

ON THE CLASSIFICATION OF SMOOTH CURVES OF GENUS $g = 3, 4, 5, 6$ WITH ONE PLACE AT INFINITY.

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§1. Introduction.

We consider a smooth affine curve $C = \{f(x, y) = 0\} \subset \mathbb{C}^2$ of degree n with one place at infinity, say at $\rho = (1; 0; 0)$ and let g be the genus of the smooth compactification of C . By the assumption, $f(x, y)$ is written as

$$(1.1) \quad f(x, y) = (y^{a_1} + \xi_1 x^{c_1})^{A_2} + (\text{lower terms}), \quad \xi_1 \in \mathbb{C}^*, c_1 < a_1, n = a_1 A_2$$

where a_1, c_1, A_2 are integers and $\gcd(a_1, c_1) = 1$.

The purpose of this note is classify the possible normal forms for a given genus g , $g \leq 6$. We use the following result of A'Campo-Oka [AO]. Let $\bar{C}$ be the projective compactification of C .

Theorem (1.2). *There is a canonical factorization $A_i = a_i a_{i+1} \cdots a_k$ and a resolution tower of $(\bar{C}, \rho), \mathcal{T}$, of toric modifications*

$$\mathcal{T} = \{ X_k \xrightarrow{p_k} X_{k-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{p_1} X_0 = \mathbb{C}^2 \}$$

with the corresponding weight vectors $P_i = {}^t(a_i, b_i)$ for $i = 1, \dots, k$ ($b_1 = a_1 - c_1$) which satisfies the following conditions. Let $h_i(x, y)$ be the A_{i+1} -th Tschirnhausen approximate polynomial of $f(x, y)$ as a polynomial of y and let $C_i = \{(x, y) \in \mathbb{C}^2; h_i(x, y) = 0\}$ for $i = 1, \dots, k$.

Note that $\deg C_i = a_1 \cdots a_i$, $h_k = f$ and $C_k = C$.

(1) For each $i = 1, \dots, k$, $\bar{C}_i$ passes through ρ and $(\bar{C}_i, \rho)$ is irreducible at ρ and $\Phi_i = p_1 \circ \cdots \circ p_i : X_i \rightarrow X_0$ gives a minimal resolution tower of $(\bar{C}_i, \rho)$.

(2) Milnor number $\mu(\bar{C}_i, \rho)$ is given by

$$(1.2.1) \quad \mu(\bar{C}_i, \rho) = 1 - A_1 + \sum_{s=1}^k (A_s - 1) b_s A_{s+1}$$

(3) The local intersection multiplicity $I(\bar{C}_i, \bar{C}; \rho)$ is given by

$$(1.2.2) \quad I(\bar{C}_i, \bar{C}; \rho) = \sum_{s=1}^{i+1} a_s b_s A_{s+1}^2 / A_{i+1}, \quad i \leq k-1$$

Using the modified Plücker formula and (1.2.1), we have the equality $((a_g), \S 8, [AO])$

$$(1.3) \quad \sum_{i=1}^k (A_i - 1) b_i A_{i+1} = (A_1 - 1)^2 - 2g$$

By Bezout theorem and (1.2.2), we have the inequality ((b), §8, [AO])

$$(1.4) \quad \sum_{i=1}^k a_i b_i A_{i+1}^2 \leq A_1^2$$

§2. Main result.

Theorem (2.1). *C: a smooth curve in $\mathbf{C}^2$, homeomorphic to a surface with one puncture of genus $g = 3, 4, 5, 6$. Then there exist an automorphism of $\mathbf{C}^2$ moving the curve C to a curve which is one of the following models.*

$g=3$: a) $n=4, P_1 = (4, 1)$, smooth at infinity, tangent to the line at infinity at a single point. An example is given by $\{y^4 + x^3 + 1 = 0\}$.

b) $n=7, P_1 = (7, 5)$. The curve has a non-degenerate cusp singularity at infinity. An example is given by $\{y^7 + x^2 + 1 = 0\}$.

c) $k=2, n=6, P_1 = (3, 1), P_2 = (2, 9)$. An example is given by $\{(y^3 + x^2)^2 + x = 0\}$.

