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On the resolvent problem for one dimensional Schrödinger operators with singular potentials

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1. Introduction

This paper is a joint work with Professor Giorgio Metafune (University of Salento) and a part of [13]. In this paper we consider the resolvent problem for one-dimensional Schrödinger operators with singular potentials:

$$H=-\frac{d^2}{dr^2}+\frac{a}{r^2} \quad \text{in } L^2(\mathbb{R}_+),$$

where $a\in(-\infty,-\frac{1}{4})$ and $\mathbb{R}_+:=(0,\infty)$.

As is well-known, $H_{\min}$ (H endowed with domain $C_0^\infty(\mathbb{R}_+)$) is nonnegative if and only if $a\geq-\frac{1}{4}$. In this case, the Friedrichs extension of $H_{\min}$ exists. This is a consequence of the one-dimensional Hardy inequality

$$\frac{1}{4}\int_0^\infty\frac{|u(r)|^2}{r^2}\,dr\leq\int_0^\infty|u'(r)|^2\,dr, \quad u\in C_0^\infty(\mathbb{R}_+).$$

In the view-point of ordinary differential equation, the solution of $Hu=0$ can be simply written as

$$u(r)=\begin{cases} c_1r^{\frac{1}{2}+\nu}+c_2r^{\frac{1}{2}-\nu} & \text{if } a>-\dfrac{1}{4},\\ c_1r^{\frac{1}{2}}+c_2r^{\frac{1}{2}}\log r & \text{if } a=-\dfrac{1}{4},\\ c_1r^{\frac{1}{2}+i\nu}+c_2r^{\frac{1}{2}-i\nu} & \text{if } a<-\dfrac{1}{4}\end{cases}$$

with $\nu=\sqrt{|b+\frac{1}{4}|}$ and an arbitrary constants $c_1,c_2\in\mathbb{C}$. This means that existence of positive solutions to $Hu=0$ holds if and only if $a\geq-\frac{1}{4}$ and every solution is oscillating if $a<-\frac{1}{4}$. We remark that $H_{\min}$ is essentially selfadjoint ($H_{\min}$ has a unique selfadjoint extension) if and only if $a\geq\frac{3}{4}$.

In N-dimensional case, by Hardy's inequality

$$\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx\leq\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx \qquad u\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$$

the operator $L_{\min}=-\Delta+b|x|^{-2}$ (endowed with domain $C_0^\infty(\mathbb{R}^N\setminus\{0\})$) is nonnegative if and only if $b\geq-(\frac{N-2}{2})^2$. We also remark that the essentially selfadjointness of $L_{\min}$

holds for $b \geq -(\frac{N-2}{2})^2 + 1$ (see [**17**, Section X.1]). Further previous works for $L_{\min}$ in L^p spaces can be found in Okazawa [**15**], Liskevich, Sobol and Vogt [**9**] and Metafune et al. [**14**]. On the other hand if $b < -\left(\frac{N-2}{2}\right)^2$, Baras and Goldstein proved in [**2**] that there exists no nonnegative (non-trivial) distributional solution of the equation

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + \frac{b}{|x|^2}u(x,t) = 0, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+. \tag{1.1}$$

This nonexistence result for nonnegative solutions has been generalized by subsequent papers ([**4**], [**7**], [**8**], [**10**] and [**6**]).

In the present paper we consider the one-dimensional case under the assumption

$$a < -\frac{1}{4}, \quad \nu := \sqrt{-a - \frac{1}{4}} > 0. \tag{1.2}$$

We characterize all realizations of operators between $H_{\min}$ and $H_{\max} := (H_{\min})^*$, given by

$$D(H_{\max}) := \{u \in L^2(\mathbb{R}_+) \cap H^2_{\mathrm{loc}}(\mathbb{R}_+) \; ; \; Hu \in L^2(\mathbb{R}_+)\},$$

having the non-empty resolvent set by introducing a boundary condition at 0 of oscillating type. Spectral properties of selfadjoint realizations of H are also considered in [**5**] when $a < -\frac{1}{4}$.

This paper is organized as follows. In Section 2, we analyze the properties of solutions to the equation $\lambda u + Hu = f$. Section 3 is devoted to show how to construct all realizations of H with non-empty resolvent set. Generation of analytic semigroup on $L^2(\mathbb{R}_+)$ by realizations of $-H$ is considered in Section 4. Finally, in Section 5 we mention generation result for realization of $-L$ in N-dimensional case.

2. Preliminaries

In this section we study the equation $\lambda u + Hu = f$.

2.1. The homogeneous equation

If $\lambda \notin (-\infty, 0]$, then the above equation with $f = 0$ has two solutions. One is exponential decaying and the other is exponential growing at ∞. The behavior of these two solutions near 0 is clarified in the next two lemmas.

Lemma 1. *Let $\omega \in \mathbb{C}_+ := \{z \in \mathbb{C} \; ; \; \mathrm{Re}\, z > 0\}$, $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \frac{\pi}{2}$. Assume that* (1.2) *is satisfied. Then there exists a solution $\varphi_{\omega,0}$ of*

$$\omega^2 \varphi(r) - \varphi''(r) + \frac{a}{r^2}\varphi(r) = 0, \quad r \in \mathbb{R}_+ \tag{2.1}$$

and a constant $R = R(b, \omega) > 0$ such that

$$|\varphi_{\omega,0}(r)| \leq 2e^{-(\mathrm{Re}\,\omega)r}, \quad r \geq R \tag{2.2}$$

and there exists $\alpha \in \mathbb{C} \setminus \{0\}$ *such that*

$$\left| r^{-\frac{1}{2}}\varphi_{\omega,0}(r) - \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}} \left(\alpha \mu^{i\nu} e^{-\xi\nu} r^{i\nu} + \overline{\alpha} \mu^{-i\nu} e^{\xi\nu} r^{-i\nu} \right) \right| \to 0 \quad \textit{as } r \downarrow 0. \tag{2.3}$$

