

An inequality for finite permutation groups

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0. Introduction

Let (G, Ω) be a permutation group of degree n . For any subset X of G , we put

$$F(X) := \{ \alpha \in \Omega \mid \forall x \in X \ \alpha^x = \alpha \}$$

$$f(X) := |F(X)|.$$

For $x \in G$, we use $f(x)$ instead of $f(\{x\})$.

(Definition 1) Let ℓ_i ($i=1, \dots, r$) be integers such that $0 \leq \ell_1 < \dots < \ell_r < n$. We say that (G, Ω) is an $\{\ell_1, \dots, \ell_r\}$ -group, if $\{f(x) \mid x \in G, x \neq 1\} \subseteq \{\ell_1, \dots, \ell_r\}$.

E. Bannai and M. Deza posed us the following conjecture ; if (G, Ω) is an $\{\ell_1, \dots, \ell_r\}$ -group of degree n , then $|G| \leq \prod_{i=1}^r (n - \ell_i)$. In §1 this conjecture is proved. In §§ 2, 3 we consider the case $|G| = \prod_{i=1}^r (n - \ell_i)$. Finally in §4, using the same method as in Theorem 1, we give a proof of the Burnside-Brauer Theorem.

1. Proof of the conjecture

Here we prove the conjecture mentioned above.

Theorem 1. [5] Let (G, Ω) be an $\{\ell_1, \dots, \ell_r\}$ -group of degree n . Then $|G|$ divides $\prod_{i=1}^r (n - \ell_i)$.

Proof. Let θ be the permutation character of G , and let 1_G be the principal character of G . Then it is well known that

$$\hat{\theta} := \prod_{i=1}^r (\theta - \ell_i 1_G)$$

is a generalized character of G . By the definition of $\hat{\theta}$, we have $\hat{\theta}(g) = 0$ for all $g \in G$, $g \neq 1$. Hence, the multiplicity of 1_G in $\hat{\theta}$ is given by

$$(\hat{\theta}, 1_G) = \frac{1}{|G|} \sum_{g \in G} \hat{\theta}(g) = \frac{1}{|G|} \hat{\theta}(1) = \frac{1}{|G|} \prod_{i=1}^r (n - \ell_i).$$

Thus, we get the desired result.

Corollary 2. Assume the hypothesis of Theorem 1. Then we have that $|G| = \prod_{i=1}^r (n - \ell_i)$ if and only if $\hat{\theta}$ is the regular character of G , where $\hat{\theta}$ is defined in the proof of Theorem 1.

2. $\{\ell_1, \dots, \ell_r\}$ -sharp groups

(Definition 2) Assume the hypothesis of Theorem 1. We say that (G, Ω) is an $\{\ell_1, \dots, \ell_r\}$ -sharp group, if $|G| = \prod_{i=1}^r (n - \ell_i)$.

We remark that $\{0, 1, \dots, r-1\}$ -sharp group is sharply r -transitive (see Corollary 4). Hence our concept is a generalization of sharply transitivity. It is natural that one hopes to classify all $\{\ell_1, \dots, \ell_r\}$ -sharp groups. But in general it seems to be difficult. So we must study special cases at first.

Now we state some examples and known results.

Example 1. $Z_\ell \wr Z_2$ is a $\{0, \ell\}$ -sharp group of degree 2ℓ .

Example 2. $\{1, 3\}$ -sharp groups

- (1) $G=S_4$; $\Omega=\Delta \cup \Gamma$, $G^\Delta=S_3$, $G^\Gamma=S_4$.
 (2) $G=PSL(2,7)$; $\Omega=\Delta \cup \Gamma$, G^Δ is 2-transitive of degree 7,
 G^Γ is 2-transitive of degree 8.

Known results. For the following $L=\{\ell_1, \dots, \ell_r\}$, L -sharp groups have been classified.

$L=\{2\}$	Iwahori [3]
$L=\{3\}$	Iwahori and Kondo [4]
$L=\{0,2\}$	Tsuzuku [6]

The following lemma is due to E. Bannai.

