

Push-forward formula for flag bundles from cohomological wall-crossing

By

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Abstract

We consider K -theoretic equivariant integrals on flag manifolds, especially Grassmannian manifolds. Using a method inspired by the theory of wall-crossing formulas by Takuro Mochizuki, we derive a push-forward formula for K -theory classes via the equivariant Riemann-Roch formula. Additionally, we compute the push-forward of cohomological Schur classes.

§ 1. Introduction

Let B be a smooth algebraic variety over the complex number field $\mathbb{C}$, and $\mathbb{T} = (\mathbb{C}^*)^s$ an algebraic torus acting on B . We take a $\mathbb{T}$ -equivariant vector bundle $\mathcal{E}$ over B . We write by $\Pi: Fl(\mathcal{E}, d) \rightarrow B$ the flag bundle of $\mathcal{E}$ consisting of d -dimensional flags, and by $\mathcal{V}$ the universal sub-bundle of $\Pi^*\mathcal{E}$.

In [9], we provided an alternative proof of the residue formula [10, (3)] motivated by wall-crossing method by Mochizuki [5]. The purpose of this note is to present some application of this formula, and prove several known results, and suggest another potential application studied in future.

§ 2. Residue formula

§ 2.1. Equivariant K -group

We consider an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^s$. For a character $\chi: \mathbb{T} \rightarrow \mathbb{C}^*$, the weight space $\mathbb{C}_\chi$ denotes the one-dimensional $\mathbb{T}$ -representation $\mathbb{C}$ with the $\mathbb{T}$ -action defined by

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$t \cdot 1 = \chi(t)$ for $t \in \mathbb{T}$. For a coordinate $(x_1, \dots, x_s)$ of $\mathbb{T}$, we write by $x_1^{m_1} \cdots x_s^{m_s}$ the character sending $t \in \mathbb{T}$ to $x_1(t)^{m_1} \cdots x_s(t)^{m_s} \in \mathbb{C}^*$. The symbol $x_1^{m_1} \cdots x_s^{m_s}$ also denotes the weight space $\mathbb{C}_{x_1^{m_1} \cdots x_s^{m_s}}$.

For a smooth algebraic variety M with a $\mathbb{T}$ -action, we consider the Grothendieck ring of the category of $\mathbb{T}$ -equivariant *locally free sheaves* on M . The multiplication is induced by the tensor product. For a morphism $M' \rightarrow M$ of algebraic varieties and a vector bundle $\mathcal{E}$ on M , we often write the pull-back $\mathcal{E}|_{M'}$ by the same letter $\mathcal{E}$. Since we always assume that M is smooth in this paper, we can identify $K_{\mathbb{T}}(M)$ with the Grothendieck group of the category of $\mathbb{T}$ -equivariant *coherent sheaves* on M by [6, Proposition 2.1]. In particular, when M is equal to the one point pt , we use the identification of $K_{\mathbb{T}}(\text{pt})$ with the $\mathbb{T}$ -representation ring $R_{\mathbb{T}} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$ by the weight space decomposition.

For a $\mathbb{T}$ -equivariant vector bundle $\mathcal{F}$ over M , we write by $\mathcal{F}^\vee$ and $\wedge^k \mathcal{F}$ the dual of $\mathcal{F}$ and the k -th exterior power of $\mathcal{F}$ respectively. For two $\mathbb{T}$ -equivariant vector bundles $\mathcal{F}$ and $\mathcal{F}'$ over M , we set $\text{Hom}(\mathcal{F}, \mathcal{F}') = \mathcal{F}^\vee \otimes \mathcal{F}'$. We set

$$\wedge_{-1} \mathcal{F} = \sum_{i=1}^{\text{rk } \mathcal{F}} (-1)^i \wedge^i \mathcal{F}$$

where $\text{rk } \mathcal{F}$ is the rank of $\mathcal{F}$. For a character $\chi: \mathbb{T} \rightarrow \mathbb{C}^*$, we set $\wedge_{-\chi} \mathcal{F} = \wedge_{-1} (\mathcal{F} \otimes \mathbb{C}_\chi)$.

We consider the fixed point set $M^\mathbb{T}$ of M , and recall the localization theorem [8, Theorem 2.1] for $K_{\mathbb{T}}(M)$ and $K_{\mathbb{T}}(M^\mathbb{T})$. We take the embedding $\iota: M^\mathbb{T} \rightarrow M$ of the fixed point set $M^\mathbb{T}$, and the decomposition

$$M^\mathbb{T} = \bigsqcup_{\mathfrak{J}} M_{\mathfrak{J}}$$

into connected components $M_{\mathfrak{J}}$. By [3, Proposition 1.3], we have the following lemma.

Lemma 2.1. *Each connected component $M_{\mathfrak{J}}$ of $M^\mathbb{T}$ is a sub-manifold of M .*

We denote by $N_{\mathfrak{J}}$ the normal bundle of $M_{\mathfrak{J}}$ in M . By the similar argument in the proof of [6, Theorem 3.2], we have the following theorem.

Theorem 2.2. *For an algebra S over $R_{\mathbb{T}}$, we assume that $\wedge_{-1}(N_{\mathfrak{J}})^\vee$ is invertible in $K_{\mathbb{T}}(M^\mathbb{T}) \otimes_{R_{\mathbb{T}}} S$ for each $\mathfrak{J}$. Then the push-forward ι_* gives an isomorphism*

$$\iota_*: K_{\mathbb{T}}(M^\mathbb{T}) \otimes_{R_{\mathbb{T}}} S \cong K_{\mathbb{T}}(M) \otimes_{R_{\mathbb{T}}} S.$$

The inverse image is given by

$$(2.1) \quad (\iota_*)^{-1} \mathcal{F} = \sum_{\mathfrak{J}} \frac{\mathcal{F}|_{M_{\mathfrak{J}}}}{\wedge_{-1}(N_{\mathfrak{J}})^\vee}$$

for a $\mathbb{T}$ -equivariant vector bundle $\mathcal{F}$ over M .