Moreover, if ω *is real, then* $\varphi_{\omega,0}(r)$ *is real.*

Proof. **(Step 1).** We consider the following equation in $\mathbb{C}_+$:

$$w(z) - \frac{d^2 w}{dz^2}(z) + \frac{a}{z^2} w(z) = 0, \quad z \in \mathbb{C}_+. \tag{2.4}$$

The indicial equation $\alpha(\alpha - 1) = a$ has roots $\alpha_1 = \frac{1}{2} + i\nu$ and $\alpha_2 = \frac{1}{2} - i\nu$. Then every solution has the form

$$w(z) = g_1(z) z^{\frac{1}{2}+i\nu} + g_2(z) z^{\frac{1}{2}-i\nu}, \tag{2.5}$$

with g_1 and g_2 which are entire functions. And therefore w is holomorphic in $\mathbb{C}\setminus(-\infty, 0]$, see [**3**, Chapter 9.6, 9.8].

Now we show that there exists a solution of (2.4) which behaves like e^{-z} in $E_R = \{z \in \mathbb{C}_+ \,;\, |z| > R\}$. Setting $h(z) := e^z w(z)$, we see that (2.4) reduces to

$$\frac{d^2 h}{dz^2}(z) - 2\frac{dh}{dz}(z) = \frac{a}{z^2} h(z), \quad z \in \mathbb{C}_+. \tag{2.6}$$

We denote X as the set of all bounded holomorphic functions in E_R, endowed with $\|h\|_X := \sup_{z \in E_R} |h(z)|$. Define

$$Th(z) = 1 + \int_{\Gamma_z} e^{2\xi} \left(\int_{\Gamma_\xi} \frac{a e^{-2\eta}}{\eta^2} h(\eta)\, d\eta \right) d\xi, \quad z \in E_R,$$

where $\Gamma_z := \{tz \,;\, t \in [1, \infty)\}$; note that a fixed point of T is not 0 and satisfies (2.6). Then $T : X \to X$ is well-defined and contractive in X when R is large enough. In fact, if $h \in X$, then Th is well-defined and holomorphic in E_R. Moreover, for $z \in E_R$,

$$\begin{aligned}
|Th(z) - 1| &= \left| \int_1^\infty e^{2tz} \left(\int_t^\infty \frac{a e^{-2sz}}{(sz)^2} h(sz) z\, ds \right) z\, dt \right| \\
&= \left| \int_1^\infty \left(\int_1^s e^{2tz}\, dt \right) \frac{a e^{-2sz}}{s^2} h(sz)\, ds \right| \\
&\le \sup_{1 \le s < \infty} \left| \frac{a(1 - e^{2(s-1)z})}{2z} \right| \left(\int_1^\infty \frac{1}{s^2}\, ds \right) \|h\|_X \\
&\le \frac{|a|}{R} \|h\|_X.
\end{aligned}$$

Similarly, we have $|Th_1(z) - Th_2(z)| \le \frac{|a|}{R} \|h_1 - h_2\|_X$ for every $h_1, h_2 \in X$ and $z \in E_R$. Therefore $T : X \to X$ is well-defined and if we choose $R = R_0 := 2|a|$, then T is contractive. By Banach's contraction mapping principle, there exists a unique fixed point $h_0 \in X$ of T. Noting that

$$|h_0(z) - 1| = |Th_0(z) - T0(z)| \le \frac{|a|}{R_0}\|h_0\|_X \le \frac{\|h_0 - 1\|_X + 1}{2},$$

we deduce $\|h_0 - 1\|_X \le 1$. Taking $w_0(z) = e^{-z}h_0(z)$ it follows that w_0 has an analytic continuation to a solution of (2.4) and

$$|e^z w_0(z)| \le 2, \quad z \in E_{R_0}.$$

Now we define

$$\varphi_{\omega,0}(r) = w_0(\omega r), \quad r \in \mathbb{R}_+.$$

Then $\varphi_{\omega,0}$ satisfies (2.1):

$$\begin{aligned}\omega^2\varphi_{\omega,0}(r) - \varphi''_{\omega,0}(r) + \frac{a}{r^2}\varphi_{\omega,0}(r) &= \omega^2\left(w_0(\omega r) - \frac{d^2 w_0}{dz^2}(\omega r) + \frac{a}{(\omega r)^2}w_0(\omega r)\right)\\ &= 0.\end{aligned}$$

Moreover, if $r > R := R_0/|\omega|$, then

$$\begin{aligned}|e^{\omega r}\varphi_{\omega,0}(r)| &= |e^{\omega r}w_0(\omega r)|\\ &\le 2\end{aligned}$$

and therefore (2.2) is satisfied.

(Step 2). Next we consider w_0 on $\mathbb{R}_+$. Note that w_0 is real on $\mathbb{R}_+$. In fact, $w_0(r)$ and $\overline{w_0}(r)$ are solutions of (2.4) on $\mathbb{R}_+$ which behave like e^{-r} near ∞. Since such a solution of (2.4) is unique, it follows that $w_0(r) = \overline{w_0}(r)$ for $r \in \mathbb{R}_+$. By (2.5) we have

$$w_0(z) = g_1(z)z^{\frac{1}{2}+i\nu} + g_2(z)z^{\frac{1}{2}-i\nu}, \quad z \in \mathbb{C}\setminus(-\infty, 0], \tag{2.7}$$

where g_1, g_2 are entire functions. Then $g_1(r) = \overline{g_2(r)}$ for $r > 0$ and $\alpha = g_1(0) = \overline{g_2(0)}$. This implies that

$$\left|z^{-\frac{1}{2}}w_0(z) - \left(\alpha z^{i\nu} + \overline{\alpha}z^{-i\nu}\right)\right| \to 0 \quad \text{as } z \to 0 \quad (z \in \mathbb{C}_+).$$

