

The behavior of solutions with singularities of linear partial differential equations in C^{n+1}

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Let $P(z, \partial)$ be a linear partial differential operator with order $m \geq 1$. Its coefficients are holomorphic in a neighbourhood of the origin $z = 0$ in C^{n+1} . K is a nonsingular complex hypersurface through $z = 0$, which is characteristic for $P(z, \partial)$. We choose the coordinate so that $K = \{z_0 = 0\}$. In the present talk we consider

$$(1) \quad P(z, \partial)u(z) = f(z),$$

where $u(z)$ and $f(z)$ are holomorphic except on K . In order to state the results we give notations and definitions: $z = (z_0, z_1, \dots, z_n) = (z_0, z')$, $\partial_i = \partial/\partial z_i$, $\partial = (\partial_0, \partial_1, \dots, \partial_n) = (\partial_0, \partial')$ and $|z| = \max\{|z_i|; 0 \leq i \leq n\}$. We write $P(z, \partial)$ in the form

$$(2) \quad \begin{cases} P(z, \partial) = \sum_{k=0}^m P_k(z, \partial) \\ P_k(z, \partial) = \sum_{l=s_k}^k A_{k,l}(z, \partial') \partial_0^{k-l} \\ A_{k,l}(z, \partial') = (z_0)^{j(k,l)} a_{k,l}(z, \partial'), \end{cases}$$

where $P_k(z, \partial)$ is the homogeneous part of order k . If $P_k(z, \partial) \not\equiv 0$, $A_{k,s_k}(z, \partial') \not\equiv 0$, and if $A_{k,l}(z, \partial') \not\equiv 0$, $a_{k,l}(0, z', \partial') \not\equiv 0$. If $P_k(z, \partial) \equiv 0$, put $s_k = +\infty$. Let us define the irregularities of K , which are closely related to the characteristic indices introduced in [1] and others. Put $d_{k,l} = l + j(k, l)$, $d_k = \min\{d_{k,l}; s_k \leq l \leq k\}$ and $e_k = d_k - k$. Put $\Sigma^* =$ the convex hull of $\bigcup_{k=0}^m \Pi(k, e_k)$, where $\Pi(a, b) = \{(x, y) \in \mathbb{R}^2; x \leq a, y \geq b\}$. The boundary of Σ^* consists of a vertical half line Σ_0^* , a horizontal half line Σ_p^* and segments Σ_i^* ($1 \leq i \leq p-1$). The set of vertices of Σ^* consists of p points (k_i, e_{k_i}) , $0 \leq k_{p-1} < k_{p-2} < \dots < k_1 < k_0 = m$ (see Figure 1). Let γ_i be the slope of Σ_i^* , $0 = \gamma_p < \gamma_{p-1} < \dots < \gamma_1 < \gamma_0 = +\infty$.

Definition 1. We call γ_i , ($0 \leq i \leq p$) the irregularities of K for $P(z, \partial)$. In particular γ_{p-1} is called the minimal irregularity and denote by $\gamma_{\min, P}$.

Let us define some functions spaces. Let $\Omega = \Omega^0 \times \Omega'$ be a polydisk: $\Omega^0 = \{z_0 \in \mathbb{C}^1; |z_0| \leq R\}$, $\Omega' = \{z' \in \mathbb{C}^n; |z'| \leq R\}$. Put $\Omega_\theta^0 = \{z_0 \in \Omega^0 - \{0\}; |\arg z_0| < \theta\}$ and $\Omega_\theta = \Omega_\theta^0 \times \Omega'$. $\mathcal{O}(\Omega)$ ($\mathcal{O}(\Omega')$) is the set of all holomorphic functions on Ω (resp. Ω'). $\mathcal{O}(\Omega_\theta)$ is the set of all holomorphic functions on Ω_θ , which contains multi-valued functions on $\Omega - K$, if $\theta > \pi$. Now we introduce

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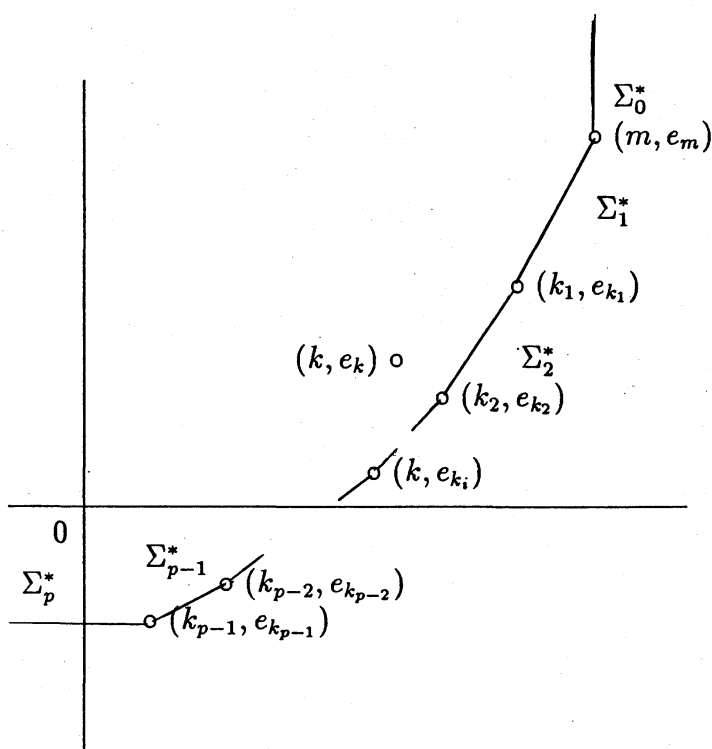


FIGURE 1. Characteristic polygon

Definition 2. $Asy_{\{\kappa\}}(\Omega_\theta)$ ($0 < \kappa \leq +\infty$) is the set of all $f(z) \in \mathcal{O}(\Omega_\theta)$ with the following asymptotic expansion: for any θ' with $0 < \theta' < \theta$ and any N

$$(3) \quad |f(z) - (\sum_{k=0}^{N-1} a_k(z') z_0^k)| \leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1) |z_0|^N \quad \text{in } \Omega_{\theta'},$$

where $a_k(z') \in \mathcal{O}(\Omega')$.

