

# Generalized Whittaker functions of degenerate principal series

Kazuki Hiroe\*

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1  
Komaba, Meguro-ku, Tokyo, 153-8914, Japan.

## Abstract

In the theory of modular forms, modular forms with weights are important objects. For automorphic forms on  $SL(2, \mathbb{R})$ , the notion of weights are translated to characters of  $SO(2)$ . Hence for general cases,  $K$ -types of admissible representations can be seen as a generalization of weights of corresponding automorphic forms. In this paper, we consider degenerate principal series representations and define a class of their  $K$ -types which are called strongly spherical (Definition 3.2). And we give a characterization of generalized Whittaker functions with strongly spherical  $K$ -types of degenerate principal series representation (Theorem 5.2). The contents in this paper will appear with concrete proofs in [2].

## 1 Notation and preliminaries

In this section we give a quick review of some definitions and well known facts in the representation theory of Lie groups.

Let  $G$  be a connected real semisimple Lie group,  $K$  a maximal compact subgroup and  $\theta$  the associated Cartan involution. Throughout this paper we assume that  $G$  is split over  $\mathbb{R}$  and has a complexification  $G_{\mathbb{C}}$ . The differentiation of  $\theta$  is also written by same symbol. The associated Cartan decomposition of Lie algebra  $\mathfrak{g}$  of  $G$  is denoted by  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ . Here  $\mathfrak{k}$  and  $\mathfrak{s}$  are eigenspaces of  $\theta$  with eigenvalues 1 and  $-1$  respectively.

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\*E-mail:kazuki@ms.u-tokyo.ac.jp

Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{s}$  and  $\Sigma$  the root system of  $(\mathfrak{g}, \mathfrak{a})$ . Its Weyl group  $W$  is isomorphic with  $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ . Fix a positive system  $\Sigma^+$  of  $\Sigma$  and denote the set of simple roots by  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Let  $\mathfrak{n}$  be the sum of the root space  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{a}\}$  for  $\alpha \in \Sigma^+$ , i.e.,  $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ . Then we have an Iwasawa decompositions  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$  and  $G = KAN$  where  $A = \exp \mathfrak{a}$  and  $N = \exp \mathfrak{n}$ . Also we define  $\bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$  and  $\bar{N} = \exp \bar{\mathfrak{n}}$ . Let us denote the Killing form on  $\mathfrak{g}$  by  $B$ . For  $\lambda \in \mathfrak{a}^*$ , we take  $H_\lambda \in \mathfrak{a}$  satisfying the equations  $\lambda(H) = B(H_\lambda, H)$  for any  $H \in \mathfrak{a}$ . We introduce an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{a}^*$  defined by  $\langle \mu, \nu \rangle = B(H_\mu, H_\nu)$  for  $\mu, \nu \in \mathfrak{a}^*$ .

We denote the centralizer of  $A$  in  $K$  by  $M$ . Then a minimal parabolic subgroup  $P$  is defined by  $P = MAN$ . Let  $\Theta \subset \Pi$  be a finite subset and define the parabolic subgroup  $P_\Theta$  associated to  $\Theta$  as follows. Let  $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for any } \alpha \in \Theta\}$  and  $\mathfrak{a}_\Theta^\perp$  the orthogonal complement of  $\mathfrak{a}_\Theta$  in  $\mathfrak{a}$  with respect to the Killing form. Furthermore let  $\mathfrak{n}_\Theta = \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Theta)} \mathfrak{g}_\alpha$  and  $\mathfrak{m}_\Theta = \mathfrak{a}_\Theta^\perp \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \text{span}(\Theta)} \mathfrak{g}_\alpha$ . Then we can define the parabolic subalgebra associated to  $\Theta$  by  $\mathfrak{p}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta$ . Let  $L_\Theta = Z_G(\mathfrak{a}_\Theta)$ ,  $K_\Theta = L_\Theta \cap K$  and  $M_\Theta = K_\Theta \exp(\mathfrak{m}_\Theta \cap \mathfrak{s})$ . Then we can define the parabolic subgroup associated to  $\Theta$  by  $P_\Theta = M_\Theta A_\Theta N_\Theta$ . If  $\Theta = \emptyset$ , the parabolic subgroup  $P_\emptyset = M_\emptyset A_\emptyset N_\emptyset$  equals to the minimal parabolic subgroup  $P = MAN$  defined above.