For the fractional algebra $S = \mathbb{Q}(x_1, \dots, x_s)$ of $R_{\mathbb{T}} = \mathbb{Z}[x_1, \dots, x_s]$, we set $K_{\mathbb{T}}^{\text{loc}}(M) = K_{\mathbb{T}}(M) \otimes_{R_{\mathbb{T}}} \mathbb{Q}(x_1, \dots, x_s)$. For a polynomial $g = g(Y_1, \dots, Y_s) \in K_{\mathbb{T}}^{\text{loc}}(M)[Y_1, \dots, Y_s]$ and a $\mathbb{T}$ -equivariant vector bundle $\mathcal{F}$ on M , we set

$$g(\wedge^{\bullet} \mathcal{F}) = g(\wedge^1 \mathcal{F}, \dots, \wedge^s \mathcal{F}) \in K_{\mathbb{T}}^{\text{loc}}(M).$$

Here for $i > \text{rk } \mathcal{F}$, we set $\wedge^i \mathcal{F} = 0$. For a symmetric polynomial $f(x_1, \dots, x_s)$, we can write

$$f(x_1, \dots, x_s) = g(e_1(x_1, \dots, x_s), \dots, e_s(x_1, \dots, x_s))$$

using the *elementary symmetric polynomials* $e_i(x_1, \dots, x_s)$ for $i = 1, \dots, s$ and a polynomial $g(Y_1, \dots, Y_s)$. Then we set $f(\mathcal{F}) = g(\wedge^{\bullet} \mathcal{F})$.

§ 2.2. Equivariant Chow ring

We recall the Riemann-Roch formula in the equivariant setting from [1]. Let $CH_{\mathbb{T}}^{\bullet}(M)$ denote the $\mathbb{T}$ -equivariant Chow group of M , $A_{\mathbb{T}}^{\bullet}(M)$ the $\mathbb{T}$ -equivariant Chow ring of M , and $K^{\mathbb{T}}(M)$ the Grothendieck group of $\mathbb{T}$ -equivariant coherent sheaves on M . Since we assume that M is smooth, we can identify $CH_{\mathbb{T}}^{\bullet}(M)$ with $A_{\mathbb{T}}^{\bullet}(M)$, and $K^{\mathbb{T}}(M)$ with the Grothendieck ring $K_{\mathbb{T}}(M)$ of $\mathbb{T}$ -equivariant vector bundles on M as stated previously.

When M is a point pt , we see that $CH_{\mathbb{T}}^{\bullet}(\text{pt}) = A_{\mathbb{T}}^{\bullet}(\text{pt})$ is equal to the polynomial ring $\mathbb{Z}[\xi_1, \dots, \xi_s]$ where $\xi_1 = c_1(x_1), \dots, \xi_s = c_1(x_s)$. We consider the ideal

$$I_{\mathbb{T}} = \langle 1 - x_1^{l_1} \cdots x_s^{l_s} \mid (l_1, \dots, l_s) \in \mathbb{Z}^s \setminus \{0\} \rangle$$

of the representation ring $R_{\mathbb{T}} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$. We see that the ideal $I_{\mathbb{T}}$ is generated by the multiplicative subset $S_{\mathbb{T}}$ of $K_{\mathbb{T}}(\text{pt})$. We write by $\widehat{R}_{\mathbb{T}}$ the completion of $R_{\mathbb{T}}$ along $I_{\mathbb{T}}$. We set $\widehat{K}_{\mathbb{T}}(M) = K_{\mathbb{T}}(M) \otimes_{R_{\mathbb{T}}} \widehat{R}_{\mathbb{T}}$, and $\widehat{CH}_{\mathbb{T}}(M) = \prod_{i \geq 0} CH_{\mathbb{T}}^i(M)$. We also set

$$A_{\mathbb{T}} = \mathbb{Z}[e^{\pm \xi_1}, \dots, e^{\pm \xi_s}] \subset \widehat{CH}_{\mathbb{T}}(\text{pt}) = \mathbb{Z}[[\xi_1, \dots, \xi_s]].$$

For a $\mathbb{T}$ -equivariant vector bundle $\mathcal{F}$ on M , we set

$$\text{Td}(\mathcal{F}) = \frac{\text{Eu}(\mathcal{F})}{\text{ch} \wedge_{-1}(\mathcal{F}^{\vee})} = \prod_{\alpha=1}^{\text{rk } \mathcal{F}} \frac{\varepsilon_{\alpha}}{1 - e^{-\varepsilon_{\alpha}}} \in \widehat{CH}_{\mathbb{T}}(M) = \prod_{i \geq 0} CH_{\mathbb{T}}^i(M)$$

and call it the *Todd class* of $\mathcal{F}$. Here ch is the Chern character, $\varepsilon_1, \dots, \varepsilon_{\text{rk } \mathcal{F}}$ are Chern roots of $\mathcal{F}$, and $\prod_{\alpha=1}^{\text{rk } \mathcal{F}} \frac{\varepsilon_{\alpha}}{1 - e^{-\varepsilon_{\alpha}}}$ is regarded as the inverse element of $\prod_{\alpha=1}^{\text{rk } \mathcal{F}} \frac{1 - e^{-\varepsilon_{\alpha}}}{\varepsilon_{\alpha}}$ in $\widehat{CH}_{\mathbb{T}}(M)$. When $\mathcal{F} = TM$, we set $\text{Td}(M) = \text{Td}(TM)$.