Consequently, we obtain (2.3):

$$\begin{aligned}&\left|r^{-\frac{1}{2}}\varphi_{\omega,0}(r) - \mu^{\frac{1}{2}}e^{i\frac{\xi}{2}}\left(\alpha e^{-\xi\nu}\mu^{i\nu}r^{i\nu} + \overline{\alpha}e^{\xi\nu}\mu^{-i\nu}r^{-i\nu}\right)\right|\\ &= \mu^{\frac{1}{2}}\left|(\omega r)^{-\frac{1}{2}}w_0(\omega r) - \left(\alpha(\omega r)^{i\nu} + \overline{\alpha}(\omega r)^{-i\nu}\right)\right| \to 0 \quad \text{as } r \downarrow 0.\end{aligned}$$

This completes the proof. □

Next we study the behavior at 0 of the exponentially growing solution.

Lemma 2. *Let $\omega \in \mathbb{C}_+$ satisfy $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \pi/2$. Assume that* (1.2) *is satisfied. Then there exist a solution $\varphi_{\omega,1}$ of* (2.1) *and constants $C'_\omega > C_\omega > 0$ and $R' > 0$ such that*

$$C_\omega e^{(\mathrm{Re}\,\omega)r} \le |\varphi_{\omega,1}(r)| \le C'_\omega e^{(\mathrm{Re}\,\omega)r} \quad \text{for } r \ge R', \tag{2.8}$$

$$\left| r^{-\frac{1}{2}}\varphi_{\omega,1}(r) - \mu^{\frac{1}{2}} e^{i\frac{\xi}{2}} \left(\alpha\mu^{i\nu} e^{-\xi\nu} r^{i\nu} - \overline{\alpha}\mu^{-i\nu} e^{\xi\nu} r^{-i\nu}\right)\right| \to 0 \quad \text{as } r \downarrow 0, \tag{2.9}$$

where α is given in Lemma 1. Moreover, if ω is real, then $i\varphi_{\omega,1}(r)$ is real.

Proof. By (2.5) there exist two solutions w_1, w_2 satisfying

$$z^{-\frac{1}{2}-i\nu} w_1(z) \to 1, \quad z^{-\frac{1}{2}+i\nu} w_2(z) \to 1 \quad \text{as } z \to 0.$$

With the same notation as in the proof of Lemma 1, we have $\varphi_{\omega,0}(r) = w_0(\omega r)$ and $w_0(z)$ is given by (2.7), $g_1(r) = \overline{g_2(r)}$ for $r > 0$ and $\alpha = g_1(0) = \overline{g_2(0)} \neq 0$. Now we take $v(z) = g_1(z) z^{\frac{1}{2}+i\nu} - g_2(z) z^{\frac{1}{2}-i\nu}$. Then w_0, v are linearly independent and $\varphi_{\omega,1}(r) = v(r\omega)$ is a solution of (2.1) which satisfies (2.9) and is imaginary when ω is real. To prove (2.8) we note that (2.1) has one solution which behaves like $\exp(-\omega r)$ (namely, $\varphi_{\omega,0}$) and one solution which behaves like $\exp(\omega r)$ at ∞, see [**12**, Proposition 4] for an elementary proof. Since $\varphi_{\omega,1}$ is independent of $\varphi_{\omega,0}$, (2.8) holds. □

Finally we consider the case where $\omega = i\mu$ with $\mu > 0$.

Lemma 3. *Assume that* (1.2) *is satisfied. Then for every $\mu > 0$, there exist two solutions $\varphi_{i\mu,0}$ and $\varphi_{i\mu,1}$ of*

$$-\mu^2\varphi(r) - \varphi''(r) + \frac{a}{r^2}\varphi(r) = 0, \quad r \in \mathbb{R}_+ \tag{2.10}$$

such that as $r \to \infty$,

$$e^{-i\mu r}\varphi_{i\mu,0}(r) \to 1, \qquad e^{i\mu r}\varphi'_{i\mu,0}(r) \to i\mu,$$
$$e^{i\mu r}\varphi_{i\mu,1}(r) \to 1, \qquad e^{i\mu r}\varphi'_{i\mu,1}(r) \to -i\mu.$$

Proof. It suffices to apply [**12**, Proposition 5], with $f(x) = -\mu^2$, to (2.10) (see also [**16**, Theorem 6.2.2]). □

2.2. The inhomogeneous equation

Lemma 4. *Let $\omega \in \mathbb{C}_+$ satisfy $\omega = \mu e^{i\xi}$ with $\mu > 0$, $|\xi| < \pi/2$. Assume that* (1.2) *is satisfied. Let $\varphi_{\omega,0}$ and $\varphi_{\omega,1}$ be as in Lemmas 1 and 2. Then for $f \in L^2(\mathbb{R}_+)$, every solution of*

$$\omega^2 u(r) - u''(r) + \frac{b}{r^2}u(r) = f(r), \quad r \in \mathbb{R}_+$$

is given by

$$u(r) = c_0\varphi_{\omega,0}(r) + c_1\varphi_{\omega,1}(r) + T_\omega f(r), \tag{2.11}$$

where $c_{\in}\mathbb{C}$ *and* $c_1 \in \mathbb{C}$ *are constants and*

$$T_\omega f(r) = \frac{1}{W(\omega)}\left(\int_0^r \varphi_{\omega,1}(s)f(s)\,ds\right)\varphi_{\omega,0}(r) + \frac{1}{W(\omega)}\left(\int_r^\infty \varphi_{\omega,0}(s)f(s)\,ds\right)\varphi_{\omega,1}(r),$$

with the Wronskian $W(\omega)$ *of* $\varphi_{\omega,0}, \varphi_{\omega,1}$. *The map* T_ω *is a bounded linear operator from* $L^2(\mathbb{R}_+)$ *to itself. Moreover, if* ω *is real, then* T_ω *is selfadjoint.*

