

On functional equations of local zeta functions of
prehomogeneous vector spaces

名大・理 鍛島 康裕 (Yasuhiro Kajima)

Introduction

In 1960's, M.Sato introduced the notion of prehomogeneous vector spaces and proved the functional equations of zeta functions associated with prehomogeneous vector spaces defined over $\mathbb{R}$ or $\mathbb{C}$

$$(|f(x)|_i^s)^{\wedge} = \sum_j \gamma_{ij}(s) |f(x)|_j^{-n/d-s},$$

where $f(x)$ is the relative invariant of the prehomogeneous vector space V , $\gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}$, $n=\dim V$, d is the degree of $f(x)$, $\wedge$ means the Fourier transform, and

$$|f(x)|_j^{-n/d-s} := \begin{cases} |f(x)|^{-n/d-s} & x \in V_j \\ 0 & \text{otherwise} \end{cases}$$

where V_j are G_{κ} -orbits in $V-S$ (for detail, see [SS]).

Similar functional equations associated with regular prehomogeneous vector spaces defined over k -adic fields have been proved by J.Igusa and F.Sato assuming some conditions of its singular set ([S],[I]).

But even when a prehomogeneous vector space (G, ρ, V) satisfies the sufficient conditions of F.Sato which assure the functional equations of zeta functions, the prehomogeneous vector space $(\tilde{G}, \tilde{\rho}, \tilde{V})$, which is obtained by the castling transform of (G, ρ, V) , does not necessarily satisfy them(cf.[S],[SK]).

Thus even if the functional equations of zeta functions of (G, ρ, V) hold, we do not know whether the functional equations of zeta

functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold or not. Since the castling transform is a standard procedure of constructing new prehomogeneous vector spaces, it is natural to ask the existence of the functional equations of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ when the functional equations of (G, ρ, V) hold.

In this paper we prove the following theorem using the results of [S0]:

Theorem *If the functional equations of zeta functions of (G, ρ, V) hold, then the functional equations of zeta functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold and vice versa (see §4).*

§1 Preliminaries

First we recall some basic notions in the theory of prehomogeneous vector spaces over a k -adic field and give some definitions following [S0].

Let k be a k -adic field and denote by $\bar{k}$ (resp. $\mathcal{O}_k$) its algebraic closure (resp. maximal order). Let G be a connected linear algebraic group defined over k and V be a finite dimensional $\bar{k}$ -vector space with k -structure V_k . Let $\rho: G \rightarrow GL(V)$ be a k -rational representation of G on V . Then the triple (G, ρ, V) is called a *prehomogeneous vector space* if there exists a proper algebraic subset S of V such that $V-S$ is a single $\rho(G)$ -orbit. The algebraic set S is called the *singular set*. It is known that $V-S$ is a single $\rho(G)$ -orbit implies that the number of $\rho(G_k)$ -orbits in V_k-S_k is finite. A nonzero rational function $P(v)$ is called a *relative invariant* of (G, ρ, V) if there exists a rational character $\chi(g)$ of G such that

$$P(\rho(g)v) = \chi(g)P(v) \quad (g \in G, v \in V).$$

Let S_i ($1 \leq i \leq l$) be the k -irreducible hypersurfaces contained in S . For each i ($1 \leq i \leq l$), take a k -irreducible polynomial $P_i(v)$ defining S_i . Then it is known that $P_i(v)$ are relative invariants and any relative invariant $P(v)$ in $k[V]$ is written uniquely as

$$P(v) = c \prod_{i=1}^l P_i(v)^{v_i} \quad (c \in k^\times, v_1, \dots, v_l \in \mathbb{Z}).$$

The polynomials $P_1, \dots, P_l$ are called the *basic relative invariants* of (G, ρ, V) , and l the k -rank of (G, ρ, V) .

A relative invariant $P(v)$ is called *nondegenerate* if the Hessian $\det\left(\frac{\partial^2 P}{\partial v_i \partial v_j}\right)$ does not vanish identically. A prehomogeneous vector space (G, ρ, V) is called *regular* if there exists a nondegenerate relative invariant; and then one can find a nondegenerate relative invariant in $k[V]$.