We define a map $\tau = \tau_M: K_{\mathbb{T}}(M) \rightarrow \widehat{CH}_{\mathbb{T}}(M)$ by

$$(2.2) \quad \tau(\mathcal{E}) = (\text{ch}(\mathcal{E}) \cdot \text{Td}(M)) \cap [M]$$

for $\mathcal{E} \in K_{\mathbb{T}}(M)$. The map τ is compatible with push-forwards, and factors through the induced isomorphism

$$(2.3) \quad \hat{\tau} = \hat{\tau}_M: \widehat{K}_{\mathbb{T}}(M) \cong \widehat{CH}_{\mathbb{T}}(M)$$

called the $\mathbb{T}$ -equivariant Riemann-Roch formula. We often use the identification of the representation ring $R_{\mathbb{T}}$ with $A_{\mathbb{T}} = \mathbb{Z}[e^{\pm \xi_1}, \dots, e^{\pm \xi_n}]$ via the exponential map τ_{pt} . In particular, the fractional algebra $S = \mathbb{Q}(x_1, \dots, x_s)$ of $R_{\mathbb{T}}$ is regarded as an $A_{\mathbb{T}}$ -algebra. Under the assumption in Theorem 2.2 we have an isomorphism

$$\widehat{CH}_{\mathbb{T}}(M) \otimes_{A_{\mathbb{T}}} S \cong \widehat{CH}_{\mathbb{T}}(M^{\mathbb{T}}) \otimes_{A_{\mathbb{T}}} S$$

via (2.3). We set $\widehat{CH}_{\mathbb{T}}^{\text{loc}}(M) = \widehat{CH}_{\mathbb{T}}(M) \otimes_{A_{\mathbb{T}}} S$

When M is projective, the $\mathbb{T}$ -equivariant Riemann-Roch formula (2.3) implies

$$\mathbb{R}\Gamma(M, \mathcal{F}) = \int_M \text{ch}(\mathcal{F}) \text{Td}(M),$$

where $\int_M$ denotes the push-forward under the projection $M \rightarrow \text{pt}$. This means the commutativity of the middle square in

$$\begin{array}{ccccccc} K_{\mathbb{T}}^{\text{loc}}(M^{\mathbb{T}}) & \xleftarrow{\cong} & K_{\mathbb{T}}^{\text{loc}}(M) & \xrightarrow{\text{ch}(\bullet) \text{Td}(M)} & \widehat{CH}_{\mathbb{T}}^{\text{loc}}(M) & \xrightarrow{\cong} & \widehat{CH}_{\mathbb{T}}^{\text{loc}}(M^{\mathbb{T}}) \\ \downarrow & & \downarrow \mathbb{R}\Gamma & & \downarrow f_M & & \downarrow \\ K_{\mathbb{T}}^{\text{loc}}(\text{pt}) & \xlongequal{\quad} & K_{\mathbb{T}}^{\text{loc}}(\text{pt}) & \xrightarrow{\text{ch}(\bullet)} & \widehat{CH}_{\mathbb{T}}^{\text{loc}}(\text{pt}) & \xlongequal{\quad} & \widehat{CH}_{\mathbb{T}}^{\text{loc}}(\text{pt}) \end{array}$$

which follows from the commutativity of the outmost square. When $M^{\mathbb{T}}$ consists of isolated points, this is checked directly as

$$\begin{aligned} \mathbb{R}\Gamma(M, \mathcal{F}) &= \text{ch} \left(\sum_{p \in M^{\mathbb{T}}} \frac{\mathcal{F}|_p}{\wedge_{-1} T_p^* M} \right) = \sum_{p \in M^{\mathbb{T}}} \frac{\text{ch}(\mathcal{F}|_p)}{\prod_{i=1}^d a_i(p)} \frac{\prod_{i=1}^d a_i(p)}{\prod_{i=1}^d (1 - e^{-a_i(p)})} \\ &= \int_M \text{ch}(\mathcal{F}) \text{Td}(M) \end{aligned}$$

where $a_1(x), \dots, a_d(x)$ are Chern roots of $T_p M$.

§ 2.3. Master space

Let $\mathcal{E}$ be a $\mathbb{T}$ -equivariant vector bundle of rank r over a smooth $\mathbb{T}$ -variety B , and consider a sequence of integers $r \geq d_1 \geq d_2 \geq \dots \geq d_m \geq 0$. We consider the flag bundle of type A_m

$$Fl(\mathcal{E}; d_1, d_2, \dots, d_m) = \{ (\mathcal{E}|_b \supset F_1 \supset \dots \supset F_m) \mid b \in B, F_k \cong \mathbb{C}^{d_k} \ (k = 1, \dots, m) \}.$$

We fix a non-negative integer d , and set $F = Fl(\mathcal{E}; d)$. For the projection

$$\Pi: F = Fl(\mathcal{E}; d) \rightarrow B,$$

we write $\Pi^*\mathcal{E}$ by the same letter $\mathcal{E}$, and by $\mathcal{V}$ the universal sub-bundle on F , that is, the sub-bundle of $\mathcal{E}$ whose fibre $\mathcal{V}_{\tilde{b}}$ over the point in $\tilde{b} \in F$ is equal to the subspace corresponding to $\tilde{b}$. For a polynomial $g(Y_1, \dots, Y_d) \in K_{\mathbb{T}}(\mathcal{M})[Y_1, \dots, Y_d]$, we consider the K -theory class $\psi = g(\wedge^\bullet \mathcal{V})$ in $K_{\mathbb{T}}(\mathcal{M})$. Using a computational method inspired by [5] and [7], we compute the push-forward of $g(\wedge^\bullet \mathcal{V})$ under the projection $\Pi: F = Fl(\mathcal{E}; d) \rightarrow B$.