Proof. By variation of parameters (2.11) easily follows. Observe that

$$T_\omega f(r) = \int_0^\infty G_\omega(r,s)f(s)\,ds,$$

where

$$G_\omega(r,s) = \begin{cases} W(\omega)^{-1}\varphi_{\omega,0}(r)\varphi_{\omega,1}(s) & \text{if } s \le r, \\ W(\omega)^{-1}\varphi_{\omega,0}(s)\varphi_{\omega,1}(r) & \text{if } s \ge r. \end{cases}$$

Using Lemmas 1 and 2 and noting that both solutions are bounded near 0, we obtain $|\varphi_{\omega,0}(r)| \le Ce^{-(\mathrm{Re}\,\omega)r}$, $|\varphi_{\omega,1}(r)| \le Ce^{(\mathrm{Re}\,\omega)r}$ for every $r > 0$. Therefore

$$|G_\omega(r,s)| \le C^2 e^{-(\mathrm{Re}\,\omega)|r-s|}, \quad r > 0,\ s > 0$$

and therefore the boundedness of T_ω follows. If ω is real, then $\varphi_{\omega,0}, i\varphi_{\omega,1}, iW(\omega)$ are real. Hence we have $\overline{G_\omega(r,s)} = G_\omega(s,r)$, that is, T_ω is selfadjoint. □

3.Realizations of H and their spectral properties

Here we characterize all extensions $H_{\min} \subset \tilde{H} \subset H_{\max}$ with non-empty resolvent set by introducing a boundary condition at 0 of oscillating type. And we study their spectral properties.

Lemma 5. *Let the operator* $\tilde{H}$ *satisfy* $H_{\min} \subset \tilde{H} \subset H_{\max}$. *Then* $[0,\infty) \subset \sigma(\tilde{H})$.

Proof. First we prove $(0,\infty) \in \sigma(-\tilde{H})$. Let $\eta_n(r)$ be a smooth function equal to 1 in $[n, 2n]$, with support contained in $[\frac{n}{2}, 3n]$ and $0 \le \eta_n \le 1$, $|\eta_n'| \le \frac{C}{n}$, $|\eta_n''| \le \frac{C}{n^2}$. Using $\varphi_{i\mu,0}$ as in Lemma 3, we consider $\psi_n = \eta_n\varphi_{i\mu,0} \in C_0^\infty(\mathbb{R}_+) \subset D(\tilde{H})$. Then we see that

$$-\mu^2\psi_n + H\psi_n = -2\eta_n'\varphi_{i\mu,0}' - \eta_n''\varphi_{i\mu,0}.$$

We have $\|\psi_n\|_2 \approx \sqrt{n}$ and, since $\varphi_{i\mu,0}$ and $\varphi_{i\mu,0}'$ are bounded near ∞,

$$\|(\mu^2 + H)\psi_n\|_2 \le Cn^{-1/2}.$$

Therefore μ^2 is the approximate point spectrum, in other words, $-\mu^2 + H$ does not have a bounded inverse. Finally, noting that $\sigma(\tilde{H})$ is closed in $\mathbb{C}$, we have $[0,\infty) \subset \sigma(\tilde{H})$. □

Lemma 6. *Let $H_{\min} \subset \tilde{H} \subset H_{\max}$. Assume that (1.2) and $\rho(\tilde{H}) \neq \emptyset$ are satisfied. Then there exists $\tilde{c} \in \mathbb{C}$ such that the domain of $\tilde{H}$ is given by*

$$(3.1)\quad D(\tilde{H}) = \left\{ u \in D(H_{\max})\,;\, \exists C \in \mathbb{C} \text{ s.t. } \lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C\left(a_1 r^{i\nu} + a_2 r^{-i\nu}\right) \right| = 0 \right\},$$

where the pair $(a_1, a_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is given by

$$(3.2)\qquad a_1 = (\tilde{c} + W(\omega)^{-1})\alpha \mu^{i\nu} e^{-\xi\nu}, \qquad a_2 = (\tilde{c} - W(\omega)^{-1})\overline{\alpha} \mu^{-i\nu} e^{\xi\nu}.$$

Proof. First we show the inclusion "$\subset$" in (3.1). Fix $\lambda \in \rho(\tilde{H})$. It follows from Lemma 5 that $\lambda \in \mathbb{C} \setminus [0, \infty)$. Let $\omega \in \mathbb{C}_+$ satisfy $-\omega^2 = \lambda$. From Lemma 4, we have

$$[(\omega^2 + \tilde{H})^{-1} f](r) = c_0(f)\varphi_{\omega,0}(r) + c_1(f)\varphi_{\omega,1}(r) + T_\omega f(r).$$

Since $\varphi_{\omega,1} \notin L^2(\mathbb{R}_+)$ and $\varphi_{\omega,0} \in L^2(\mathbb{R}_+)$, it follows that $c_1(f)$ is 0 and that $c_0(f)$ is a bounded linear functional in $L^2(\mathbb{R}_+)$. Riesz's representation theorem yields that there exists $v \in L^2(\mathbb{R}_+)$ such that

$$c_0(f) = \int_0^\infty f(s) v(s)\, ds.$$

If we choose $f = \omega^2 u + Hu$ for $u \in C_0^\infty(\mathbb{R}_+)$, then, for r small enough, by integration by parts we see that

$$\begin{aligned} 0 &= u(r) \\ &= c_0(f)\varphi_{\omega,0}(r) + \frac{1}{W(\omega)}\left(\int_0^\infty \varphi_{\omega,0}(s) f(s)\, ds\right)\varphi_{\omega,1}(r) \\ &= c_0(f)\varphi_{\omega,0}(r). \end{aligned}$$