We write  $\mathfrak{g}_\mathbb{C}$ ,  $\mathfrak{k}_\mathbb{C}$  etc. as the complexifications of  $\mathfrak{g}$ ,  $\mathfrak{k}$  etc. Let  $U(\mathfrak{g})$ ,  $U(\mathfrak{k})$  etc. be the universal enveloping algebras of complexifications of  $\mathfrak{g}$ ,  $\mathfrak{k}$ , etc. Also let  $Z(\mathfrak{g})$ ,  $Z(\mathfrak{k})$  be the centers of universal enveloping algebras  $U(\mathfrak{g})$ ,  $U(\mathfrak{k})$  respectively. As it is well-known, there is an inclusion

$$Z(\mathfrak{g}) \subset U(\mathfrak{a}) \oplus \bar{\mathfrak{n}}_\mathbb{C} U(\mathfrak{g}).$$

Let  $\sigma: Z(\mathfrak{g}) \rightarrow U(\mathfrak{a})$  be the projection map along this decomposition. Put  $\rho = \text{tr}(\text{Ad}|_{\mathfrak{n}}) \in \mathfrak{a}_\mathbb{C}^*$ , then we can define the  $\rho$ -shifted map  $\sigma': Z(\mathfrak{g}) \rightarrow U(\mathfrak{a})$  by  $\sigma'(X)(\lambda) = \sigma(X)(\lambda - \rho)$  for  $X \in Z(\mathfrak{g})$  and  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . It is well known that this map gives an algebra isomorphism

$$\sigma': Z(\mathfrak{g}) \longrightarrow U(\mathfrak{a})^W,$$

which is called Harish-Chandra isomorphism. For  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ , we can define a character of  $Z(\mathfrak{g})$  by

$$\chi_\lambda: \begin{array}{ccc} Z(\mathfrak{g}) & \longrightarrow & \mathbb{C} \\ X & \longmapsto & \sigma'(X)(\lambda). \end{array}$$

For  $C^\infty(G, E)$ , the space of smooth functions from  $G$  to a finite dimensional vector space  $E$ , we can consider natural actions of  $G$  and  $\mathfrak{g}$  by left (right)

translations and left (right) derivations, i.e.,

$$L_g f(x) = f(g^{-1}x), \quad R_g f(x) = f(xg), \quad (1.1)$$

$$L_X f(x) = \frac{d}{dt} L_{(\exp tX)} f(x)|_{t=0}, \quad R_X = \frac{d}{dt} R_{(\exp tX)} f(x)|_{t=0}, \quad (1.2)$$

where  $x, g \in G$ ,  $X \in \mathfrak{g}$  and  $f \in C^\infty(G, E)$ .

Let  $(\pi, E)$  be a continuous representation of  $G$  where  $E$  is a Hausdorff locally convex complete topological vector space. We write the space of  $K$ -finite vectors of  $E$  by  $E_K$ .

## 2 Poisson transform on vector bundle.

The Poisson transform is a continuous  $G$ -homomorphism from a spherical principal series representation to the space of right  $K$ -invariant functions on  $G$ . As a generalization of this, we will define a vector-valued Poisson transform and determine its image.

Let  $(\tau, V_\tau)$  be an irreducible unitary representation of  $K$  and  $\lambda$  an element of  $\mathfrak{a}_\mathbb{C}^*$ . Then we consider the induced representation  $\pi_{\tau, \lambda}$  realized as follows. The representation space is

$$\mathcal{H}_{\tau, \lambda}^\infty = \{f \in C^\infty(G, V_\tau) \mid f(gman) = \tau(m)^{-1} a^{\lambda-\rho} f(g) \text{ for } (m, a, n, g) \in M \times A \times N \times G\}$$

and  $G$  acts on this space by left translation, i.e.,  $\pi_{\tau, \lambda}(g)f(x) = L_g f(x) = f(g^{-1}x)$  for  $f \in \mathcal{H}_{\tau, \lambda}^\infty$  and  $g \in G$ . This is an admissible representation of  $G$  with infinitesimal character  $\chi_\lambda$ . Also we denote the space of  $K$ -finite vectors of  $\mathcal{H}_{\tau, \lambda}^\infty$  by  $H_{\tau, \lambda}$  which becomes a  $(\mathfrak{g}_\mathbb{C}, K)$ -module naturally.