$g=4$: a) $k=1, n=5, P_1 = (5, 2)$. $\{y^5 + x^3 + 1 = 0\}$.

b) $k=1, n=9, P_1 = (9, 7)$. $\{y^9 + x^2 + 1 = 0\}$.

c) $k=2, n=6, P_1 = (3, 1), P_2 = (2, 7)$. $\{(y^3 + x^2)^2 + xy + 1 = 0\}$.

d) $k=2, n=9, P_1 = (3, 1), P_2 = (3, 16)$. $\{(y^3 + x^2)^3 + y = 0\}$.

$g=5$: a) $k=1, n=11, P_1 = (11, 9)$. $\{y^{11} + x^2 + 1 = 0\}$.

d) $k=2, n=6, P_1 = (3, 1), P_2 = (2, 5)$. $\{(y^3 + x^2)^2 + xy^2 + 1 = 0\}$.

$g=6$: a) $k=1, n=5, P_1 = (5, 1)$. $\{y^5 + x^4 + 1 = 0\}$.

b) $k=1, n=7, P_1 = (7, 4)$. $\{y^7 + x^3 + 1 = 0\}$.

c) $k=1, n=13, P_1 = (13, 11)$. $\{y^{13} + x^2 + 1 = 0\}$.

d) $k=2, n=6, P_1 = (3, 1), P_2 = (2, 3)$. $\{(y^3 + x^2)^2 + x^3 + 1 = 0\}$.

e) $k=2, n=10, P_1 = (5, 3), P_2 = (2, 15)$. $\{(y^5 + x^2)^2 + x = 0\}$.

f) $k=2, n=9, P_1 = (3, 1), P_2 = (3, 14)$. $\{(y^3 + x^2)^3 + y^2 + 1 = 0\}$.

Proof. If necessary, applying the Jung automorphisms:

$$\phi : \mathbf{C}^2 \rightarrow \mathbf{C}^2, \quad \phi(x, y) = (y^{a_1} + \xi_1 x, y),$$

we can assume that $a_1 > c_1 \geq 2$. If $k=1$, then we have $(a_1 - 1)(c_1 - 1) = 2g$ by (1.3), hence,

$$a_1 > c_1 = 1 + \frac{2g}{a_1 - 1}.$$

Using the above inequality and $\gcd(a_1, b_1) = 1$, we can get the preceding results in the case of $k = 1$. So, we consider the case $k \geq 2$. In this case, using that $(1 - A_2) \times (1.4) + A_2 \times (1.3)$, we can get the following inequality ((*), §8, [AO]):

$$(2.2) \quad A_2 \leq \frac{2g - 1}{(a_1 - 1)(c_1 - 1) - 1} \leq 2g - 1$$

$g = 3$: (The result of this case is given in [AO] without proof.) By (2.2), $A_2 = 2, 3, 4, 5$. ($A_2 = 1$ if and only if $k = 1$)

If $A_2 = 5$, $(a_1 - 1)(c_1 - 1) - 1 = 1$ by (2.2). Hence, $k = 2, a_1 = 3, c_1 = 2, b_1 = 1, a_2 = A_2 = 5, n = A_1 = 15$. By (1.3), we have $b_2 = 30$. This contradicts $\gcd(a_2, b_2) = 1$.

If $A_2 = 4$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

- (i) $k = 2, a_2 = A_2 = 4, n = A_1 = 12$. By (1.3), $b_2 = 71/3$. This contradicts the fact that b_2 is a integer.
- (ii) $k = 3, n = A_1 = 12, a_2 = 2, a_3 = A_3 = 2$. By (1.3),

$$6b_2 + b_3 = 71, \quad (1)$$

hence,

$$b_2 = \frac{71 - b_3}{6} < \frac{71}{6} < 12. \quad (2)$$

By (b),

$$4b_2 + b_3 \leq 48. \quad (3)$$

Using (1) and (3), we get $2b_2 \geq 23$, hence, $b_2 \geq 12$. This contradicts (2).