Let us introduce the flag bundle $\Pi': F' = Fl(\mathcal{E}; d-1) \rightarrow B$ of $(d-1)$ -dimensional sub-spaces, and write by $\mathcal{V}'$ the universal sub-bundle on F' tentatively. We take another torus $\mathbb{C}_u^* = \mathbb{C}^*$ with the coordinate $u \in \mathbb{C}_u^*$. We consider the $\mathbb{T} \times \mathbb{C}_u^*$ -equivariant line bundle $L = \mathcal{O}_{F'} \otimes \mathbb{C}_u^*$ over F' , and the projective bundle

$$\mathcal{M} = \mathbb{P}_{F'} \left((\mathcal{E}/\mathcal{V}') \oplus L \right)$$

over F' . We call $\mathcal{M}$ the *master space*. We consider the fibre-wise $\mathbb{C}_u^*$ -action on $\mathcal{M}$ induced by $\mathbb{C}_u^*$ -action on L and the trivial action on $\mathcal{E}/\mathcal{V}'$. We set $\tilde{F} = Fl(\mathcal{E}; d, d-1)$.

For the fixed point set $\mathcal{M}^{\mathbb{C}_u^*}$, we have an obvious decomposition $\mathcal{M}^{\mathbb{C}_u^*} = \mathbb{P}_{F'} \left(\mathcal{E}/\mathcal{V}' \right) \sqcup \mathbb{P}_{F'} (L)$ of connected components, and hence an isomorphism

$$(2.4) \quad \mathcal{M}^{\mathbb{C}_u^*} \cong \tilde{F} \sqcup F'$$

by identifications $\tilde{F} = \mathbb{P}_{F'} \left(\mathcal{E}/\mathcal{V}' \right)$ and $F' = \mathbb{P}_{F'} (L)$. The first component $\tilde{F}$ recovers F via the projection $\tilde{F} = Fl(\mathcal{E}; d, d-1) \rightarrow F = Fl(\mathcal{E}; d)$ forgetting the $d-1$ -dimensional flags. We denote the pull-back of the bundle $\mathcal{V}'$ on F' under the projection $\mathcal{M} \rightarrow F'$ and the one under $\tilde{F} \rightarrow F'$ by the same letter $\mathcal{V}'$. We also denote the pull-back of $\mathcal{V}$ on F under $\tilde{F} \rightarrow F$ by the same letter $\mathcal{V}$.

For the tautological bundle $\mathcal{O}_{\mathcal{M}}(-1)$ of the projective bundle $\mathcal{M} = \mathbb{P}_{F'} \left((\mathcal{E}/\mathcal{V}') \oplus L \right)$, we consider the inclusion of $\mathcal{O}_{\mathcal{M}}(-1)$ into $(\mathcal{E}/\mathcal{V}') \oplus L$. We take the inverse image of $\mathcal{O}_{\mathcal{M}}(-1)$ by the quotient map $\mathcal{E} \oplus L \rightarrow \mathcal{E}/\mathcal{V}' \oplus L$, and write it by the same letter $\mathcal{V}$. We have isomorphisms $\mathcal{V}|_{\tilde{F}} \cong \mathcal{V}$ and $\mathcal{V}|_{F'} \cong \mathcal{V}' \oplus L$, where $\mathcal{V}$ and $\mathcal{V}'$ have the trivial $\mathbb{C}_u^*$ -actions. We see that $\mathcal{O}_{\mathcal{M}}(-1) = \mathcal{V}/\mathcal{V}'$ restricts to $\mathcal{O}_{\mathcal{M}}(-1)|_{\tilde{F}} = \mathcal{V}/\mathcal{V}'$ on $\tilde{F}$ and $\mathcal{O}_{\mathcal{M}}(-1)|_{F'} = L$ on F' . Hence the normal bundles $\tilde{N} \in K_{\mathbb{T} \times \mathbb{C}_u^*}(\tilde{F})$ of $\tilde{F}$ and $N' \in K_{\mathbb{T} \times \mathbb{C}_u^*}(F')$ of F' in $\mathcal{M}$ are written as

$$(2.5) \quad \tilde{N} = \mathcal{O}_{\mathcal{M}}(1) \otimes L|_{\tilde{F}} = \mathbb{C}_u \otimes \mathcal{V}^\vee - \mathbb{C}_u \otimes \mathcal{V}'^\vee,$$

$$(2.6) \quad N' = \mathcal{O}_{\mathcal{M}}(1)|_{F'} \otimes \mathcal{E}/\mathcal{V}' = \mathbb{C}_{u^{-1}} \otimes \mathcal{E} - \mathbb{C}_{u^{-1}} \otimes \mathcal{V}'.$$

Here $\mathcal{V}^\vee$ and $\mathcal{V}'^\vee$ are the dual of $\mathcal{V}$ and $\mathcal{V}'$ respectively. Furthermore we set $\Theta = \mathcal{H}om(\mathcal{V}', \mathcal{O}_{\mathcal{M}}(-1))$ on $\mathcal{M}$ so that we have the restrictions

$$(2.7) \quad \Theta|_{\tilde{F}} = \mathcal{H}om(\mathcal{V}', \mathcal{V}/\mathcal{V}'),$$

$$(2.8) \quad \Theta|_{F'} = \mathcal{H}om(\mathcal{V}', \mathcal{O}_{F'} \otimes \mathbb{C}_u) = \mathcal{V}'^\vee \otimes \mathbb{C}_u.$$

We write $\Theta|_{\tilde{F}}$ in (2.7) by Θ . This is equal to the relative tangent bundle of the projection $\mathcal{M} \rightarrow F$.