Also we consider another induced representation. The representation space is

$$C_\tau^\infty(G/K; \chi_\lambda) = \left\{ f \in C^\infty(G, V_\tau) \mid \begin{array}{l} f(gk) = \tau(k)^{-1} f(g), \quad (k, g) \in K \times G, \\ R_X f = \chi_\lambda(X) f \text{ for } X \in Z(\mathfrak{g}) \end{array} \right\}$$

and  $G$  acts on this space by left translation. We denote the space of its  $K$ -finite vectors by  $C_\tau^\infty(G/K; \chi_\lambda)_K$ .

We define the generalized Harish-Chandra  $C$ -function as follows,

$$C(\lambda, \tau) = \int_{\overline{N}} \tau(k(\overline{n})) e^{-(\lambda+\rho)H(\overline{n})} d\overline{n}.$$

Here  $g = k(g) \exp H(g)n(g)$  for  $k(g) \in K$ ,  $H(g) \in \mathfrak{a}$  and  $n(g) \in N$ . It is known that this integral is absolutely convergent by the operator norm of  $\text{End}(V_\tau)$  in  $\{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \text{Re}\langle \lambda, \alpha \rangle > 0 \text{ for any } \alpha \in \Sigma^+\}$ . It is meromorphically continued in all  $\mathfrak{a}_\mathbb{C}^*$  (cf. [4]).

Since  $M$  is the finite abelian group,  $V_\tau$  can be decomposed as the direct sum of 1-dimensional representations of  $M$ . Therefore we can take a basis  $\{v_1, \dots, v_l\}$  of  $V_\tau$  so that there exist 1-dimensional representation  $\sigma_i$  ( $i = 1, \dots, l$ ) of  $M$  such that  $\tau(m)v_i = \sigma_i(m)v_i$  ( $i = 1, \dots, l$ ) for  $m \in M$ . Also we take the dual basis  $\{v_1^*, \dots, v_l^*\}$  of  $V_\tau^* = \text{Hom}_\mathbb{C}(V_\tau, \mathbb{C})$ , i.e., each  $v_i$  satisfies  $v_i^*(v_j) = \delta_{ij}$  for  $i, j = 1, \dots, l$ . We regard  $V_\tau^*$  as a representation space of  $M$  by the contragredient representation.

**Definition 2.1** (Poisson transform). *We define the  $G$ -homomorphism  $\mathcal{P}_{\tau, \lambda}$  from  $\mathcal{H}_{\tau, \lambda}^\infty$  to  $C_\tau^\infty(G/K; \chi_\lambda)$  by*

$$\begin{aligned} \mathcal{P}_{\tau, \lambda}: \mathcal{H}_{\tau, \lambda}^\infty &\longrightarrow C_\tau^\infty(G/K; \chi_\lambda) \\ f &\longmapsto \int_K \tau(k) f(gk) dk \end{aligned}$$

*This is called the Poisson transform.*

We see that  $\mathcal{P}_{\tau, \lambda}$  gives a bijection between the  $K$ -finite subspaces for generic  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ .

**Theorem 2.2.** *We put following assumptions.*

1.  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  is regular and dominant, i.e.,

$$2 \frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \{0, -1, -2, \dots\} \text{ for any } \beta \in \Sigma^+.$$

2. The determinant of  $C(\tau, \lambda) \in \text{End}(V_\tau)$  is nonzero.

Then  $\mathcal{P}_{\tau, \lambda}$  gives a  $(\mathfrak{g}_\mathbb{C}, K)$ -isomorphism,

$$\mathcal{P}_{\tau, \lambda}: H_{\tau, \lambda} \xrightarrow{\sim} C_\tau^\infty(G/K; \chi_\lambda)_K.$$

**Remark 2.3.** *This theorem is first proved by An Yang [5] in more general settings. However Yang put a stronger assumption*

$$2 \frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \mathbb{Z} \text{ for any } \beta \in \Sigma.$$

*This is too strong for our purpose in this paper. Therefore we need a refined theorem under the weaker condition as above.*

### 3 Strongly spherical $K$ -types and vector valued Poisson transforms of degenerate principal series representations

Our purpose of this note is to give a characterization of the vector-valued generalized Whittaker functions of degenerate principal series. To do this, we need the Poisson transforms on degenerate principal series representations. Hence we need to restrict the vector-valued Poisson transform to degenerate principal series representations and determine their images.