Let V^* be the vector space dual to V and $\rho^*: G \rightarrow GL(V^*)$ the rational representation of G contragredient to ρ . The vector space V^* has a k -structure canonically defined by the k -structure of V . Then the representation ρ^* is defined over k .

Let m and n be positive integers with $m > n \geq 1$. We consider a rational representation $\rho_0: H \rightarrow GL(m)$ of a connected linear algebraic group H . We assume that H and ρ_0 are defined over the field k . Put $G = H \times GL(n)$ and $V = M(m, n)$. Also put $\tilde{G} = H \times GL(m-n)$ and $\tilde{V} = M(m, m-n)$. Let $\rho: G \rightarrow GL(V)$ (resp. $\tilde{\rho}: \tilde{G} \rightarrow GL(\tilde{V})$) be a rational representation of G on V (resp. $\tilde{G}$ on $\tilde{V}$) defined by

$$\rho(h, g_n)v = \rho_0(h)vg_n^{-1} \quad ((h, g_n) \in G = H \times GL(n))$$

$$\text{(resp. } \tilde{\rho}(h, g_{m-n})w = {}^t\rho_0(h)^{-1}w{}^tg_{m-n} \text{ } ((h, g_{m-n}) \in \tilde{G} = H \times GL(m-n)) \text{)}.$$

The triple $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is called the *castling transform* of (G, ρ, V) and vice versa. Then we have the following lemma:

Lemma 1 (Sato-Kimura [SK1]) *The triplet (G, ρ, V) is a prehomogeneous vector space if and only if so is its castling transform $(\tilde{G}, \tilde{\rho}, \tilde{V})$.*

In the following, we assume that (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$ are prehomogeneous vector spaces with k -structure

$$\begin{aligned} G_k &= H_k \times GL(n; k), & V_k &= M(m, n; k), \\ \tilde{G}_k &= H_k \times GL(m-n; k), & \tilde{V}_k &= M(m, m-n; k). \end{aligned}$$

Put $N = \binom{m}{n}$ and let $\Delta_1(v), \dots, \Delta_N(v)$ (resp. $\tilde{\Delta}_1(w), \dots, \tilde{\Delta}_N(w)$) be the minor determinants of $v \in V$ (resp. $w \in \tilde{V}$) of size n (resp. $m-n$).

Let V_0 be the vector space of column vectors of m entries and V_0^* the vector space dual to V_0 . We identify V (resp. $\tilde{V}$) with the direct product of n (resp. $m-n$) copies of V_0 (resp. V_0^*) in the standard manner. Let $\Delta : V \longrightarrow \bigwedge^n V_0$ and $\tilde{\Delta} : \tilde{V} \longrightarrow \bigwedge^{m-n} V_0^*$ be the mapping defined by

$$\Delta(v) = \Delta(v_1, \dots, v_n) = v_1 \wedge \dots \wedge v_n \quad (v_1, \dots, v_n \in V_0)$$

and

$$\tilde{\Delta}(w) = \tilde{\Delta}(w_1, \dots, w_{m-n}) = w_1 \wedge \dots \wedge w_{m-n} \quad (w_1, \dots, w_{m-n} \in V_0^*),$$

respectively. We identify $\bigwedge^n V_0$ with $\bigwedge^{m-n} V_0^*$ via the canonical isomorphism:

$$\bigwedge^n V_0 \xrightarrow{\simeq} (\bigwedge^{m-n} V_0)^* \xrightarrow{\simeq} \bigwedge^{m-n} V_0^*.$$

By taking the standard basis, we may identify $\bigwedge^n V_0$ and $\bigwedge^{m-n} V_0^*$ with k^N , so that the mappings Δ and $\tilde{\Delta}$ are given by

$$\begin{aligned}\Delta(v) &= (\Delta_1(v), \dots, \Delta_N(v)) \\ \tilde{\Delta}(w) &= (\tilde{\Delta}_1(w), \dots, \tilde{\Delta}_N(w)).\end{aligned}$$