We write by $\hat{\Pi}: \mathcal{M} \rightarrow B$ the projection, and set $\tilde{\Pi} = \hat{\Pi}|_{\tilde{F}}$ and $\Pi' = \hat{\Pi}|_{F'}$. We use the following commutative diagram:

$$\begin{array}{ccc} K_{\mathbb{T} \times \mathbb{C}_u^*}(\mathcal{M}) & \xrightarrow{\tau_{\mathcal{M}}} & \widehat{CH}_{\mathbb{T} \times \mathbb{C}_u^*}(\mathcal{M}) \\ \hat{\Pi}_* \downarrow & & \downarrow \hat{\Pi}_* \\ K_{\mathbb{T}}(B)[u^{\pm 1}] \equiv K_{\mathbb{T} \times \mathbb{C}_u^*}(B) & \xrightarrow{\tau_B} & \widehat{CH}_{\mathbb{T} \times \mathbb{C}_u^*}(B) \end{array}$$

We set $c_1(u) = \hbar$. For an arbitrary $\gamma \in K_{\mathbb{T} \times \mathbb{C}_u^*}(\mathcal{M})$, we have $\hat{\Pi}_* \gamma \in K_{\mathbb{T}}(B)[u^{\pm 1}]$, and get the entire holomorphic function $\tau_B(\gamma) = \text{ch}(\hat{\Pi}_* \gamma) \cdot \text{Td}(B)$ of $\hbar$ by substituting $\text{ch}(u) = e^{\hbar}$ and multiplying the constant $\text{Td}(B) \in K_{\mathbb{T}}(B)$ in $\hbar$. Hence we have

$$(2.9) \quad \oint d\hbar \hat{\Pi}_* (\tau_{\mathcal{M}} \gamma) = \oint d\hbar \tau_B (\hat{\Pi}_* \gamma) = 0$$

for any closed path. By (2.1) in Theorem 2.2 and (2.4), we have

$$(\iota_*)^{-1} \gamma = \frac{\gamma|_{\tilde{F}}}{\wedge_{-1}(\tilde{N})^\vee} + \frac{\gamma|_{\tilde{F}'}}{\wedge_{-1}(\tilde{N}')^\vee}$$

for the embedding $\iota: \mathcal{M}^{\mathbb{C}_u^*} \rightarrow \mathcal{M}$. Using $\hat{\Pi}_* \iota_* = \tilde{\Pi}_* + \Pi'_*$ and (2.9), this implies

$$(2.10) \quad \oint d\hbar \tilde{\Pi}_* \text{ch} \left(\frac{\gamma|_{\tilde{F}}}{\wedge_{-1}(\tilde{N})^\vee} \right) \text{Td}(\tilde{F}) = - \oint d\hbar \Pi'_* \text{ch} \left(\frac{\gamma|_{\tilde{F}'}}{\wedge_{-1}(\tilde{N}')^\vee} \right) \text{Td}(F').$$

Here we used the compatibility of the isomorphisms (2.3) with push-forwards.

§ 2.4. Push-forward formula

For a symmetric polynomial $f(x_1, \dots, x_d) = g(e_1)$ and a positive integer $d \leq r = \text{rk } \mathcal{E}$, we take a K -theory class

$$\gamma = f(\mathcal{V}) \cdot \frac{\wedge_{-1} \mathcal{H}om(\mathcal{V}/\mathcal{V}', \mathcal{V}')}{d} \in K_{\mathbb{T} \times \mathbb{C}_u^*}(\mathcal{M}).$$

Taking the contour of integration for $\hbar$ suitably in the left hand side of (2.10), we get

$$\begin{aligned} \oint d\hbar \tilde{\Pi}_* \frac{\text{ch}(\gamma)|_{\tilde{F}} \text{Td}(\tilde{F})}{\text{ch} \wedge_{-1}(\tilde{N})^\vee} &= \oint d\hbar \tilde{\Pi}_* \frac{\text{ch}(\gamma)|_{\tilde{F}} \text{Td}(\tilde{F})}{(1 - e^{-\hbar + c_1(\mathcal{V})})} \\ &= \tilde{\Pi}_* \frac{\text{ch}\left(f(\mathcal{V}) \cdot \wedge_{-1} \Omega_{\tilde{F}/F}\right) \text{Td}(\tilde{F})}{d} \\ &= \Pi_* \text{ch}(f(\mathcal{V})) \cdot \text{Td}(F) \end{aligned}$$

by (2.5). Here $\Omega_{\tilde{F}/F}$ is the quotient bundle of $T^*\tilde{F}$ by the pull back $T^*F|_{\tilde{F}}$. On the other hand, the right hand side of (2.10) is equal to

$$- \oint d\hbar \Pi'_* \text{ch} \left(\frac{f(u \oplus \mathcal{V}')}{d} \cdot \frac{\wedge_{-1}(u^{-1} \otimes \mathcal{V}') \cdot \wedge_{-1}(u \otimes \mathcal{V}'^\vee)}{\wedge_{-1}(u \otimes \mathcal{E}^\vee)} \right) \text{Td}(F')$$

by (2.6). Thus we have

$$(2.11) \quad \tau_B(\Pi_* f(\mathcal{V})) = - \oint d\hbar \tau_B \left(\Pi'_* \frac{f(u \oplus \mathcal{V}')}{d} \cdot \frac{\wedge_{-1}(u^{-1} \otimes \mathcal{V}') \cdot \wedge_{-1}(u \otimes \mathcal{V}'^\vee)}{\wedge_{-1}(u \otimes \mathcal{E}^\vee)} \right),$$

where we use the same symbol τ_B for the suitably localized version of (2.2).