Take a finite subset  $\Theta \subset \Pi$  and let  $P_\Theta$  be the corresponding parabolic subgroup of  $G$ . For  $\lambda \in (\mathfrak{a}_\Theta^*)_{\mathbb{C}}$ , we define a character  $\lambda_\Theta$  of  $\mathfrak{p}_\Theta$  by

$$\lambda_\Theta: \begin{array}{ccc} \mathfrak{p}_\Theta & \longrightarrow & \mathbb{C} \\ X + H & \longmapsto & \lambda(H), \end{array}$$

where  $X \in \mathfrak{m}_\Theta + \mathfrak{n}_\Theta$  and  $H \in \mathfrak{a}_\Theta$ . We take a character  $\Lambda_\Theta$  of  $P_\Theta$  whose differentiation is  $\lambda_\Theta$ . Then we define a degenerate principal series representation of  $G$  as follows. The representation space is  $C^\infty(G/P_\Theta; \Lambda_\Theta) = \{f \in C^\infty(G) \mid f(gp) = \Lambda_\Theta(p)f(g) \text{ for } p \in P_\Theta, g \in G\}$ . The action of  $G$  on this space is defined by left translation. We denote the space of  $K$ -finite vectors of  $C^\infty(G/P_\Theta; \Lambda_\Theta)$  by  $H_{\Theta, \lambda}$ .

**Definition 3.1** (annihilator ideal). *We define a left ideal of  $U(\mathfrak{g})$  by*

$$J_\Theta(\lambda) = \sum_{X \in (\mathfrak{p}_\Theta)_{\mathbb{C}}} U(\mathfrak{g})(X - \lambda_\Theta(X))$$

*and also define a two-sided ideal*

$$I_\Theta(\lambda) = \bigcap_{g \in G} \text{Ad}(g)J_\Theta(\lambda).$$

This two-sided ideal  $I_\Theta(\lambda)$  is studied by H. Oda and T. Oshima in [3] and they give explicit generators of  $I_\Theta(\lambda)$ . This ideal is very important tool to investigate  $C^\infty(G/P_\Theta; \Lambda_\Theta)$ , because we can show that for any  $X \in I_\Theta(\lambda)$  and  $f \in C^\infty(G/P_\Theta; \Lambda_\Theta)$ , we have  $R_X f = 0$ , i.e.,  $I_\Theta(\lambda)$  is the annihilator ideal of  $C^\infty(G/P_\Theta; \Lambda_\Theta)$ . Also it is known that  $I_\Theta(\lambda)$  is the annihilator of the generalized Verma module  $U(\mathfrak{g})/J_\Theta(\lambda)$ .

We define the notion of strongly spherical  $K$ -types.

**Definition 3.2** (Strongly spherical  $K$ -type). Let  $(\tau, V_\tau)$  be a irreducible unitary representation of  $K$  such that  $\dim \text{Hom}_K(V_\tau, H_{\Theta, \lambda}) \neq 0$ . We call this representation  $\tau$  a strongly spherical  $K$ -type of  $H_{\Theta, \lambda}$  if the dimension of  $V_\tau^{\mathfrak{m}_\Theta \cap \mathfrak{k}} = \{v \in V_\tau \mid \tau(X)v = 0 \text{ for } X \in \mathfrak{m}_\Theta \cap \mathfrak{k}\}$  is equal to 1.

**Remark 3.3.** If  $\Theta = \emptyset$ , i.e.,  $P_\Theta$  is minimal parabolic subgroup, this condition says  $V_\tau$  is 1-dimensional because  $\mathfrak{m}_\Theta$  is trivial. On the other hand, if  $(K, M_\Theta \cap K)$  is a symmetric pair, it is easy to see that every irreducible unitary representation of  $K$  is strongly spherical.