If $A_2 = 3$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2). and $a_2 = A_2 = 3, n = A_1 = 9$. By (1.3), $b_2 = 17$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (3, 17)$. We shall show that there is no polynomial $f(u, v)$ of degree 9 with the weight vectors above. Let

$$f(u, v) = (v^3 + u)^3 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 9, \quad 9 < 3\alpha + \beta. \quad (4)$$

We consider an admissible toric modification $p : X_1 \rightarrow \mathbb{C}^2$. We may assume that $\sigma = (E_1, P_1), E_1 = (1, 0)$, is the left toric cone of the divisor $E(P_1)$ and let (s, t) be the toric coordinates. Then $u = st^3, v = t$. Hence,

$$\begin{aligned} \pi_\sigma^* f(s, t) &= t^9(1 + s)^3 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta} \\ &= t^9 \left\{ (1 + s)^3 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta - 9} \right\}. \end{aligned}$$

By (4), there is no (α, β) such that $3\alpha + \beta - 9 = 17$. Therefore $P_2 = (3, 17)$ is not the second weight vector for $f(u, v)$. Thus this case does not occur.

If $A_2 = 2$, then $(a_1 - 1)(c_1 - 1) - 1 = 1$ or 2 by (2.2).

- (i) If $(a_1 - 1)(c_1 - 1) - 1 = 2$, then $a_1 = 4, c_1 = 2, b_1 = 2$. This contradicts $\gcd(a_1, b_1) = 1$.
- (ii) If $(a_1 - 1)(c_1 - 1) - 1 = 1$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 2, n = A_1 = 6$. By $(a_g), b_2 = 9$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (2, 9)$. Let

$$f(u, v) = (v^3 + u)^2 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 6, \quad 6 < 3\alpha + \beta.$$

Using the preceding admissible toric modification,

$$\begin{aligned} \pi_\sigma^* f(s, t) &= t^6(1 + s)^2 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta} \\ &= t^6 \left\{ (1 + s)^2 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta - 6} \right\}. \end{aligned}$$

If $\alpha = 5, \beta = 0$, then $3\alpha + \beta - 6 = 9$. So if $c_{5,0} \neq 0$, the second weight vector for $f(u, v)$ can be $P_2 = (2, 9)$. For example, let $f(u, v) = (v^3 + u)^2 + u^5$. Then $F(x, y) = (y^3 + x^2)^2 + x$, which is non-singular in $\mathbb{C}^2$.

$g = 4$: By (2.2), $A_2 = 2, 3, 4, 5, 6, 7$. If $A_2 = 7$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2), and $a_2 = A_2 = 7, n = A_1 = 21$. By (1.3), $b_2 = 42$. This contradicts $\gcd(a_2, b_2) = 1$.

If $A_2 = 6$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

- (i) $k = 2, a_2 = A_2 = 6, n = A_1 = 18$. By (1.3), $b_2 = 179/5$. This contradicts the fact that b_2 is a integer.
- (ii) $k = 3, n = A_1 = 18, a_2 = 2$ or 3 .
If $a_2 = 2$, then $a_3 = A_3 = 3$. By (1.3),

$$15b_2 + 2b_3 = 179, \quad (5)$$

hence,

$$b_2 = \frac{179 - 2b_3}{15} < \frac{179}{15} < 12. \quad (6)$$

By (b),

$$6b_2 + b_3 \leq 72. \quad (7)$$

Using (5) and (7), $3b_2 \geq 35$, hence, $b_2 \geq 12$. This contradicts (6).

