Categorical entropy and Periodic points

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1. Introduction

Let X be a quasi-projective variety over a filed k and f a finite endomorphism of X. Then, we can define a (algebraic) dynamical system (X, f). By categorification of the dynamics: $(X, f) \rightsquigarrow (\mathrm{Perf}(X), \mathbb{L}f^*)$, where $\mathrm{Perf}(X)$ is the triangulated category of perfect complexes on X and $\mathbb{L}f^*$ is the derived pull-back of f, we estimate the entropy of categorical dynamics motivated by the Gromov-Yomdin's fundamental theorem, and apply the categorical entropy to periodic points of f.

The entropy of categorical dynamics is also related to the "dimension" of triangulated categories and the mass growth of Bridgeland stability conditions.

2. Entropy of categorical dynamics

Let Φ be a Fourier-Mukai type endofunctor of $\mathrm{Perf}(X)$ and G a split-generator of $\mathrm{Perf}(X)$. For a very ample line bundle $\mathcal{L}$ on X, $\oplus_{i=0}^{\dim X}\mathcal{L}^i$ is a fundamental example of split-generators of $\mathrm{Perf}(X)$.

Definition 1. The complexity $\delta(G, M)$ of $M \in \mathrm{Perf}(X)$ with respect to G is defined as follows:

$$\delta(G,M) := \min\left\{ p\in\mathbb{Z}_{>0} \;\middle|\; \begin{tikzcd}[column sep=small] 0 \arrow[rr] & & A_1 \arrow[dl] & \cdots & A_{p-1} \arrow[rr] & & A_p\oplus M \arrow[dl] \\ & G[n_1] \arrow[ul, dashed] & & \cdots & & G[n_p] \arrow[ul, dashed] & \end{tikzcd} \right\} \in \mathbb{Z}_{>0}.$$

Definition 2 (Dimirtrov–Haiden–Katzarkov–Kontsevich [1])**.** The entropy $h(\Phi)$ of categorical dynamics $(\mathrm{Perf}(X), \Phi)$ is defined as follows:

$$h(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \delta(G, \Phi^n G) \in \mathbb{R}_{\geq 0}.$$

The entropy of categorical dynamics is similar to the topological entropy $h_{top}(f)$ in the sense of sharing many standard properties:

Proposition 3. We have the followings.

(i) $h(\Phi^l) = lh(\Phi)$.
(ii) $\Phi_1\Phi_2 = \Phi_2\Phi_1 \Rightarrow h(\Phi_1\Phi_2) \leq h(\Phi_1) + h(\Phi_2)$.
(iii) The entropy $h(F)$ is invariant in conjugacy classes of $\mathrm{Auteq}(\mathrm{Perf}(X))$.

The above definitions and properties hold for more general triangulated categories. Especially, (iii) is generalized as an inequality for semi-conjugate categorical dynamics. In smooth projective cases, we have the following theorem:

Theorem 4 (DHKK [1])**.** When X is smooth projective over k, the following holds:

$$h(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{m\in\mathbb{Z}} \dim_k \mathrm{Hom}_{\mathrm{Perf}(X)}(G, \Phi^n G[m])\Big).$$

This theorem enables us to compute many examples easily.

3. Gromov-Yomdin type equality

In this section, suppose that X is smooth projective varieties over $k = \mathbb{C}$.

Theorem 5 (K-Takahashi [4], Ouchi [7])**.** We have the following:

$$h(\mathbb{L}f^*) = \log \rho(\mathcal{N}(\mathbb{L}f^*)),$$

where $\rho(\mathcal{N}(\mathbb{L}f^*))$ is the spectral radius of $\mathcal{N}(\mathbb{L}f^*)$ on $\mathcal{N}(\mathrm{Perf}(X))$.

