\eta} h(x, \xi + \eta, x + y, \xi) dy d\eta.$$

Then,

$$1). \quad h_L(x, \xi) \in S^{m+m'}(\kappa, \rho'), \quad \rho' < \rho.$$

$$2). \quad h(x, D_x, x', D_{x'})u = h_L(x, D_x)u, \quad u \in S.$$

3). For any positive integer N ,

$$(7) \quad h_L(x, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} h_{(0, \alpha)}^{(\alpha, 0)}(x, \xi, x, \xi) + r_N(x, \xi),$$

where $h_{(0, \alpha)}^{(\alpha, 0)}(x, \xi, x, \xi) \in S^{m+m'-|\alpha|}(\kappa, \rho')$, $\rho' < \rho$,

$$r_N(x, \xi) = N \int_0^1 (1-\theta)^{N-1} d\theta Os - \iint \sum_{|\alpha|=N} \frac{1}{\alpha!} e^{-iy\eta} h_{(0, \alpha)}^{(\alpha, 0)}(x, \xi + \theta\eta, x + y, \xi) dy d\eta$$

and $r_N(x, \xi) \in S^{m+m'-N}(\kappa, \rho')$.

To prove this theorem, the following lemma is important.

Lemma A.1. Let $h(x, \xi, x', \xi') \in S^{m, m'}(\kappa, \rho)$ and $\alpha, \beta, \alpha', \beta'$ be arbitrary multi-indices, and put

$$q(x, \xi, x', \xi') = h_{(\beta, \beta')}^{(\alpha, \alpha')}(x, \xi, x', \xi'),$$

$$q_\theta(x, \xi) = Os - \iint e^{-iy\eta} q(x, \xi + \theta\eta, x+y, \xi) dy d\eta,$$

where θ is a parameter with $|\theta| \leq 1$. Then,

$$|q_\theta(x, \xi)| \leq C_{\alpha, \alpha'} \rho^{-|\beta+\beta'|} |\beta|!^\kappa |\beta'|!^\kappa \langle \xi \rangle^{m+m'-|\alpha|-|\alpha'|},$$

$(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $\rho' < \rho$, where $C_{\alpha, \alpha'}$ is a constant independent of β, β' and θ .

Lemma A.1 and Theorem A.2 are easily verified by consulting [17], so we omit these proofs. At the end, we give the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $H \in S^m(\kappa, \rho_1)$, $\sigma(H) = h(x, \xi)$, $m=0$ or 1 , and $r \geq 0$. Then by Theorem A.2,

$$(8) \quad \sigma(\Lambda^r H) = \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha \langle \xi \rangle^r h_{(\alpha)}(x, \xi) + r_N(x, \xi)$$

where $N=r+m$ and

$$r_N(x, \xi) = N \int_0^1 (1-\theta)^{N-1} d\theta Os - \iint e^{-iy\eta} \sum_{|\alpha|=N} \frac{1}{\alpha!} \partial_\xi^\alpha \langle \xi + \theta\eta \rangle^r h_{(\alpha)}(x+y, \xi) dy d\eta.$$

It is well known that if $P \in S^0$, $\sigma(P) = p(x, \xi)$, then

$$(9) \quad \|Pu\| \leq C \|p\|_\ell \|u\|,$$

where C is a constant, ℓ is an integer, both of which depend only on n , and

$$(9') \quad |p|_{\ell} = \max_{|\alpha+\beta| \leq \ell} \sup_{x, \xi} |p_{(\beta)}^{(\alpha)}(x, \xi)| \langle \xi \rangle^{-|\alpha|}.$$

Besides, we see easily that

$$(10) \quad \sum_{|\alpha|=j} \frac{1}{\alpha!} |\partial_{\xi}^{\alpha} \langle \xi \rangle^r| \leq \begin{cases} \binom{r^*}{j} C^j \langle \xi \rangle^{r-j}, & \text{if } j=1, \dots, r^*, \\ C^j \langle \xi \rangle^{r-j}, & \text{if } j > r^*, \end{cases}$$

where C is a constant independent of r and j .