If $a_2 = 3, a_3 = A_3 = 2$. By (1.3),

$$10b_2 + b_3 = 179, \quad (8)$$

hence,

$$b_2 = \frac{179 - b_3}{10} < 18. \quad (9)$$

By (b),

$$6b_2 + b_3 \leq 108. \quad (10)$$

Using (8) and (10), $4b_2 \geq 71$, hence, $b_2 \geq 18$. This contradicts (9).

If $A_2 = 5$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2), and $a_2 = A_2 = 5, n = A_1 = 15$. By (1.3), $b_2 = 59/2$. This contradicts the fact that b_2 is a integer.

If $A_2 = 4$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

- (i) $k = 2, a_2 = A_2 = 4, n = A_1 = 12$. By (1.3), $b_2 = 23$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (4, 23)$. Let

$$f(u, v) = (v^3 + u)^4 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 12, \quad 12 < 3\alpha + \beta. \quad (11)$$

Using the preceding admissible toric modification: $u = st^3, v = t$,

$$\begin{aligned} \pi_\sigma^* f(s, t) &= t^{12}(1 + s)^4 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta} \\ &= t^{12} \left\{ (1 + s)^4 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta - 12} \right\}. \end{aligned}$$

By (11), there is no (α, β) such that $3\alpha + \beta - 12 = 23$. Therefore $P_2 = (4, 23)$ is not the second weight vector for $f(u, v)$. Thus this case does not occur.

- (ii) $k = 3, n = A_1 = 12, a_2 = 2, a_3 = A_3 = 2$. By (1.3),

$$6b_2 + b_3 = 69, \quad (12)$$

hence,

$$b_2 = \frac{69 - b_3}{6} < \frac{69}{6} < 12. \quad (13)$$

By (b),

$$4b_2 + b_3 \leq 48. \quad (14)$$

Using (12) and (14), $2b_2 \geq 21$, hence, $b_2 \geq 11$. By this inequality and (13), we can conclude that $b_2 = 11$. And $b_3 = 3$ by (12). Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (2, 11), P_3 = (2, 3)$. Let

$$f(u, v) = (v^3 + u)^4 + (\text{higher terms}).$$

Then

$$h_1(u, v) = (v^3 + u) + (\text{higher terms}),$$

$$h_2(u, v) = (v^3 + u)^2 + (\text{higher terms}),$$

where h_i is A_{i+1} -th Tschirnhausen approximate polynomial of f . Since h_1 is the 2-th Tschirnhausen approximate polynomial of h_2 , $h_2(u, v)$ is written as

$$h_2(u, v) = h_1(u, v)^2 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\beta \leq 2, \alpha + \beta \leq 6, 6 < 3\alpha + \beta. \quad (15)$$

Using the preceding admissible toric modification: $u = st^3, v = t$,

$$\pi_\sigma^* h_1(s, t) = t^3 \{(1 + s) + \dots\}.$$

Hence

$$p_1^* h_1(u_1, v_1) = u_1^3 v_1,$$

$$p_1^* h_2(u_1, v_1) = u_1^6 v_1^2 + p_1^* \left(\sum c_{\alpha, \beta} u^\alpha v^\beta \right).$$

And now, by $P_2 = (2, 11)$ we have

$$p_1^* h_2(u_1, v_1) = u_1^6 (v_1^2 + u_1^{11}) + (\text{higher terms}).$$

Therefore, the monomial u_1^{17} must exist in $p_1^* \left(\sum c_{\alpha, \beta} u^\alpha v^\beta \right)$. Though

$$\pi_\sigma^* \left(\sum c_{\alpha, \beta} u^\alpha v^\beta \right) = \sum c_{\alpha, \beta} s^\alpha t^{3\alpha + \beta},$$

by (15) there is no (α, β) such that $3\alpha + \beta = 17$. Therefore we find that $P_2 = (2, 11)$ is not the second weight vector for $f(u, v)$. Thus this case does not occur.