Lemma 3. Let G be a $\{0, \ell_2, \dots, \ell_r\}$ -sharp group on Ω . Then G is transitive on Ω , and G_α is an $\{\ell_2-1, \dots, \ell_r-1\}$ -sharp group on $\Omega-\{\alpha\}$, where α is any element of Ω .

Applying Theorem 1 to G_α , we can easily get the proof of Lemma 3.

Corollary 4. Let G be a $\{0, 1, \dots, r-1\}$ -sharp group. Then G is sharply r -transitive.

The following Theorem, due to T. Ito, is an extension of Corollary 4.

Theorem 5. [2] Let G be an $\{\ell, \ell+1, \dots, \ell+r-1\}$ -sharp group on Ω ($r \geq 2$). Then $f(G)=\ell$ and G is sharply r -transitive on $\Omega-F(G)$.

Remark. It looks very likely that every $\{\ell_1, \dots, \ell_r\}$ -sharp group has ℓ_1+1 orbits. Note that Lemma 3 is a special case where $\ell_1=0$.

3. The case $r=2$

Now we consider the case $r=2$ i.e. $\{\ell, \ell+s\}$ -sharp groups.

In this case we can show that $f(G)$ is considerably large and that $\ell-f(G)$ is bounded by a function of s . Hence the essential parameter is s alone. More precisely we have

Theorem 6. [2] Let G be an $\{\ell, \ell+s\}$ -sharp group.

Put $s' := \max \left\{ 1, \left\lfloor \frac{s-1}{2} \right\rfloor \right\}$, $m := \ell + (1-s)s' + s'^2 - 1$.

Then we have $f(G) \geq m$.

For $s=1,2,3,4$ this inequality is best possible. For $s \geq 5$ we guess that $f(G)=m$ does not occur. But I can not prove it yet.

Using Theorem 6, we can classify all $\{\ell, \ell+s\}$ -sharp groups for $s=1,2,3,4$. For example, the $\{\ell, \ell+2\}$ -sharp groups are the following groups ; $G=D_8, S_4, GL(2,3), PSL(2,7)$. These groups are determined up to permutation isomorphism. For more details see [2]. The case $s \geq 5$ is very difficult.

4. Final remark

We give another example which can be proved by the same method as in the proof of Theorem 1. Let G be a finite group, and let θ be a faithful character of G . Let $\theta(1) = \alpha_1, \alpha_2, \dots, \alpha_m$

be the distinct values taken by θ . We put $\hat{\theta} := \prod_{i=2}^m (\theta - \alpha_i)$. Since θ is faithful, we have

$$\hat{\theta} = \alpha \cdot \rho_G = \sum_{\chi \in \text{Irr}(G)} \alpha \chi(1) \chi, \text{ where } \alpha = \frac{1}{|G|} \hat{\theta}(1) \in \mathbb{C}.$$

Since $\hat{\theta}(1) \neq 0$, we have $\alpha \neq 0$. On the other hand $\hat{\theta}$ is a $\mathbb{C}$ -linear combination of θ^j for $0 \leq j < m$, as it can be seen from the definition of $\hat{\theta}$. Then every $\chi \in \text{Irr}(G)$ must be a constituent of some θ^j . Thus we obtain

Theorem. (Burnside-Brauer cf. [1] p49) Let θ be a faithful character of G and suppose $\theta(g)$ takes exactly m different values for $g \in G$. Then every $\chi \in \text{Irr}(G)$ is a constituent of one of the characters θ^j for $0 \leq j < m$.

If some $\alpha_i = 0$, then $\hat{\theta}$ is a $\mathbb{C}$ -linear combination of θ^j for $0 < j < m$. Thus we obtain

Corollary. Assume the hypothesis of the Theorem. Suppose that $\theta(g) = 0$ for some $g \in G$. Then every $\chi \in \text{Irr}(G)$ is a constituent of one of the characters θ^j for $0 < j < m$.

We remark that every non-linear faithful irreducible character of G satisfies the hypothesis of the Corollary.

References

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