Definition 3. $\tilde{\mathcal{M}} - Asy_{\{\kappa\}}(\Omega_\theta)$ ($0 < \kappa \leq +\infty$) is the set of all $f(z) \in \mathcal{O}(\Omega_\theta)$ with the following asymptotic expansion: for any θ' with $0 < \theta' < \theta$ and any N

$$(4) \quad |f(z) - (\sum_{k=0}^{N-1} a_k(z') z_0^k \log z_0 - (\sum_{k=-H}^{N-1} b_k(z') z_0^k))| \leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1) |z_0|^N |\log z_0| \quad \text{in } \Omega_{\theta'},$$

and

$$(5) \quad |f(z) - (\sum_{k=0}^N a_k(z') z_0^k \log z_0 - (\sum_{k=-H}^{N-1} b_k(z') z_0^k))| \leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1) |z_0|^N \quad \text{in } \Omega_{\theta'},$$

where $H \in \mathbb{N}$ and $a_k(z'), b_k(z') \in \mathcal{O}(\Omega')$.

Definition 4. $\mathcal{M}(\Omega)$ is the set of all $f(z) \in \mathcal{O}(\Omega_{+\infty})$ with the form $f(z) = a(z) \log z_0 + b(z) z_0^{-H}$, where $H \in \mathbb{N}$ and $a(z), b(z) \in \mathcal{O}(\Omega)$.

Definition 5. $\mathcal{O}_{(\kappa)}(\Omega_\theta)$ ($\kappa > 0$) is the set of all $f(z) \in \mathcal{O}(\Omega_\theta)$ with the following bound: for any θ' with $0 < \theta' < \theta$ and any $\epsilon > 0$

$$(6) \quad |f(z)| \leq C_\epsilon \exp(\epsilon |z_0|^{-\kappa}) \quad \text{in } \Omega_{\theta'}.$$

Now we suppose that $P(z, \partial)$ satisfies the following condition:

Condition

$$(7) \quad \begin{cases} (a) \gamma_{\min, P} \neq +\infty & (b) d_{k_{p-1}} = 0 \\ (c) d_{k_i} = s_{k_i} & \text{for } 0 \leq i \leq p-1. \end{cases}$$

Put $\gamma = \gamma_{\min, P}$. Then the main results are the following.

Theorem 6. If $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta)$ is a solution of

$$(8) \quad P(z, \partial)u(z) = f(z) \in \text{Asy}_{\{\kappa\}}(\Omega_\theta) \quad (0 < \kappa \leq \gamma),$$

then $u(z) \in \text{Asy}_{\{\kappa\}}(\Omega_\theta)$.

Corollary 7. Suppose that $f(z) \in \mathcal{O}(\Omega)$ and $\theta > (\pi/2\gamma) + \pi$ in Theorem 6. Then $u(z) \in \mathcal{O}(\Omega)$.

Theorem 8. If $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta)$ is a solution of

$$(9) \quad P(z, \partial)u(z) = f(z) \in \tilde{\mathcal{M}} - \text{Asy}_{\{\kappa\}}(\Omega_\theta) \quad (0 < \kappa \leq \gamma),$$

then $u(z) \in \tilde{\mathcal{M}} - \text{Asy}_{\{\kappa\}}(\Omega_\theta)$.

Corollary 9. Suppose that $f(z) \in \tilde{\mathcal{M}}(\Omega)$ and $\theta > (\pi/2\gamma) + 2\pi$ in Theorem 8. Then $u(z) \in \tilde{\mathcal{M}}(\Omega)$.

Condition (a) means that K is an irregular characteristic surface in the sense in [1] and it is equivalent to $p > 1$. Condition (b) means that the $(p-1)$ -th localization on K of $P(z, \partial)$, which is defined in [1], is a function. Condition (c) is an assumption imposed on the vertices of the characteristic polygon Σ^* . Theorems 6 and 8 are shown by the detailed analysis of the integral representation of solutions singular on K ([2, 3]) and for this purpose we assume (c).

A simple example satisfying the conditions in Theorems is

$$(10) \quad P(z, \partial) = a(z) \partial_1^m + b(z) \partial_1' \partial_0^{k'-l'} + \partial_0^k, \quad z = (z_0, z_1) \quad m > k' > k,$$

where we assume $a(0)b(0) \neq 0$ and $l' > k' - k$. For this $P(z, \partial)$, we have $\gamma = \min\{k/(m-k), (l' - k' + k)/(k' - k)\}$.

We can also obtain results similar to Theorems 6 and 8 of the following type for other $\mathcal{F}(\Omega_\theta) \subset \mathcal{O}(\Omega_\theta)$:

$$\begin{cases} u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta) \\ P(z, \partial)u(z) = f(z) \in \mathcal{F}(\Omega_\theta) \end{cases} \implies u(z) \in \mathcal{F}(\Omega_\theta).$$

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