Thus $c_0(f) = 0$ for every $f \in (\omega^2 + H)(C_0^\infty(\mathbb{R}_+))$. This yields that $(\omega^2 + H)v = 0$ and hence we see that $v = \tilde{c}\varphi_{\omega,0}$. Therefore

$$(3.3)\qquad c_0(f) = \tilde{c} \int_0^\infty \varphi_{\omega,0}(s) f(s)\, ds \quad \text{for some } \tilde{c} \in \mathbb{C},$$

Consequently, for every $f \in L^2(\mathbb{R}_+)$, $u = (\omega^2 + \tilde{H})^{-1} f$ satisfies

$$(3.4)\qquad \lim_{r \downarrow 0} r^{-\frac{1}{2}} \left| u(r) - \left(\int_0^\infty \varphi_{\omega,0}(s) f(s)\, ds\right)\left(\tilde{c}\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)\right) \right| = 0.$$

Using (2.3) and (2.9) (with the same notation), we obtain "$\subset$" with $(a_1, a_2) \neq (0,0)$ given by (3.2) and $\tilde{c}$ given by (3.3).

Conversely, we prove the inclusion "$\supset$" in (3.1). Let $u \in D(H_{\max})$ satisfy

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C'\left(a_1 r^{i\nu} + a_2 r^{-i\nu}\right) \right| = 0,$$

where the pair (a_1, a_2) is defined in (3.2) and $\tilde{c}$ in (3.3). By (2.3) and (2.9) we have

$$\lim_{r\downarrow 0} r^{-\frac{1}{2}} \left|u(r) - C\left(\tilde{c}\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)\right)\right| = 0.$$

Set $\tilde{u} = (\omega^2+\tilde{H})^{-1}(\omega^2+H_{\max})u$ and $w = u-\tilde{u}$. Then $(\omega^2+H)w = 0$. Since $w \in L^2(\mathbb{R}_+)$, we see that $w = c'\varphi_{\omega,0}$ for some $c' \in \mathbb{C}$. Noting that

$$\lim_{r\downarrow 0} r^{-\frac{1}{2}} \left|\tilde{u}(r) - \tilde{C}\left(\tilde{c}\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)\right)\right| = 0,$$

we obtain

$$\lim_{r\downarrow 0} r^{-\frac{1}{2}} \left|c'\varphi_{\omega,0}(r) - (C-\tilde{C})\left(c\varphi_{\omega,0}(r) + W(\omega)^{-1}\varphi_{\omega,1}(r)\right)\right| = 0,$$

or equivalently,

$$\lim_{r\downarrow 0} r^{-\frac{1}{2}} \left|\left(c' - \tilde{c}(C-\tilde{C})\right)\varphi_{\omega,0}(r) - (C-\tilde{C})W(\omega)^{-1}\varphi_{\omega,1}(r)\right| = 0.$$

By (2.3) and (2.9) again we deduce that $c' = 0$, hence $u = \tilde{u} \in D(\tilde{H})$. □

In view of Lemma 6, we define realizations between $H_{\min}$ and $H_{\max}$ as follows.

Definition 1. Let $A = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Then

$$\begin{cases} D(H_A) := \left\{u \in D(H_{\max})\ ;\ \exists C \in \mathbb{C} \text{ s.t. } \lim_{r\downarrow 0}\left|r^{-\frac{1}{2}}u(r) - C\left(a_1 r^{i\nu} + a_2 r^{-i\nu}\right)\right| = 0\right\}, \\ H_A u = Hu. \end{cases}$$

Remark 3.1. All functions in $D(H_{\max})$ satisfies Dirichlet boundary condition at 0. For fixed A, we consider an additional boundary condition $r^{-\frac{1}{2}}u(r) \approx a_1 r^{i\nu} + a_2 r^{-i\nu}$ near $r \ll 1$. This can be regarded as a boundary condition of oscillating type.

Remark 3.2. If $\tilde{H}$ satisfies $H_{\min} \subset \tilde{H} \subset H_{\max}$ and $\rho(\tilde{H}) \neq \emptyset$, then by Lemma 6 there exists a pair $A = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\tilde{H}$ coincides with H_A. Moreover, if $a_1' = ca_1$ and $a_2' = ca_2$ for some $c \in \mathbb{C}\setminus\{0\}$, then $H_A = H_{A'}$. This implies that the map

$$A \in \mathbb{C}P_1 \mapsto H_A \in \{\tilde{H}\ ;\ H_{\min} \subset \tilde{H} \subset H_{\max}\ \&\ \rho(\tilde{H}) \neq \emptyset\}$$

is well-defined and one to one, where $\mathbb{C}P_1$ denotes the Riemann sphere (or the one-dimensional complex projective space). Note that it is known in a field of mathematical physics that there exists a bijective map

$$\mathbb{R}P_1(\cong S^1) \to \{\tilde{H}\ ;\ H_{\min} \subset \tilde{H} \subset H_{\max}\ \&\ \tilde{H} \text{ is selfadjoint}\}.$$

See Proposition 3.1 for more explanation.

In order to clarify the spectrum of H_A, we need the following preliminary result.