Here the minor determinants are indexed such that

$$\det(v, w) = \sum_{i=1}^N \Delta_i(v) \tilde{\Delta}_i(w).$$

Then it is easy to see that

$$\Delta(\rho(h, g_n)v) = \det g_n^{-1} \cdot \left(\bigwedge^n \rho_0(h) \right) (\Delta(v))$$

and

$$\tilde{\Delta}(\tilde{\rho}(h, g_{m-n})w) = (\det g_{m-n} / \det \rho_0(h)) \cdot \left(\bigwedge^n \rho_0(h) \right) (\tilde{\Delta}(w)).$$

Thus we consider that G (resp. $\tilde{G}$) operates on $\Delta(V)$ (resp. $\Delta(\tilde{V})$).

Now put

$$\begin{aligned}V' &= \{v \in V \mid \text{rank } v = n\} \quad (\text{resp. } \tilde{V}' = \{w \in \tilde{V} \mid \text{rank } w = m-n\}) \\ V'_k &= V' \cap V_k \quad (\text{resp. } \tilde{V}'_k = \tilde{V}' \cap \tilde{V}_k).\end{aligned}$$

Then we have

$$\begin{aligned}Y &= \Delta(V') = \tilde{\Delta}(\tilde{V}') \subset \bar{k}^N \\ Y_k &= \Delta(V'_k) = \tilde{\Delta}(\tilde{V}'_k) \subset k^N.\end{aligned}$$

Moreover we have the following lemma:

Lemma 2 (cf. [S0] Lemma 1.2)

- (1) The k -rank of (G, ρ, V) is equal to the k -rank of $(\tilde{G}, \tilde{\rho}, \tilde{V})$.
- (2) There exist irreducible homogeneous polynomials $Q_1, \dots, Q_l \in k[y_1, \dots, y_n]$ ($l = \text{the } k\text{-rank of } (G, \rho, V)$) such that

$$P_1(v) = Q_1(\Delta(v)), \dots, P_l(v) = Q_l(\Delta(v))$$

are the basic relative invariants of (G, ρ, V) over k and

$$\tilde{P}_1(w) = Q_1(\tilde{\Delta}(w)), \dots, \tilde{P}_l(w) = Q_l(\tilde{\Delta}(w))$$

are the basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ over k .

- (3) Put $d_i = \deg Q_i$. Then there exist k -rational characters $\psi_1, \dots, \psi_l$ of H such that

$$\begin{aligned}
P_i(\rho(h, g_n)v) &= \chi_i(h, g_n)P_i(v), \quad \chi_i(h, g_n) = (\det g_n)^{-d_i} \cdot \psi_i(h) \\
\tilde{P}_i(\tilde{\rho}(h, g_{m-n})w) &= \tilde{\chi}_i(h, g_{m-n})\tilde{P}_i(w), \\
\tilde{\chi}_i(h, g_{m-n}) &= (\det g_{m-n} / \det \rho_0(h))^{d_i} \cdot \psi_i(h).
\end{aligned}$$

Let X be a subset of V_k (resp. $\tilde{V}_k$) to which G_k (resp. $\tilde{G}_k$) operates. Denote by $\mathcal{G}(X)$ the Schwartz-Bruhat space of the topological space X and by $\mathcal{G}'(X)$ its dual space as a $\mathbb{C}$ -vector space. We call an element of $\mathcal{G}'(X)$ a *distribution* on X . And for a subset X' of X , we denote the characteristic function of X' by $Ch(X')$. For $f \in \mathcal{G}(X)$, $T \in \mathcal{G}'(X)$, $g \in G_k$, we define $f^g(x) \in \mathcal{G}(X)$ and $gT \in \mathcal{G}'(X)$ as follows:

$$\begin{aligned}
f^g(x) &= f(\rho(g)x), \\
gT(f(x)) &= T(f^g(x)) = T(f(\rho(g)x)).
\end{aligned}$$

In this paper, we always mean by a character ω of G_k a character of G_k of the form

$$\omega = \phi(\chi),$$

where $\chi: G \rightarrow k^\times$ is a k -rational character of G , and $\phi: k^\times \rightarrow \mathbb{C}^\times$ is a continuous homomorphism.