Corollary 6. We have the following:

$$h_{top}(f) = h(\mathbb{L}f^*).$$

Motivated by the Gromov-Yomdin theorem: $h_{top}(f) = \log \rho(N(f))$ and Thm5, it is natural to compare the categorical entropy and the spectral radius on the numerical Grothendieck group. Then we show the lower-bound:

Theorem 7 (K [5])**.** We have the following:

$$h(\Phi) \geq \log \rho(\mathcal{N}(\Phi)).$$

Theorem 8. $h(\Phi) = \log \rho(\mathcal{N}(\Phi))$ holds for curves (K [3]), varieties with (anti-) ample K_X (K-Takahashi [4]), abelian surfaces and simple abelian varieties (Yoshioka [8]), where Φ is an autoequivalence.

Remark 9. In general, the upper-bound of Theorem 7 does NOT hold for some autoequivalences of even-dimensional Calabi-Yau hypersurfaces (Y.-W.Fan [2]) and all K3 surfaces (Ouchi [7]). It is an interesting and important problem to characterize such autoequivalences.

4. Local entropy

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and $\phi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ a local homomorphism of finite length. (e.g. finite morphisms are of finite length.)

Definition 10 (Majidi–Zolbanin-Miasnikov-Szpiro [6])**.** The local entropy $h_{loc}(\phi)$ of dynamics $(R, \mathfrak{m}, \phi)$ is defined as follows.

$$h_{loc}(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \ell_R(\phi^n) \in \mathbb{R}_{\geq 0}, \text{ where } \ell_R(\phi^n) := \ell_R(R/\phi^n(\mathfrak{m})R).$$

Theorem 11 (MZMS [6])**.** Suppose char $R = p > 0$ and let F be the Frobenius morphism of R. Then we have the following:

$$h_{loc}(F) = d \log p \quad (\Leftrightarrow p = \exp(h_{loc}(F)/d)).$$

Theorem 12 (MZMS [6]: An analogue of the Kunz's regularity criterion)**.** In the following properties, (i)⇒(ii)⇒(iii) holds. Moreover, if ϕ is contracting, then the converses hold.

(i) R is regular.
(ii) $\phi : R \to R$ is flat.
(iii) $\ell_R(\phi) = p_\phi^d$, where $p_\phi := \exp(h_{loc}(\phi)/d)$.

Proposition 13. Let C be a smooth projective curve over $k = \bar{k}$, f a separable k-endomorphism with a fixed (closed) point $x \in C$. Then $h_{loc}(f_x) = \log e_x$ hold, where e_x is the ramification index of f at x.

5. Applications to periodic points

We compare the categorical entropy and the local entropy of dynamics on a periodic point via semi-conjugate categorical dynamics:

Theorem 14 (K)**.** Suppose f has periodic points. Then we have the following:

$$h(\mathbb{L}f^*) \geq \sup\left\{ h_{loc}((f^l)_x)/l \;\middle|\; l \in \mathbb{Z}_{>0}, x : l\text{-periodic point of } f \right\}.$$

Corollary 15. We have the followings.

(i) If f has a contracting periodic point, then $h(\mathbb{L}f^*) > 0$.
(ii) l-periodic point x of f satisfying $\ell_{\mathcal{O}_{X,x}}((f^l)_x)/l > h(\mathbb{L}f^*)$ is singular.

Corollary 16. Suppose char $k = p > 0$ and let F be the absolute Frobenius morphism of X. Then the inequality $h(\mathbb{L}F^*) \geq d \log p$ holds. Moreover, The equality holds in smooth projective cases.

Remark 17. For local rings on periodic (closed) points, the contractivity is equivalent to the super-attractivity in the theory of complex dynamics.

Example 18. Set $f(z) := z^m - z^n$ $(m > n > 1)$, which is a separable k-endomorphism of $\mathbb{P}^1_k$ over $k = \bar{k}$. Then, $z = 0$ is a super-attracting fixed point of f, and $h(\mathbb{L}f^*) = \log m > \log n = \log e_0 = h_{loc}(f_0)$.

Zariski-density of the set of periodic points of polarized endomorphism due to Fakhruddin is well-known. As an analogy of complex dynamics, it is natural to ask whether the set of contracting (isolated) periodic points is finite.

References

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