Lemma 7. *Let $\omega = \mu e^{i\xi} \in \mathbb{C}_+$ satisfy $|\xi| < \pi/2$. Then $(\omega^2 + H_A)$ is invertible if and only if $\varphi_{\omega,0} \notin D(H_A)$.*

Proof. Assume that $\varphi_{\omega,0} \notin D(H_A)$ and therefore $\omega^2 + H_A$ is injective. By (2.3) this is equivalent to

$$\begin{vmatrix} \alpha\mu^{i\nu}e^{-\xi\nu} & \overline{\alpha}\mu^{-i\nu}e^{\xi\nu} \\ a_1 & a_2 \end{vmatrix} \neq 0. \tag{3.5}$$

Let $f \in L^2(\mathbb{R}_+)$ and $u = c_0(f)\varphi_{\omega,0} + T_\omega f$, where $c_0(f)$ is defined in (3.3). Then (3.4) holds, and hence $u \in D(H_B)$, where $B = (b_1, b_2)$ and

$$b_1 = (\tilde{c} + W(\omega)^{-1})\alpha\mu^{i\nu}e^{-\xi\nu},$$
$$b_2 = (\tilde{c} - W(\omega)^{-1})\overline{\alpha}\mu^{i\nu}e^{\xi\nu}.$$

The system $b_1 = \kappa a_1, b_2 = \kappa a_2$ has a unique solution $(\tilde{c}, \kappa)$ because of (3.5). With this choice, $u \in D(H_B) = D(H_A)$ and $(\omega^2 + H_A)^{-1}f = c_0(f)\varphi_{\omega,0} + T_\omega f$ is bounded by (3.3) and Lemma 4. □

To formulate the assertion for spectrum of realizations of H, we introduce the set

$$\begin{aligned} S(\kappa) &= \left\{-\rho e^{i\theta} \in \mathbb{C} : \rho^{-i\nu}e^{\theta\nu} = \kappa e^{2i\eta}\right\} \\ &= \left\{-\rho_j e^{i\theta} \in \mathbb{C} : \theta = \frac{\log|\kappa|}{\nu},\ \rho_j = e^{\frac{\eta+2j\pi}{\nu}},\ j \in \mathbb{Z}\right\}, \end{aligned} \tag{3.6}$$

where $\kappa \in \mathbb{C}\setminus\{0\}$ and $\alpha = |\alpha|e^{i\eta}$ is defined in Lemma (1). Note that $S(\kappa)$ consists of double-ended sequence $\{(z_j), j \in \mathbb{Z}\}$ lying on the half line $\{z = -\rho e^{i\theta}\}$, such that $|z_j| \to \infty$ as $j \to +\infty$ and $|z_j| \to 0$ as $j \to -\infty$. The above angle θ is independent of α and the moduli of the points z_j depend only on ν and $\eta = \arg(\alpha)$.

Theorem 3.1. *The following assertions hold:*

(i) Assume $a_1 \neq 0$, $a_2 \neq 0$ and let $\kappa = \frac{a_1}{a_2}$. If

$$|\kappa| \in \left(e^{-\nu\pi}, e^{\nu\pi}\right), \tag{3.7}$$

then

$$\sigma(H_A) = [0,\infty) \cup S(\kappa),$$

where $S(\kappa)$ is given by (3.6). Moreover, $S(\kappa)$ coincides with the set of all eigenvalues of H_A.

(ii) If A does not satisfy condition in (i), then

$$\sigma(H_A) = [0,\infty).$$

Proof. Lemma 5 yields $[0,\infty) \subset \sigma(H_A)$. If $\omega = \mu e^{i\xi} \in \mathbb{C}_+$, $|\xi| < \pi/2$, then Lemma 7 asserts that $\lambda = -\omega^2 \in \sigma(H_A)$ if and only if $\varphi_{\omega,0} \in D(H_A)$. By (3.5) this happens if and only if

$$a_1\overline{\alpha} = a_2\alpha\mu^{2i\nu}e^{-2\xi\nu}$$

or $\lambda \in S(\kappa)$. Since $|2\xi| < \pi$, this relation holds when (3.7) holds. Finally, the assertion for eigenvalues follows from Lemmas 3 and 7 (it suffices to prove that 0 is not an eigenvalue of H_A). This is easy verified since every solution of $Hu = 0$ is given by $u = c_1 r^{\frac{1}{2}+i\nu} + c_2 r^{\frac{1}{2}-i\nu}$ and never belongs to $L^2(1,\infty)$. □

Finally, we characterize the adjoint of H_A.

Proposition 3.1. *Let $A = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Then $(H_A)^* = H_B$ where $B = (b_1, b_2)$ and $b_1 = \overline{a}_2$, $b_2 = \overline{a}_1$. H_A is selfadjoint if and only if $|a_1| = |a_2|$.*

Proof. Theorem 3.1 yields the existence of $\omega > 0$ such that $\omega^2 + H_A$ is invertible. From the proof of Lemma 6 we see that

$$(\omega^2 + H_A)^{-1}f = c\left(\int_0^\infty \varphi_{\omega,0}(s)f(s)\,ds\right)\varphi_{\omega,0} + T_\omega f$$

for a suitable $c \in \mathbb{C}$ and then (3.2) with $\mu = \omega$ and $\xi = 0$ yields

$$a_1 = (c + W(\omega)^{-1})\alpha\omega^{i\nu} \qquad a_2 = (c - W(\omega)^{-1})\overline{\alpha}\omega^{-i\nu}.$$

By Lemma 4, T_ω is selfadjoint. Thus we obtain

$$(\omega^2 + (H_A)^*)^{-1}f = \overline{c}\left(\int_0^\infty \varphi_{\omega,0}(s)f(s)\,ds\right)\varphi_{\omega,0} + T_\omega f$$

and therefore $(H_A)^* = H_B$, where

$$b_1 = (\overline{c} + W(\omega)^{-1})\alpha\omega^{i\nu} = \overline{a}_2 \qquad b_2 = (\overline{c} - W(\omega)^{-1})\overline{\alpha}\omega^{-i\nu} = \overline{a}_1$$

since $W(\omega)$ is purely imaginary. Finally, H_A is selfadjoint if and only if $\overline{a}_2 = ca_1$, $\overline{a}_1 = ca_2$ for a suitable $c \in \mathbb{C} \setminus \{0\}$ and this happens if and only if $|a_1| = |a_2|$. □

Remark 3.3. Four cases appear in the description of $\sigma(H_A)$.