We take a symmetric polynomial $f_k(x_1, \dots, x_{d-k})$ such that their coefficients belong to $K_{\mathbb{T}}(Fl(\mathcal{E}; d-k))[u_1^{\pm 1}, \dots, u_k^{\pm 1}, \wedge_{-1}(u_1 \otimes \mathcal{E}^\vee)^{-1} \cdots \wedge_{-1}(u_k \otimes \mathcal{E}^\vee)^{-1}]$ and

$$\begin{aligned} f_k(\mathcal{V}) &= \frac{f(u_1 \oplus \cdots \oplus u_k \oplus \mathcal{V})}{d(d-1) \cdots (d-k+1)} \cdot \prod_{1 \leq i < j \leq k} (1 - u_j/u_i)(1 - u_i/u_j) \\ &\quad \cdot \left(\prod_{i=1}^k \frac{\wedge_{-1}(u_i^{-1} \otimes \mathcal{V}) \wedge_{-1}(u_i \otimes \mathcal{V}^\vee)}{\wedge_{-1}(u_i \otimes \mathcal{E}^\vee)} \right) \end{aligned}$$

for the tautological bundle $\mathcal{V}$ on $Fl(\mathcal{E}; d-k)$. Set $\text{ch}(u_1) = e^{\hbar_1}, \dots, \text{ch}(u_k) = e^{\hbar_k}$. In (2.11), we substitute $f(\mathcal{V}) = f_k(\mathcal{V})$, $u = u_{k+1}$, $F = Fl(\mathcal{E}; d-k)$, $F' = Fl(\mathcal{E}; d-k-1)$, and $d = d-k$. Then the last factors in $f_k(u_{k+1} \oplus \mathcal{V}')$ give $\prod_{1 \leq i < k+1} (1 - u_{k+1}/u_i)(1 - u_i/u_{k+1})$. Hence for $k = 0, 1, \dots, d-1$, we get

$$(2.12) \quad \tau_B(\Pi_*^{d-k} f_k(\mathcal{V})) = - \oint d\hbar_{k+1} \tau_B(\Pi_*^{d-k-1} f_{k+1}(\mathcal{V})),$$

where $\Pi^{d-k}: Fl(\mathcal{E}; d-k) \rightarrow B$ and $\Pi^{d-k-1}: Fl(\mathcal{E}; d-k-1) \rightarrow B$ are the projections.

The residue map $\text{Res}_{u=0,\infty}$ is defined using the split principle applied to $\mathcal{E}$. We can directly compute to get

$$(2.13) \quad \text{Res}_{u=0,\infty} \frac{u^l}{\wedge_{-u}(\mathcal{E}^\vee)} \frac{du}{u} = \begin{cases} h_{-l}(\mathcal{E}^\vee) & l \leq 0 \\ (-1)^{r-1} \det(\mathcal{E}) h_{l-r}(\mathcal{E}) & l > 0 \end{cases}$$

Here h_k is the complete homogeneous symmetric function of degree k

$$h_k(x_1, \dots, x_r) = \begin{cases} \sum_{\ell_1 + \dots + \ell_r = k} x_1^{\ell_1} \cdots x_r^{\ell_r} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

In [9, Theorem 1], we provided an alternative proof of the following theorem, which was originally proved in [10, (3)] by a different method.

Theorem 2.3. *For the projection $\Pi: F = Fl(\mathcal{E}; d) \rightarrow B$ of a flag bundle, we have*

$$(2.14) \quad \Pi_* g(\wedge^\bullet \mathcal{V}) = \operatorname{Res}_{u_1, \dots, u_d=0, \infty} \psi_d(\mathcal{E}) \cdot \frac{du_1 \cdots du_d}{u_1 \cdots u_d} \in K_{\mathbb{T}}(B)$$

where

$$(2.15) \quad \psi_d(\mathcal{E}) = \frac{f(u_1, \dots, u_d) \cdot \prod_{1 \leq i \neq j \leq d} (1 - u_i/u_j)}{d! \cdot \prod_{i=1}^d \wedge_{-u_i}(\mathcal{E}^\vee)}.$$

The operation $\operatorname{Res}_{u_1, \dots, u_d=0, \infty}$ is the iterated residue $\operatorname{Res}_{u_1=0, \infty} \operatorname{Res}_{u_2=0, \infty} \cdots \operatorname{Res}_{u_d=0, \infty}$ where the symbol $\operatorname{Res}_{u=0, \infty}$ denotes taking the sum of residues at 0 and ∞ .

Proof. It is enough to show

$$\tau_B(\Pi_* f(\mathcal{V})) = (-1)^d \oint d\hbar_1 \cdots \oint d\hbar_d \tau_B(f_d(u_1, \dots, u_d)).$$

We choose the contour of path of integration as follows. Possible poles are the sums $\sum_{\alpha \in I} a_\alpha$ over nonempty subsets $I \subset \{1, 2, \dots, r\}$. We take a_α small enough such that $0 < \Re \alpha < \varepsilon$, and $0 < \Im \alpha < \varepsilon$, where $\varepsilon \ll 1/r$ and $r = \operatorname{rk} \mathcal{E}$. Then we restrict the range of imaginary part of $\hbar$ smaller than 2π , and go around these poles. Then the assertion follows from (2.12). $\square$

§ 3. Application

By a similar argument using the localization formula [2, (1)] for the $\mathbb{T}$ -equivariant Chow ring $CH_{\mathbb{T}}^\bullet(M)$, we get the residue formula [11, Corollary 1] for cohomology classes. We re-write weight spaces by $u_1 = e^{z_1}, \dots, u_d = e^{z_d}$.

Theorem 3.1. *For a cohomology class $g(c_\bullet(\mathcal{V})) = g(c_1(\mathcal{V}), \dots, c_d(\mathcal{V}))$ and the fundamental cycle $[F]$ of $F = Fl(\mathcal{E}; d)$, we have*

$$(3.1) \quad \Pi_* (g(c_\bullet(\mathcal{V})) \cap [F]) = \operatorname{Res}_{z_1 \cdots z_d = \infty} \frac{f(z_1, \dots, z_d) \prod_{i \neq j} (z_i - z_j)}{d! \prod_{i=1}^d \operatorname{Eu}(e^{-z_i} \otimes \mathcal{E})} dz_1 \cdots dz_d$$

in $CH_{\mathbb{T}}^\bullet(B)$, where $f(x_1, \dots, x_d) = g(e_1(x_1, \dots, x_d), \dots, e_d(x_1, \dots, x_d))$, and $\operatorname{Res}_{z_1 \cdots z_d = \infty}$ is taking iterations of residues $\operatorname{Res}_{z_1 = \infty} \cdots \operatorname{Res}_{z_d = \infty}$.