Now for a group G_k and a character $\omega: G_k \rightarrow \mathbb{C}^\times$, define $\xi(X, G_k, \omega)$ as follows:

$$\xi(X, G_k, \omega) = \{T \in \mathcal{G}'(X) \mid gT = \omega(g)^{-1}T \text{ for all } g \in G_k\}.$$

And for a subgroup H of G_k we define the *modular function* $\delta(H)$ by $d(h_0^{-1}hh_0) = \delta(H)(h_0) \cdot dh$, where dh is a left invariant measure on H .

Now we quote the following lemma which will be used in the next chapter:

Lemma 3 ([I] p.1015) *Let G be a linear algebraic group defined over*

k , G_k be the set of its k -rational points and H_k be a closed subgroup of G_k . Let $x = G_k/H_k$, and ω as above; then $\xi(x, G_k, \omega) \neq 0$ if and only if

$$\omega \cdot \delta(G_k)|_{H_k} = \delta(H_k),$$

and in that case $\dim_{\mathbb{C}} \xi(x, G_k, \omega) = 1$

Put

$$S(1) = \{v \in V_k \mid \text{rank } v < n\}.$$

Then we have

$$\rho(G_k)S(1) = S(1).$$

Thanks to results of [S0], we know that our theorem is valid if we restrict ourselves to the case where $x \in V_k \setminus S(1)$. Therefore we study the distributions $\xi(S(1), G_k, \omega)$ in §2 and prove the main theorem in §3.

§2 We keep the notations in §1. The aim of this section is to prove the following proposition:

Proposition 1

If $\xi(S(1), G_k, \omega) \neq 0$, there exist a finite number of subgroups H_i , $1 \leq i \leq q$ of $GL(n; k)$ such that

$$\omega|_{H_i} = \delta(H_i) \quad \text{for some } i,$$

and $\delta(H_i)$ is not identically equal to 1 if $H_i \neq GL(n; k)$.

Let V_k be as before and G' the set of k -rational points of a connected algebraic group which acts on V_k by a k -rational representation ρ .

Let A be a G' -stable subset of V_k such that for any $x, y \in A$, the isotropy subgroups G'_x and G'_y are conjugate in G' . Then there exists a complete system V_A of representatives of G' -orbits in A such that $G'_x = G'_y$ for any $x, y \in V_A$. We put $G'_A = G'_x$ for $x \in V_A$. Then we have a bijection

$$\Phi: G'/G'_A \times V_A \longrightarrow A$$

defined by $\Phi(g \cdot G'_A, x) = \rho(g)x$.

Now we assume that

Φ is a homeomorphism.

Lemma 4 Under the conditions above, we have

$$\xi(A, G', \omega) \neq 0 \text{ if and only if } \omega \cdot \delta(G')|_{G'_{iso}} = \delta(G'_{iso}).$$

Proof. It is easy to see that if there exists a G' -orbit $\emptyset$ in A such that $\xi(\emptyset, G', \omega) \neq 0$, then we have $\xi(A, G', \omega) \neq 0$. Therefore "if" part is trivial from Lemma 3. Now we prove "only if" part.

Notice that any compact open subset of totally disconnected topological space $G'/G'_A \times V_A$ can be written as a disjoint union of compact open subsets of the form $U_o \times U_v$, where U_o (resp. U_v) is a compact open subset of G'/G'_A (resp. V_A).

Thus, if $\xi(A, G', \omega) \neq 0$, then there exists

$$T \in \xi(A, G', \omega) \text{ such that } T(Ch(\Phi(U))) \neq 0$$

where $U = U_o \times U_v$. We fix these U_o and U_v . Now for each compact

open subset $U(\sigma)$ of G'/G'_A , we define a mapping τ by

$$\tau: U(\sigma) \longrightarrow \Phi(U(\sigma) \times U_\nu).$$

Then we have

$$\tau(U_\sigma) = \Phi(U).$$

It is easy to see that τ is G' -admissible i.e.,

$$\tau(g \cdot U(\sigma)) = \Phi(g \cdot U(\sigma) \times U_\nu) = g \cdot \Phi(U(\sigma) \times U_\nu) = g \cdot \tau(U(\sigma)). \quad (1)$$