If $A_2 = 3$, then $(a_1 - 1)(c_1 - 1) - 1 = 1$ or 2 by (2.2). Since $\gcd(a_1, b_1) = 1$, $(a_1 - 1)(c_1 - 1) - 1 \neq 2$. Therefore $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 3, n = A_1 = 9$, By (1.3), $b_2 = 16$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (3, 16)$. Let

$$f(u, v) = (v^3 + u)^3 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 9, 9 < 3\alpha + \beta.$$

Using the preceding admissible toric modification,

$$\begin{aligned}\pi_\sigma^* f(s, t) &= t^9(1+s)^3 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha+\beta} \\ &= t^9 \left\{ (1+s)^3 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha+\beta-9} \right\}.\end{aligned}$$

If $\alpha = 8, \beta = 1$, then $3\alpha + \beta - 9 = 16$. So if $c_{8,1} \neq 0$, the second weight vector for $f(u, v)$ can be $P_2 = (3, 16)$. For example, let $f(u, v) = (v^3 + u)^3 + u^8 v$. Then $F(x, y) = (y^3 + x^2)^3 + y$, which is non-singular in $\mathbb{C}^2$.

If $A_2 = 2$, then $(a_1 - 1)(c_1 - 1) - 1 = 1, 2, 3$ by (2.2). Since $\gcd(a_1, b_1) = 1$, $(a_1 - 1)(c_1 - 1) - 1 \neq 2$.

- (i) If $(a_1 - 1)(c_1 - 1) - 1 = 1$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 2, n = A_1 = 6$. By (1.3), $b_2 = 7$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (2, 7)$. Let

$$f(u, v) = (v^3 + u)^2 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 6, \quad 6 < 3\alpha + \beta.$$

Using the preceding admissible toric modification: $u = st^3, v = t$,

$$\begin{aligned}\pi_\sigma^* f(s, t) &= t^6(1+s)^2 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha+\beta} \\ &= t^6 \left\{ (1+s)^2 + \sum c_{\alpha, \beta} s^\alpha t^{3\alpha+\beta-6} \right\}.\end{aligned}$$

If $\alpha = 4, \beta = 1$, then $3\alpha + \beta - 6 = 7$. So if $c_{4,1} \neq 0$, the second weight vector for $f(u, v)$ can be $P_2 = (2, 7)$. For example, let $f(u, v) = (v^3 + u)^2 + u^4 v + u^6$. Then $F(x, y) = (y^3 + x^2)^2 + xy + 1$, which is non-singular in $\mathbb{C}^2$.

- (ii) If $(a_1 - 1)(c_1 - 1) - 1 = 3$, then $k = 2, a_1 = 5, c_1 = 2, b_1 = 3$, and $a_2 = A_2 = 2, n = A_1 = 10$. By (1.3), $b_2 = 19$. Thus the tower has the weight vectors $P_1 = (5, 3), P_2 = (2, 19)$. Let

$$f(u, v) = (v^5 + u^3)^2 + \sum_{\alpha, \beta} c_{\alpha, \beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 10, \quad 30 < 5\alpha + 3\beta. \quad (16)$$

We may assume that $\sigma = (Q_1, P_1), Q_1 = (2, 1)$, is the left toric cone of the divisor $E(P_1)$ and let (s, t) be the toric coordinates. Then $u = s^2 t^5, v = st^3$. Hence

$$\begin{aligned}\pi_\sigma^* f(s, t) &= s^{10} t^{30} (1+s)^2 + \sum c_{\alpha, \beta} s^{2\alpha+\beta} t^{5\alpha+3\beta} \\ &= t^{30} \left\{ s^{10} (1+s)^2 + \sum c_{\alpha, \beta} s^{2\alpha+\beta} t^{5\alpha+3\beta-30} \right\}.\end{aligned}$$

By (16), there is no (α, β) such that $5\alpha + 3\beta - 30 = 19$. Therefore $P_2 = (2, 19)$ is not the second weight vector for $f(u, v)$. Thus this case does not occur.

The cases $g = 5, 6$ are proved likewise. $\square$

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