Case I. Assume that H_A is selfadjoint. By Proposition 3.1, we have $|\kappa| = 1$ and $\theta = 0$. It follows from Theorem 3.1 that every selfadjoint extension of $H_{\min}$ has infinitely many eigenvalues and its spectrum is unbounded both from above and below.

Case II. Next we consider the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}}\right].$$

that is, $\theta \in [-\pi/2, \pi/2]$. In this case, $\rho(-H_A)$ does not contain $\overline{\mathbb{C}_+} \setminus \{0\}$. Therefore, $-H_A$ does not generate an analytic semigroup on $L^2(\mathbb{R}_+)$.

Case III. In the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in \left(e^{-\nu\pi}, e^{\nu\pi}\right) \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}}\right],$$

we have $\theta \in (-\pi, \pi)\setminus[-\pi/2, \pi/2]$. Hence one can expect that $-H_A$ generates an analytic semigroup on $L^2(\mathbb{R}_+)$. Indeed, we prove in Proposition 4.1 that $-H_A$ generates a bounded analytic semigroup of angle $\pi/2 - |\theta|$.

Case IV. Finally we consider the case

$$|\kappa| = \frac{|a_2|}{|a_1|} \in [0, \infty] \setminus \left(e^{-\nu\pi}, e^{\nu\pi}\right).$$

Here we use $|\kappa| = \infty$ if $a_1 = 0$ and $|\kappa| = 0$ if $a_2 = 0$. By Theorem 3.1 (ii) we have $\sigma(H_A) = [0, \infty)$, see Figure 4. As in Case III, we prove that $-H_A$ generates a bounded analytic semigroup on $L^2(\mathbb{R}_+)$ of angle $\pi/2$.

4. Generation of analytic semigroups

In this section we characterize the cases when $-H_A$ generates an analytic semigroup.

Theorem 4.1. *Let H_A be defined in Definition 1. Then $-H_A$ generates a bounded analytic semigroup $\{T_A(z)\}$ on $L^2(\mathbb{R}_+)$ if and only if a_1 and a_2 satisfy*

$$|\kappa| = \frac{|a_2|}{|a_1|} \in [0, \infty] \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}}\right]. \tag{4.1}$$

Moreover, if $\theta = \frac{\log|\kappa|}{\nu}$, the maximal angle of analyticity θ_A of $\{T_A(z)\}$ is given by

$$\theta_A := \begin{cases} |\theta| - \dfrac{\pi}{2} & \textit{if } |\kappa| \in \left(e^{-\nu\pi}, e^{\nu\pi}\right) \setminus \left[e^{-\frac{\nu\pi}{2}}, e^{\frac{\nu\pi}{2}}\right], \\ \dfrac{\pi}{2} & \textit{otherwise.} \end{cases}$$

Setting

$$\Sigma(\theta) = \{z \in \mathbb{C} \setminus \{0\} \, ; \, |\mathrm{Arg}\, z| < |\theta|\},$$

from Theorem 3.1, we obtain

Lemma 8. $\Sigma(\pi/2 + \theta_A) \subset \rho(-H_A)$. *In particular, $\overline{\mathbb{C}}_+ \setminus \{0\} \subset \rho(-H_A)$ if and only if a_1 and a_2 satisfy* (4.1).

To prove Theorem (4.1), we use a scaling argument. It worth noticing that if $a_1 \neq 0$ and $a_2 \neq 0$, then $D(H_A)$ is not invariant under scaling $u(r) \mapsto u(sr)$ for some $s > 0$ in spite of the scale invariant property of $D(H_{\min})$ and $D(H_{\max})$. This means that the scale symmetry of H_A (with $s \in (0, \infty)$) is broken. However, there exists a subgroup G of $(0, \infty)$ such that the scale symmetry of H_A with $s \in G$ is still true.

Lemma 9. *For $\nu > 0$, we define*

$$G(\nu) = \left\{ e^{\frac{m\pi}{\nu}} \; ; \; m \in \mathbb{Z} \right\}.$$

Assume that $a_1 \neq 0$ and $a_2 \neq 0$. Then $D(H_A)$ is invariant under the scaling $u(r) \mapsto u(sr)$ if and only if $s \in G(\nu)$. On the other hand, if $a_1 = 0$ or $a_2 = 0$, then $D(H_A)$ is invariant under the scaling $u(r) \mapsto u(sr)$ for every $s \in (0, \infty)$.

Proof. Fix $A = (a_1, a_2)$ with $a_1 \neq 0$ and $a_2 \neq 0$ and let $u \in D(H_A)$ satisfy

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(r) - C \left(a_1 r^{i\nu} + a_2 r^{-i\nu} \right) \right| = 0$$

for some $C \neq 0$. Then $u(sr) \in D(H_A)$ if and only if

$$\lim_{r \downarrow 0} \left| r^{-\frac{1}{2}} u(sr) - C' \left(a_1 r^{i\nu} + a_2 r^{-i\nu} \right) \right| = 0$$

for some C'. This is equivalent to

$$\lim_{r \downarrow 0} \left| C \left(a_1 (sr)^{i\nu} + a_2 (sr)^{-i\nu} \right) - C' \left(a_1 r^{i\nu} + a_2 r^{-i\nu} \right) \right| = 0,$$

or

$$C s^{i\nu} = C' = C s^{-i\nu}.$$

We deduce $\log s \in (\pi/\nu)\mathbb{Z}$, or equivalently, $s \in G(\nu)$. The cases $a_1 = 0$ or $a_2 = 0$ are similar. □