Now τ induces a linear mapping

$$\bar{\tau}: \mathcal{G}(G'/G'_A) \longrightarrow \mathcal{G}(A).$$

Using (1), we have

$$g \cdot (\bar{\tau}(f(x))) = \bar{\tau}(f(\rho(g)x))$$

for all $f \in \mathcal{G}(x)$. Now we define $F \in \mathcal{G}'(G'/G'_A)$ by

$$F(f) = T(\bar{\tau}(f)).$$

Then we have

$$\begin{aligned} g \cdot F(f) &= T(\bar{\tau}(f^g)) = T((\bar{\tau}(f))^g) \\ &= g \cdot T(\bar{\tau}(f)) = \omega^{-1}(g) T(\bar{\tau}(f)) = \omega(g)^{-1} F(f), \end{aligned}$$

where $g \in G'$.

Therefore we have $F \in \xi(G'/G'_A, G', \omega)$. Moreover we have

$$F(f) = T(Ch(U(\sigma))) \neq 0$$

for $f = Ch(U(\sigma))$, which implies

$$\xi(G'/G'_A, G', \omega) \neq 0.$$

Thus, from Lemma 3, we have

$$\omega \cdot \delta(G')|_{G'_A} = \delta(G'_A).$$

□

Now put

$$S_r = \{v \in S(1) \mid \text{rank } v = r\} \quad \text{for } 0 \leq r \leq n-1.$$

Then $\rho(1, GL(n; k)) S_r = S_r$ for all $0 \leq r \leq n-1$ and

$$S(1) = S_{n-1} \cup \dots \cup S_0 \quad (\text{disjoint union}).$$

Now the following lemma holds.

Lemma 5

Each S_r can be decomposed as follows:

$$S_r = \rho(1, GL)S_{r1} \cup \dots \cup \rho(1, GL)S_{rl} \quad (\text{disjoint union}),$$

where $l = \binom{m}{r}$, $GL = GL(n; k)$, and by putting $G' = GL$, $A = \rho(1, GL)S_{rh}$ and $V_A = S_{rh}$, these G' , A , and V_A satisfy the conditions (a), (b), and (c) above for all $1 \leq h \leq l$.

Proof. For a matrix $x = (x_{ij})$ and $j = 1, \dots, n$, we denote by $I(j, x)$ the smallest i for which $x_{ij} \neq 0$. We denote by $Rep(r)$ the set of matrices of the form

$$x = \begin{pmatrix} w & 0 \end{pmatrix} \in M(m, n), \quad w = (w_{ij}) \in M(m, r)$$

with the condition that

$$w_{I(j, x)h} = \delta_{jh} \quad (\text{Kronecker's symbol}) \quad \text{for all } 1 \leq j \leq r, 1 \leq h \leq r \quad \text{and} \\ I(j, x) \leq I(j+1, x) - 1 \quad \text{for all } 1 \leq j \leq r-1.$$

It is trivial that $Rep(r)$ is a complete system of representatives of $\rho(1, GL)$ -orbits in S_r . Also define $\alpha(x) \in \mathbb{Z}^r$ by

$$\alpha(x) = (I(1, x), \dots, I(r, x)).$$

Then, for $x \in Rep(r)$, we have

$$1 \leq I(1, x) < \dots < I(r, x) \leq m.$$

Now put

$$I = \{(i_1, \dots, i_r) \mid 1 \leq i_1 < \dots < i_r \leq m\}.$$

It is clear that $|I| = \binom{m}{r}$, and we number the elements of the set I in an arbitrary order and write I as follows:

$$I = \{I_i \mid (1 \leq i \leq \binom{m}{r})\}.$$

For any $I_t = (I(1), \dots, I(r)) \in I$ we define S_{ri} by

$$S_{ri} = \alpha^{-1}(I_t).$$

Now S_{ri} is homeomorphic to the k -vector space of dimension

$$(n - I(r)) \times r + \sum_{j=1}^{r-1} (I(j+1) - I(j) - 1) \times j,$$

and it is easy to see that if we put $G' = GL$, $A = \rho(1, GL)S_{rh}$, $V_A = S_{rh}$, these G' , A , V_A satisfy the conditions (a), (b), and (c). Moreover the condition

$$S_r = \rho(1, GL)S_{r1} \cup \dots \cup \rho(1, GL)S_{rl} \quad (\text{disjoint union})$$

is now trivial. □

Now we can prove Proposition 1.