Thus, the proof of Lemma 3.1 will be over, if we show

$$(11) \quad \begin{aligned} |h_{(\alpha)}(x, \xi) \langle \xi \rangle^{-m}|_{\ell} &\leq C \rho_2^{-|\alpha|+m} (|\alpha|-m)!^K, \\ |r_N(x, \xi)|_{\ell} &\leq C \rho_2^{-r} r!^K, \end{aligned}$$

where C and ρ_2 ($< \rho_1$) are constants independent of α and r .

It is easy to show the former, so we show only the latter.

Put

$$\tilde{r}_N(x, \xi) = 0s - \iint e^{-iy\eta} \sum_{|\alpha|=N} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} \langle \xi + \theta\eta \rangle^r h_{(\alpha)}(x+y, \xi) dy d\eta.$$

Then,

$$\begin{aligned} (\tilde{r}_N)_{(\mu)}^{(\gamma)} &= \sum C_{\gamma_1, \gamma_2} 0s - \iint e^{-iy\eta} \langle y \rangle^{-2k} \langle D_{\eta} \rangle^{2k} \langle \eta \rangle^{-2k} \langle D_y \rangle^{2k} \left\{ \sum \frac{1}{\alpha!} \right. \\ &\quad \left. \times \partial_{\xi}^{\alpha+\gamma_1} \langle \xi + \theta\eta \rangle^r h_{(\alpha+\mu)}^{(\gamma_2)}(x+y, \xi) \right\} dy d\eta \end{aligned}$$

$$= \sum C_{\gamma_1, \gamma_2} \sum_{\substack{|\nu_1 + \nu_2| \leq 2k' \\ |\mu'| \leq 2k}} \theta^{|\nu_2|} C_{\nu_1, \nu_2, \mu'} 0s - \iint e^{-iy\eta} \sum \frac{1}{\alpha!} \langle y \rangle^{-2k'} \\ \times D_n^{\nu_1} \langle \eta \rangle^{-2k} \partial_{\xi}^{\alpha + \gamma_1 + \nu_2} \langle \xi + \theta \eta \rangle^{r_h(\gamma_2)}_{(\alpha + \mu + \mu')} (x+y, \xi) dy d\eta.$$

Because $(\alpha + \gamma_1 + \nu_2)! / \alpha! \leq (N + \ell + 2k')^{\ell + 2k'}$,

$$|\dot{r}_N|_{\ell} \leq C_{\ell, k, k'} (N + \ell + 2k')^{\ell + 2k'} C^{N + \ell + 2k'} \rho_1^{-N - \ell - 2k} (N + \ell + 2k)!^{\kappa_K} \\ \times \langle \xi \rangle^m \int \langle y \rangle^{-2k'} dy \int \langle \eta \rangle^{-2k} \langle \xi + \theta \eta \rangle^{r-N} d\eta.$$

Here we used (10).

If $2k' > n$, then $\langle y \rangle^{-2k'}$ is summable. If $2k > n$, then

$$\int_{|\eta| \leq \frac{1}{2} \langle \xi \rangle} \langle \eta \rangle^{-2k} \langle \xi + \theta \eta \rangle^{r-N} d\eta \leq C_k \langle \xi \rangle^{r-N}, \\ \int_{|\eta| \geq \frac{1}{2} \langle \xi \rangle} \langle \eta \rangle^{-2k} \langle \xi + \theta \eta \rangle^{r-N} d\eta \leq C_k \langle \xi \rangle^{-2k+n}$$

Thus, if we put $k' = ((n+1)/2)^*$ and $k = ((n+1+m)/2)^*$, then

$$|\dot{r}_N|_{\ell} \leq C_{n, m, \ell} (r^* + m + \ell + 2k')^{\ell + 2k'} C^{r^* + m + \ell + 2k' - r} \rho_1^{-r^* - m - 2k - \ell + r} \\ \times \frac{(r^* + m + \ell + 2k)!^{\kappa}}{r!^{\kappa}} \left(\frac{\rho_1}{\rho_1} \right)^r K(\rho_1'/C)^{-r} r!^{\kappa} \\ \leq C' \rho_2^{-r} r!^{\kappa},$$

where $\rho_1' < \rho_1$, $\rho_2 = \frac{\rho_1'}{C}$, and C' depends on n, m, ℓ, ρ, ρ' and κ but does not depend on r and θ .

Thus the latter of (11) has been proved. (q.e.d.)

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