For these strongly spherical  $K$ -types, we can consider vector valued Poisson transform of degenerate principal series. And we can determine its image. For an irreducible representation  $(\tau, V_\tau)$  of  $K$ , we define a space

$$C_\tau^\infty(G/K; I_\Theta(\lambda)) = \{f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k^{-1})f(g), R_X f = 0 \text{ for } g \in G, k \in K, X \in I_\Theta(\lambda)\}.$$

This is a  $G$ -representation by the left translation.

**Theorem 3.4.** We use the notations as above. For  $\lambda \in (\mathfrak{a}_\Theta^*)_{\mathbb{C}}$ , we assume that

1.  $\lambda + \rho$  is regular and dominant.

2.  $\det C(\tau, \lambda + \rho) \neq 0$ .

Let  $(\tau, V_\tau)$  be a strongly spherical  $K$ -type of  $H_{\Theta, \lambda}$ . Then the restriction of  $\mathcal{P}_{\tau, \lambda}$  to  $H_{\Theta, \lambda}$  gives a following  $(\mathfrak{g}_{\mathbb{C}}, K)$ -isomorphism,

$$\begin{aligned} \mathcal{P}_{\Theta, \lambda}: \quad H_{\Theta, \lambda} &\longrightarrow C_\tau^\infty(G/K; I_\Theta(\lambda))_K \\ \phi &\longmapsto \int_K \tau(k) \phi(gk) dk. \end{aligned}$$

Here we note that we can see  $\mathfrak{a}_\Theta^* \subset \mathfrak{a}^*$  by the Killing form  $B$ .

*Proof.* By the assumption, we have the  $(\mathfrak{g}_{\mathbb{C}}, K)$ -isomorphism

$$\begin{aligned} \mathcal{P}_{\tau, \lambda}: \quad H_{\tau, \lambda} &\longrightarrow C_\tau^\infty(G/K; \chi_\lambda)_K \\ \phi &\longmapsto \int_K \tau(k) \phi(gk) dk. \end{aligned}$$

Since  $H_{\Theta, \lambda}$  is a  $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of  $H_{\tau, \lambda}$ , we have

$$\mathcal{P}_{\tau, \lambda}(H_{\Theta, \lambda}) \subset C_\tau^\infty(G/K; I_\Theta(\lambda))_K.$$

Here we notice that since it is easy to show that  $\sum_{X \in Z(\mathfrak{g})} U(\mathfrak{g})(X - \chi_\lambda(X)) \subset I_\Theta(\lambda)$ , we have  $C_\tau^\infty(G/K; I_\Theta(\lambda)) \subset C_\tau^\infty(G/K; \chi_\lambda)$ . It remains to show that  $H_{\Theta, \lambda} \supset \mathcal{P}_{\tau, \lambda}^{-1}(C_\tau^\infty(G/K; I_\Theta(\lambda))_K)$ . To show this, we take an arbitrary element  $u \in C_\tau^\infty(G/K; I_\Theta(\lambda))$ . We can see  $\lambda \in (\mathfrak{a}_\Theta^*)_C$  as an element of  $\mathfrak{a}_C^*$ , hence we denote this by  $\lambda_\Theta \in \mathfrak{a}_C^*$ . We define a character of the Borel subalgebra of  $\mathfrak{g}_C$ ,  $\mathfrak{b} = \mathfrak{a}_C + \mathfrak{n}_C$  as follows,

$$\begin{aligned} \lambda_{\mathfrak{b}}: \quad \mathfrak{b} &\longrightarrow \mathbb{C} \\ H + X &\longmapsto \lambda(H) \end{aligned}$$