Proof of Proposition 1. We keep the notation in Lemma 5.

Since $\xi(S(1), G_k, \omega) \neq 0$ implies $\xi(S(1), GL(n; k), \omega) \neq 0$, we have only to prove that the conditions in our proposition are necessary if $\xi(S(1), GL(n; k), \omega) \neq 0$.

The isotropy subgroup of $GL(n; k)$ for $x \in \text{Rep}(r)$, which we denote by H_x , is given by

$$H_x = \left\{ \begin{pmatrix} I_r & 0 \\ * & * \end{pmatrix} \in GL(n; k) \right\}, \text{ where } I_r \text{ is the identity matrix of size } r.$$

Therefore $\delta(H_x)$ is not identically equal to 1 if $H_x \neq GL(n; k)$

($H_x = GL(n; k)$ if and only if $x = 0$). Thus using Lemma 4 and Lemma 5,

$S(1)$ is decomposed as follows:

$$S(1) = \cup S_{rh} \quad (\text{disjoint union}),$$

where the union runs through $0 \leq r \leq n-1$, $1 \leq h \leq \binom{n}{r}$, and

$\rho(1, GL(n; k))S_{rh} = S_{rh}$ for all $1 \leq r \leq n-1$, $1 \leq h \leq \binom{n}{r}$, and if

$\xi(S_{rh}, GL(n; k), \omega) \neq 0$, then there exist $H_{rh} (= H_x \text{ for } x \in \text{Rep}(r)) \subset GL(n; k)$

such that

$$\delta(GL(n; k)) \cdot \omega|_{H_{rh}} = \delta(H_{rh})$$

and $\delta(H_{rh})$ is not identically equal to 1 if $H_{rh} \neq GL(n; k)$. Now since $GL(n; k)$ is unimodular, we have

$$\omega|_{H_{rh}} = \delta(H_{rh}).$$

Thus we have proved the proposition. $\square$

§3 Main result

For $\sigma = (\sigma_1, \dots, \sigma_l) \in \mathbb{C}^l$, we define the character ω_σ of G_k by

$$\omega_\sigma = (\phi_1 \omega_{\sigma_1}(\chi_1), \dots, \phi_l \omega_{\sigma_l}(\chi_l)),$$

where $\omega_{\sigma_i}(\alpha) = |\alpha|_k^{\sigma_i}$ ($\alpha \in k^\times$, $1 \leq i \leq l$), ϕ_i ($1 \leq i \leq l$) are dual of $\mathcal{O}_k^\times$ and l

is the k -rank of (G, ρ, V) . Also define $\tilde{\omega}_\sigma$ for $\sigma = (\sigma_1, \dots, \sigma_l)$ by

$$\tilde{\omega}_\sigma = (\phi_1 \omega_{\sigma_1}(\tilde{\chi}_1), \dots, \phi_l \omega_{\sigma_l}(\tilde{\chi}_l)),$$

and put $\Omega = \{\sigma \in \mathbb{C}^l \mid \xi(S, G_k, \omega_\sigma) \neq 0\}$.

Now we recall some facts in the theory of prehomogeneous vector spaces roughly.