If we write by $a_1, \dots, a_r$ the Chern roots of $\mathcal{E}$, then we have

$$(3.2) \quad \operatorname{Res}_{z=\infty} \frac{z^k dz}{\prod_{\alpha=1}^r -z + a_\alpha} = \begin{cases} (-1)^{r-1} h_{k-r+1}(a_1, \dots, a_r) & k \geq r \\ 0 & k < r. \end{cases}$$

It can be checked directly, and also by applying (3.1) for $\mathbb{P}(W) = Fl_{\text{pt}}(W; 1)$ and $\psi(\mathcal{V}) = c_1(\mathcal{V})^k$. Using (3.2), we can compute the linear map $\operatorname{Res}_{z_1 \cdots z_d = \infty}$ in the right hand side of (3.1).

§ 3.1. Determinantal formula

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with the length $\ell \leq d$, the Schur polynomial s_λ is defined by

$$s_\lambda = s_\lambda(x_1, \dots, x_d) = \frac{\det(x_i^{\lambda_j + d - j})_{1 \leq i, j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)}.$$

Theorem 3.2. *We have*

$$\Pi_* s_\lambda(\mathcal{V}) = (-1)^{dr + d(d+1)/2} \det(h_{\lambda_i - i - j + 2d - r + 1}(\mathcal{E})).$$

Proof. We take the Chern roots $a_1, \dots, a_r$ of $\mathcal{E}$. By (3.1), we have

$$\begin{aligned} \Pi_* s_\lambda(\mathcal{V}) &= \operatorname{Res}_{x_1 \cdots x_d = \infty} \frac{s_\lambda(x_1, \dots, x_d) \prod_{i \neq j} (x_i - x_j)}{d! \prod_{i=1}^d \prod_{\alpha=1}^r (-x_i + a_\alpha)} dx_1 \cdots dx_d \\ &= \operatorname{Res}_{x_1 \cdots x_d = \infty} \frac{(-1)^{d(d-1)/2} \det(x_i^{\lambda_j + d - j})_{1 \leq i, j \leq d} \det(x_i^{d-j})_{1 \leq i, j \leq d}}{d! \prod_{i=1}^d \prod_{\alpha=1}^r (-x_i + a_\alpha)} dx_1 \cdots dx_d \\ &= \operatorname{Res}_{x_1 \cdots x_d = \infty} \frac{(-1)^{d(d-1)/2} \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^d x_{\sigma(i)}^{\lambda_j + d - j} \det(x_{\sigma(i)}^{d-j})_{1 \leq i, j \leq d}}{d! \prod_{i=1}^d \prod_{\alpha=1}^r (-x_i + a_\alpha)} dx_1 \cdots dx_d \\ &= \operatorname{Res}_{x_1 \cdots x_d = \infty} \frac{(-1)^{d(d-1)/2} \prod_{i=1}^d x_i^{\lambda_j + d - j} \det(x_i^{d-j})_{1 \leq i, j \leq d}}{\prod_{i=1}^d \prod_{\alpha=1}^r (-x_i + a_\alpha)} dx_1 \cdots dx_d \\ &= (-1)^{d(r-1) + d(d-1)/2} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) \prod_{j=1}^d h_{\lambda_j - j - \sigma(j) + 2d - r + 1}(\mathcal{E}). \end{aligned}$$

□

When $B = \text{pt}$, and $\mathcal{E} = W$ is regarded as a vector bundle of rank r over the point pt , we get the Grassmannian $Fl(W; d)$ of d -dimensional sub-spaces in $W = \mathbb{C}^r$. In particular when $d = r$, we have

$$(3.3) \quad s_\lambda(a_1, \dots, a_r) = (-1)^{r(r-1)/2} \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) \prod_{j=1}^r h_{\lambda_j - j - \sigma(j) + r + 1}(a_1, \dots, a_r).$$

For a partition λ of length $\ell \leq d$, we set $\bar{\lambda} = \lambda - (r - d)^d$. We set $h_\ell = 0$ for $\ell < 0$, and hence $s_{\bar{\lambda}} = 0$ if $\bar{\lambda}$ contains negative integers. By the above theorem, we have

$$s_{\bar{\lambda}}(\mathcal{E}) = (-1)^{r(r-1)/2} \sum_{\sigma \in \mathfrak{S}_r} \text{sgn}(\sigma) \prod_{j=1}^r h_{\bar{\lambda}_j - j - \sigma(j) + r + 1}(\mathcal{E}).$$

The last summand is equal to zero unless we have $\tau \in \mathfrak{S}_d$ such that

$$\sigma = \sigma_\tau = \begin{pmatrix} 1 & \cdots & d & d+1 \cdots r \\ r-d+\tau(1) \cdots r-d+\tau(d) & r-d \cdots 1 \end{pmatrix}$$

since $s_{-j-\sigma(j)+r+1} = 0$ unless $-j - \sigma(j) + r + 1 \geq 0$ for $j = n+1, \dots, r$.

$$s_{\bar{\lambda}}(a_1, \dots, a_r) = (-1)^{r(r-1)/2} \sum_{\tau \in \mathfrak{S}_d} \frac{\text{sgn}(\sigma_\tau)}{\text{sgn}(\tau)} \text{sgn}(\tau) \prod_{j=1}^d s_{\bar{\lambda}_j - j - r + d - \tau(j) + r + 1}.$$

Lemma 3.3. *We have $\frac{\text{sgn}(\sigma_\tau)}{\text{sgn}(\tau)} = (-1)^{\frac{r(r-1)+d(d-1)}{2}}$.*

Proof. We compute the sign of

$$\sigma_\tau = \begin{pmatrix} r-d+\tau(1) \cdots r-d+\tau(n) & r-d \cdots 1 \end{pmatrix}$$

as follows.