Proof of Theorem 4.1. Assume that (4.1) is satisfied. For $0 < \varepsilon < \theta_A$, let

$$\Sigma_\varepsilon = \left\{ \lambda \in \overline{\Sigma(\pi/2 + \theta_A - \varepsilon)} \; ; \; 1 \leq |\lambda| \leq e^{\frac{2\pi}{\nu}} \right\} \subset \rho(-H_A).$$

Since Σ_ε is compact in $\mathbb{C}$, $\|(\lambda + H_A)^{-1}\|$ is bounded in Σ_ε. Therefore we have

$$\|(\lambda + H_A)^{-1}\| \leq \frac{M_\varepsilon}{|\lambda|}, \qquad \lambda \in \Sigma_\varepsilon.$$

Observe that by Lemma 9 the dilation operator $(I_s u)(x) := s^{\frac{1}{2}} u(sx)$ satisfies $\|I_s u\|_{L^2(\mathbb{R}_+)} = \|u\|_{L^2(\mathbb{R}_+)}$ and

$$H_A I_s = s^2 I_s H_A, \qquad s \in G(\nu). \tag{4.2}$$

Let $\lambda \in \Sigma(\pi/2 + \theta_A - \varepsilon)$. Taking $s_0 \in G(\nu)$ as

$$\log s_0 \in \left[-\frac{\log|\lambda|}{2}, \frac{\pi}{\nu} - \frac{\log|\lambda|}{2} \right) \cap \frac{\pi}{\nu}\mathbb{Z} \neq \emptyset,$$

we see that $s_0^2 \lambda \in \Sigma_\varepsilon$, and hence, we have

$$\|(s_0^2 \lambda + H_A)^{-1}\| \leq \frac{M_\varepsilon}{|s_0^2 \lambda|}. \tag{4.3}$$

Using (4.2) with (4.3), we obtain

$$\begin{aligned}\|(\lambda+H_A)^{-1}\| &= \|(\lambda+s_0^{-2}I_{s_0^{-1}}H_AI_{s_0})^{-1}\| \\ &= s_0^2\|I_{s_0^{-1}}(s_0^2\lambda+H_A)^{-1}I_{s_0}\| \\ &\le \frac{s_0^2M_\varepsilon}{|s_0^2\lambda|} \\ &= \frac{M_\varepsilon}{|\lambda|}.\end{aligned}$$

Therefore $-H_A$ generates a bounded analytic semigroup on $L^2(\mathbb{R}_+)$ of angle θ_A. The optimality of θ_A follows from Theorem 3.1.

On the other hand, if (4.1) is violated, then Lemma 8 implies that $-H_A$ does not generates an analytic semigroup on $L^2(\mathbb{R}_+)$. $\square$

Remark 4.1. In the case $|\kappa| = e^{\frac{\nu\pi}{2}}$ or $|\kappa| = e^{-\frac{\nu\pi}{2}}$, we do not know whether the operator $-H_A$ generates a C_0-semigroup on $L^2(\mathbb{R}_+)$. We point out that if $-H_A$ generates a C_0-semigroup, then it cannot be (quasi) contractive because Hardy's inequality does not hold on $C_0^\infty(\mathbb{R}_+)$, since $a < -\frac{1}{4}$.

5 Remarks on the N-dimensional case

Here we give a result for the N-dimensional Schrödinger operators

$$L = -\Delta + \frac{b}{|x|^2} \quad \text{in } L^2(\mathbb{R}^N),$$

where $N \ge 2$ and $b \in (-\infty, -(\frac{N-2}{2})^2)$. As in one dimension we define

$$\begin{aligned}D(L_{\min}) &= C_0^\infty(\mathbb{R}^N \setminus \{0\}), \\ D(L_{\max}) &= \{u \in L^2(\mathbb{R}^N) \cap H^2_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})\ ;\ Lu \in L^2(\mathbb{R}^N)\}.\end{aligned}$$

As mentioned in Introduction, Hardy's inequality implies the existence of a nonnegative selfadjoint extension of $L_{\min}$, namely the Friedrichs extension, for $b \ge -(\frac{N-2}{2})^2$. Therefore in this section we assume $b < -(\frac{N-2}{2})^2$. Using Proposition 4.1 we can derive the following result.

Proposition 5.1. *Assume $b < -(\frac{N-2}{2})^2$. Then there exist infinitely many intermediate operators between $L_{\min}$ and $L_{\max}$ which are negative generators of analytic semigroups on $L^2(\mathbb{R}^N)$.*

To prove Proposition 5.1 we use the following expansion of $f \in L^2(\mathbb{R}^N)$ by spherical harmonics

$$f = \sum_{j=0}^{\infty} F_j(G_jf).$$

where $F_j : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}^N)$ and $G_j : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}_+)$ are defined by

$$F_j g(x) = |x|^{-\frac{N-1}{2}} g(|x|) Q_j(\omega), \quad g \in L^2(\mathbb{R}_+),$$
$$G_j f(r) = r^{\frac{N-1}{2}} \int_{S^{N-1}} f(r,\omega) Q_j(\omega)\, d\omega, \quad f \in L^2(\mathbb{R}^N).$$

Here $\{Q_j \,;\, j \in \mathbb{N}\}$ is a orthonormal basis of $L^2(S^{N-1})$ consisting of spherical harmonics Q_j of order n_j. Q_j is an eigenfunction of Laplace-Beltrami operator $\Delta_{S^{N-1}}$ with respect to the eigenvalue $-\lambda_j = -n_j(N-2+n_j)$, see e.g., [**20**, Chapter IX] and also [**18**, Chapter 4, Lemma 2.18]. For detail, see [**13**].

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