where  $H \in \mathfrak{a}_C$  and  $X \in \mathfrak{n}_C$ . We define a left ideal of  $U(\mathfrak{g})$  by  $J(\lambda_{\mathfrak{b}}) = \sum_{X \in \mathfrak{b}} U(\mathfrak{g})(X - \lambda_{\mathfrak{b}}(X))$ . Then for any  $X \in J(\lambda_{\mathfrak{b}})$  and  $f \in H_{\tau, \lambda}$  we have  $R_X f = 0$ . Hence  $\mathcal{P}_{\tau, \lambda}^{-1}u$  satisfies that  $R_X \mathcal{P}_{\tau, \lambda}^{-1}u = 0$  for any  $J_\Theta(\lambda)$  because  $J_\Theta(\lambda) = I_\Theta(\lambda) + J(\lambda_{\mathfrak{b}})$  by the result of Oda and Oshima (Theorem 3.12 in [3]). This implies that there exists a representation  $\sigma$  of  $M_\Theta$  which satisfies that  $\text{Hom}_{M_\Theta \cap K}(\sigma, \tau) \neq \{0\}$  and differentiation of  $\sigma$  is trivial. And  $\mathcal{P}_{\tau, \lambda}^{-1}u \in C^\infty\text{-Ind}_{P_\Theta}^G(\sigma \otimes e^{-\lambda} \otimes 1_{N_\Theta})$ . However since  $\dim V_\tau^{\mathfrak{m}_\Theta \cap \mathfrak{n}} = 1$ ,  $\sigma$  must be equal to  $\Lambda_\Theta|_{M_\Theta}$ .  $\square$

## 4 Maximal globalization

The vector-valued Poisson transform gives a  $(\mathfrak{g}_C, K)$ -isomorphism from the degenerate principal series  $H_{\Theta, \lambda}$  to  $C_\tau^\infty(G/K; I_\Theta(\lambda))_K$  if  $\tau$  is a strongly spherical  $K$ -type of  $H_{\Theta, \lambda}$ . Furthermore, we see that this  $(\mathfrak{g}_C, K)$ -isomorphism extends to the continuous  $G$ -isomorphism.

Let  $X$  be an admissible  $(\mathfrak{g}_C, K)$ -module with finite length. We consider the space of  $(\mathfrak{g}_C, K)$ -homomorphisms from the dual  $(\mathfrak{g}_C, K)$ -module  $X^*$  to  $C^\infty(G)$ ,  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$  where  $G$  acts on  $C^\infty(G)$  by left translation. Since  $C^\infty(G)$  has a uniformly convergent topology and  $X^*$  has a countably many basis, we can define the complete locally convex topology on  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ . On the other hand,  $G$  can also act on  $C^\infty(G)$  by right translation. This action is continuous on the topology of  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ . the space of  $K$ -finite elements of  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$  can be identified with  $(X^*)^* \cong X$  by the evaluation at the origin, i.e., for  $I \in \text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$ ,  $X^* \ni v \mapsto I(v)(e) \in \mathbb{C}$  is a linear form of  $X^*$ . Hence  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$  is a continuous  $G$  representation and its  $K$ -finite subspace is  $X$ , i.e.,  $\text{Hom}_{(\mathfrak{g}_C, K)}(X^*, C^\infty(G))$  is a globalization of  $X$ . This is called the maximal globalization [1].

Let us return to our setting. In the previous section we see that there is a

$(\mathfrak{g}_{\mathbb{C}}, K)$ -isomorphism

$$\begin{aligned} \mathcal{P}_{\Theta, \lambda}: \quad H_{\Theta, \lambda} &\longrightarrow C_{\tau}^{\infty}(G/K; I_{\Theta}(\lambda))_K \\ \phi &\longmapsto \int_K \tau(k) \phi(gk) dk. \end{aligned}$$

This  $(\mathfrak{g}_{\mathbb{C}}, K)$ -isomorphism can be extend to  $G$ -isomorphism as follows. If  $(\tau, V_{\tau})$  is a strongly spherical  $K$ -type of  $(\mathfrak{g}_{\mathbb{C}}, K)$ -module  $H_{\Theta, \lambda}$ , it is multiplicity free by definition. We fix a  $K$ -projection  $p_{\tau}: H_{\Theta, \lambda} \rightarrow V_{\tau}$ . We define a  $K$ -embedding  $\iota_{\tau}: V_{\tau}^* \hookrightarrow H_{\Theta, \lambda}^*$  as the dual map of  $p_{\tau}$ .

**Theorem 4.1.** *We assume that*

1.  $\lambda_{\Theta} + \rho$  is regular and dominant.

2.  $\det C(\lambda + \rho, \tau) \neq 0$ .