$$\begin{aligned} (-1)^{(r-d)d} &= \text{sgn} \begin{pmatrix} r-d+1 \cdots r & 1 \cdots r-d \end{pmatrix} \\ &= (-1)^{(r-d)(r-d-1)/2} \cdot \text{sgn} \begin{pmatrix} r-d+1 \cdots r & r-d \cdots 1 \end{pmatrix} \\ &= (-1)^{(r-d)(r-d-1)/2} \cdot \text{sgn}(\tau) \cdot \text{sgn}(\sigma_\tau) \end{aligned}$$

Therefore we have $\frac{\text{sgn}(\sigma_\tau)}{\text{sgn}(\tau)} = (-1)^{(r-d)d + (r-d)(r-d-1)/2} = (-1)^{\frac{r(r-1)+d(d-1)}{2}}$. □

By this lemma, we have

$$s_{\bar{\lambda}}(\mathcal{E}) = (-1)^{d(d-1)/2} \sum_{\tau \in \mathfrak{S}_d} \text{sgn}(\tau) \prod_{j=1}^d h_{\bar{\lambda}_j - j - \tau(j) + d + 1}(\mathcal{E}).$$

Since $\bar{\lambda}_j = \lambda_j - r + d$ for $j = 1, \dots, d$,

$$\begin{aligned} s_{\bar{\lambda}}(\mathcal{E}) &= (-1)^{d(d-1)/2} \cdot \sum_{\tau \in \mathfrak{S}_d} \text{sgn}(\tau) \prod_{j=1}^d h_{\lambda_j - j - \tau(j) + 2d - r + 1}(\mathcal{E}) \\ &= (-1)^{d(d-1)/2} \cdot (-1)^{dr + d(d+1)/2} \Pi_* s_\lambda(\mathcal{V}) \\ (3.4) \quad &= (-1)^{(r-d)d} \Pi_* s_\lambda(\mathcal{V}). \end{aligned}$$

§ 3.2. Plücker formula

Let $\mathcal{Q}$ be the universal quotient bundle of Π^*E on $Fl(\mathcal{E}, d)$. We denote by θ the Chern class $c_1(\mathcal{Q}) = c_1(\det \mathcal{Q})$, and call it the *Plücker class* of $Fl(\mathcal{E}, d)$.

We can identify $\mathcal{Q}^\vee$ with the universal sub-bundle on $Fl(\mathcal{E}^\vee; r-d)$. Hence, using (3.4), we have

$$\begin{aligned} \Pi_* s_\lambda(\mathcal{Q}) &= (-1)^{|\lambda|} \Pi_* s_\lambda(\mathcal{Q}^\vee) \\ &= (-1)^{|\lambda|+d(r-d)} s_{\bar{\lambda}}(\mathcal{E}^\vee) \\ &= (-1)^{|\lambda|+d(r-d)+|\lambda|-d(r-d)} s_{\bar{\lambda}}(\mathcal{E}), \end{aligned}$$

where $|\lambda| = \sum_{i=1}^d \lambda_i$. Thus we have

$$(3.5) \quad \Pi_* s_\lambda(\mathcal{Q}) = s_{\bar{\lambda}}(\mathcal{E}).$$

This is used to show the Plücker formula in [4].

Theorem 3.4. *For each integer $N \geq d(r-d)$, we have*

$$\Pi_* \theta^N = \sum_{|\lambda|=N-d(r-d)} f^{\lambda+(r-d)^d} s_\lambda(\mathcal{E})$$

where $f^{\lambda+(r-d)^d}$ is the number of standard Young tableaux with shape $\lambda + (r-d)^d$.

Proof. Here we reproduce the proof in [4]. Using Pieri's formula repeatedly, we obtain that

$$\theta^N = \sum_{|\mu|=N} f^\mu s_\mu(\mathcal{Q}).$$

By (3.5), we have

$$\Pi_* \theta^N = \sum_{|\mu|=N} f^\mu \Pi_* s_\mu(\mathcal{Q}) = \sum_{|\lambda|=N-d(r-d)} f^{\lambda+(r-d)^d} s_{\bar{\lambda}}(\mathcal{E}).$$

□

References

- [1] D. Edidin and W. Graham, *Riemann-Roch for equivariant Chow groups*, Duke Math. J. **102** (2000), no.3, 567–594.
- [2] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.

- [3] B. Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.
- [4] H. Kaji and T. Terasoma, Degree formula for Grassmann bundles, J. Pure Appl. Algebra **219** (2015), no. 12, 5426–5428.
- [5] T. Mochizuki, *Donaldson Type Invariants for Algebraic Surfaces: Transition of Moduli Stacks*, Lecture Notes in Math. **1972**, Springer, Berlin, 2009.
- [6] A. Nielsen, *Diagonalizably linearized coherent sheaves*, Bull. Soc. Math. France **219** (1974), 85–97.
- [7] H. Nakajima and K. Yoshioka, *Perverse coherent sheaves on blowup. III. Blow-up formula from wall-crossing*, Kyoto J. Math. **51** (2011), no. 2, 263–335.
- [8] R. Thomason, *Une formule de Lefschetz en K -théorie équivariante algébrique*, Duke Math. J. **68** (1992), no. 3, 447–462.
- [9] R. Ohkawa, *Residue formula for flag manifold of type A from wall-crossing*, J. Algebraic Combin. **61** (2024), no. 1, 13, 12 pages.
- [10] A. Weber and M. Zielenkiewicz, *Residues formulas for the push-forward in K -theory, the case of G_2/P* , J. Algebraic Combin. **49** (2019), no. 3, 361–380.
- [11] M. Zielenkiewicz, *Integrations over homogeneous spaces for classical Lie groups using iterated residues at infinity*, Cent. Eur. J. Math. **12** (2014), no. 4, 574–583.