The fundamental theorem in the theory of regular prehomogeneous vector spaces states that there exist $u \in GL(n; \mathbb{Z})$, $\lambda \in \mathbb{C}^l$, and meromorphic functions $\gamma_{ij}(\sigma)$ such that

$$F(x, \sigma)_l = (|P(x)|_{k, l}^\sigma)^\wedge - \sum_j^v \gamma_{ij}(\sigma) \cdot |P^*(x)|_{k, j}^{\lambda + u\sigma}$$

vanishes for all $x \in V_k$, $\sigma \in \mathbb{C}^l$, and $1 \leq i, j \leq v$, i.e.,

$$(|P(x)|_{k, l}^\sigma)^\wedge = \sum_j^v \gamma_{ij}(\sigma) \cdot |P^*(x)|_{k, j}^{\lambda + u\sigma}, \quad (2)$$

where $^\wedge$ means the Fourier transform, v is the number of

$\rho(G_k)$ -orbits in $V_k - S_k$, and $|P(x)|_{k,i}^{\sigma} = \prod_{h=1}^l |P_h(x)|_{k,i}^{\sigma_h}$ (functional equations of zeta functions, see [S]).

For $x \in V_k - S$, the theorem above is proved by the uniqueness of relatively invariant distributions of homogeneous space (cf. Lemma 3). Moreover, thanks to results of [S0], we know that if the functional equations of zeta functions of (G, ρ, V) hold for $x \in V_k - S(1)$, then the functional equations of zeta functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold for $x \in \tilde{V}_k - \tilde{S}(1)$. On the other hand, it is known that $F(x, \sigma)_i \neq 0$ for some $x \in S(1)$ implies that there exists non-zero $T \in \xi(S(1), G_k, \omega_{\sigma})$ such that T is meromorphic with respect to $\sigma \in \mathbb{C}^l$. Therefore we can prove $F(x, \sigma)_i = 0$ for all $x \in S(1)$, $\sigma \in \mathbb{C}^l$ by showing that $\mathbb{C}^l \setminus \Omega$ is dense in $\mathbb{C}^l$. Hence, for the proof of the functional equations, it is enough to show that $\mathbb{C}^l \setminus \Omega$ is dense in $\mathbb{C}^l$.

Now we prove the main theorem.

Theorem *If the functional equations of zeta functions (2) of (G, ρ, V) hold, then the functional equations of zeta functions (2) of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold and vice versa.*

Proof. Put $\Omega(1) = \{\sigma \in \mathbb{C}^l \mid \xi(S(1), GL(n; k), \omega_{\sigma}) \neq 0\}$,

$\tilde{\Omega}(1) = \{\sigma \in \mathbb{C}^l \mid \xi(\tilde{S}(1), GL(m-n; k), \tilde{\omega}_{\sigma}) \neq 0\}$.

For a subset Ω' of $\mathbb{C}^l$, we denote $\mathbb{C}^l \setminus \Omega'$ by $C(\Omega')$.

Now from Proposition 1 of §2 and the fact that ω_{σ} is not trivial on $GL(n; k)$, we know that $C(\Omega(1))$ and $C(\tilde{\Omega}(1))$ are dense subsets of $\mathbb{C}^l$.

Now using the remarks above, we have proved the theorem. $\square$

References

- [I] J.Igusa, Some results on p-adic complex powers. *Amer.J.Math.*, 106(1984) 1013-1032.
- [S] F.Sato, On functional equations of zeta-distributions. *Adv. Studies in pure Math.* 15(1989) 465-508.
- [SO] F.Sato and H.Ochiai, Castling transforms of prehomogeneous vector spaces and functional equations. *Comment. Math. Univ. St. Pauli.* 40:1(1991) 61-82.
- [SK] M.Sato and T.Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya. Math. J.* 65(1977) 1-155.
- [SS] M.Sato and T.Shintani, On zeta functions associated with prehomogeneous vector spaces. *Ann. of Math.* 100(1974) 131-170.

(注) 以上は 投稿予定の論文を、若干省略したものです。 応用等は省きましたが 例えば既約概均質ベクトル空間については、その少なくとも一つの k -form においては関数等式が成たつことがわかります。

また、以前 preprint を お渡ししたかたには、 (G_ρ, V) の関数等式から その castling transform について示すのでなく、 (G_ρ, V) において [S] の sufficient condition が成立するという仮定から その castling transform について 関数等式の成立を示していた事を 記しておきます (既約 p.v.については これで十分ですが)。