*Let  $(\tau, V_{\tau})$  be a strongly shperical  $K$ -type of  $H_{\Theta, \lambda}$ . Then we have the following topological  $G$ -isomorphism.*

$$\begin{aligned} \Phi: \quad \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(H_{\Theta, \lambda}^*, C^{\infty}(G)) &\longrightarrow C_{\tau}^{\infty}(G/K; I_{\Theta}(\lambda)) \\ I &\longmapsto \sum_{i=1}^l I(\iota_{\tau}(v_i^*))(g)v_i \end{aligned}$$

## 5 Generalized Whittaker models

Finally we give the main theorem of this note. We can give a characterization of vector-valued generalized Whittaker functions as solutions of system of differential equations which comes from  $I_{\Theta}(\lambda)$ .

Let  $U$  be a closed subgroup of  $N$  and  $(\eta, V_{\eta})$  an irreducible unitary representation of  $U$ . We consider a representation of  $G$  induced from  $\eta$ . The representation space is

$$C_{\eta}^{\infty}(U \backslash G) = \{f: G \rightarrow V_{\eta}^{\infty} \text{ smooth} \mid f(ug) = \eta(u)f(g) \text{ for all } u \in U, g \in G\}.$$

Here  $V_{\eta}^{\infty}$  stands for the space of smooth vectors of  $V_{\eta}$ . We note that  $V_{\eta}^{\infty}$  has a Hausdorff complete locally convex topology and we can define the derivation of  $f: G \rightarrow V_{\eta}^{\infty}$  by the convergence on the topology of  $V_{\eta}^{\infty}$ .

**Definition 5.1** (Generalized Whittaker model). *Let  $X$  be an admissible  $(\mathfrak{g}_{\mathbb{C}}, K)$ -module with finite length. Let  $U$  be a closed subgroup of  $N$  and  $(\eta, V_{\eta})$  an*

irreducible unitary representation of  $U$ . We consider the space of  $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphisms from  $X$  to  $C_{\eta}^{\infty}(U \backslash G)$ ,

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X, C_{\eta}^{\infty}(U \backslash G)).$$

If  $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X, C_{\eta}^{\infty}(U \backslash G)) \neq \{0\}$ , we say  $X$  has generalized Whittaker models.

We consider genelarized Whittaker models of  $H_{\Theta, \lambda}$ . Let  $(\tau, V_{\tau})$  be a strongly spherical  $K$ -type of  $H_{\Theta, \lambda}$ . Take a irreducible unitary representation  $(\eta, V_{\eta})$  of  $N$ . For the algebraic tensor product  $V_{\eta}^{\infty} \otimes V_{\tau}$ , we can define a natural topology comes from  $V_{\eta}^{\infty}$  because  $V_{\tau}$  is finite dimensional. Hence we can consider the following space of smooth functions from  $G$  to  $V_{\eta}^{\infty} \otimes V_{\tau}$ ,

$$C_{\eta, \tau}^{\infty}(U \backslash G / K) = \{f: G \rightarrow V_{\eta}^{\infty} \otimes V_{\tau} \text{ smooth} \mid f(ugk) = \eta(u) \otimes \tau(k^{-1})f(g) \text{ for } u \in U, g \in G, k \in K, \}.$$

Also we define

$$C_{\eta, \tau}(U \backslash G / K; I_{\Theta}(\lambda)) = \{f \in C_{\eta, \tau}^{\infty}(U \backslash G / K) \mid R_X f = 0 \text{ for } X \in I_{\Theta}(\lambda)\}.$$

As a colloraly of Theorem 4.1, we have the follwing characterization of genelarized Whittaker models.

**Theorem 5.2.** *We use the same notations as Theorem 4.1. We assume that*

1.  $\lambda_{\Theta} + \rho$  is regular and dominant.

2.  $\det C(\lambda + \rho, \tau) \neq 0$ .

*Let  $(\tau, V_{\tau})$  be a strongly shperical  $K$ -type of  $H_{\Theta, \lambda}$ . Then we have the following linear isomorphism.*

$$\begin{aligned} \Phi: \quad \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(H_{\Theta, \lambda}^*, C_{\eta}^{\infty}(U \backslash G)) &\longrightarrow C_{\eta, \tau}^{\infty}(U \backslash G / K; I_{\Theta}(\lambda)) \\ I &\longmapsto \sum_{i=1}^l I(\iota_{\tau}(v_i^*))(g)v_i \end{aligned}$$

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