

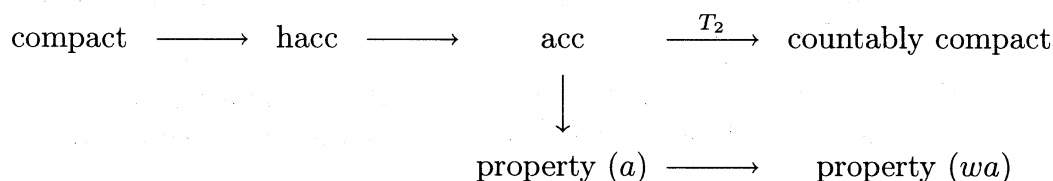
ABSOLUTELY COUNTABLY COMPACT SPACES AND RELATED SPACES

YAN-KUI SONG (宋延奎)

Faculty of Science, Shizuoka University

§1. INTRODUCTION

By a space, we mean a topological space. Matveev [7] defined a space X to be *absolutely countably compact* ($= acc$) if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subset X$, there exists a finite subset $F \subset D$ such that $\text{St}(F, \mathcal{U}) = X$ and defined a space X to be *hereditarily absolutely countably compact* ($= hacc$) if all closed subspaces of X are acc . In [8], he also defined a space X to have the *property (a)* (resp. *property (wa)*) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X , there exists a discrete closed subspace (resp. discrete subspace) $F \subset D$ such that $\text{St}(F, \mathcal{U}) = X$. By the definitions, all compact spaces are $hacc$, all $hacc$ spaces are acc , all acc spaces have the property (a) and all spaces having the property (a) have the property (wa). Moreover, it is known [7] that all acc spaces are countably compact (cf. also [4]). Thus, we have the following diagram:



In the above diagram, the converse of each arrow does not hold, in general (cf. [7], [8]). For an infinite cardinality κ , a space X is called *initially κ -compact* if every open cover of X with cardinality $\leq \kappa$ has a finite subcover. The main theorems of this paper are Theorems 1, 2 and 3 below. We prove only Theorem 2 here and leave the details of the proofs of Theorems 1 and 3 to elsewhere.

Theorem 1. *Let κ be an infinite cardinal. Let X be an initially κ -compact T_3 -space, Y a compact T_2 -space with $t(Y) \leq \kappa$ and A a closed subspace of $X \times Y$. Assume that $A \cap (X \times \{y\})$ is acc for each $y \in Y$ and the projection $\pi_Y : X \times Y \rightarrow Y$ is a closed map. Then, the subspace A is acc .*

Vaughan [12] proved that

- (i) if X is an acc T_3 -space and Y is a sequential, compact T_2 -space, then $X \times Y$ is acc , and

1991 *Mathematics Subject Classification.* 54D20, 54B10, 54D55.

Key words and phrases. compact, countably compact, absolutely countably compact, hereditarily absolutely countably compact, topological group.

- (ii) if X is an ω -bounded, acc T_3 -space and Y is a compact T_2 -space with $t(Y) \leq \omega$, then $X \times Y$ is acc.

Further, Bonanzinga [1] proved that

- (iii) if X is an hacc T_3 -space and Y is a sequential, compact T_2 -space, then $X \times Y$ is hacc, and
 (iv) if X is an ω -bounded, hacc T_3 -space and Y is a compact T_2 -space with $t(Y) \leq \omega$, then $X \times Y$ is hacc.

In Section 2, we show that Vaughan's theorems (i), (ii) and Bonanzinga's theorems (iii), (iv) are deduced from Theorem 1. Matveev [8] asked if there exists a Tychonoff space which has not the property (wa). In Section 3, we answer the question by proving the following theorem:

Theorem 2. *There exists a 0-dimensional, first countable, Tychonoff space without the property (wa).*

Matveev [9] also asked if there exists a separable, countably compact, topological group which is not acc. Vaughan [11] asked the same question and showed that the answer is positive if there is a separable, sequentially compact T_2 -group which is not compact. From this point of view, he also asked if there is a separable, sequentially compact T_2 -group which is not compact. The final theorem below, which is a joint work with Ohta, answers the questions. Let $\mathfrak{s}$ denote the splitting number, i.e., $\mathfrak{s} = \min\{\kappa : \text{the power } 2^\kappa \text{ is not sequentially compact}\}$ (cf. [2 Theorem 6.1]).

Theorem 3. (Ohta-Song). *There exists a separable, countably compact T_2 -group which is not acc. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact T_2 -group which is not acc.*

It was shown in the proof [2 Theorem 5.4] that the assumption that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$ is consistent with ZFC. Theorem 3 will be proved in Section 4.

Remark 1. Theorem 2 was proved independently by Just, Matveev and Szeptycki [5]. Matveev kindly informed Ohta that a similar theorem to Theorem 3 above was also proved independently by W. Pack in his Ph. D thesis at the University of Oxford (1997).

For a set A , $|A|$ denotes the cardinality of A . As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Other terms and symbols will be used as in [3].

§2. THEOREM 1 AND ITS COROLLARIES

Throughout this section, κ stands for an infinite cardinal. For a set A , let $[A]^{\leq \kappa} = \{B : B \subseteq A, |B| \leq \kappa\}$ and $[A]^{< \kappa} = \{B : B \subseteq A, |B| < \kappa\}$. For a subset A of a space X , we define the κ -closure of A in X by $\kappa\text{-cl}_X A = \bigcup\{\text{cl}_X B : B \in [A]^{\leq \kappa}\}$ and say that A is κ -closed in X if $A = \kappa\text{-cl}_X A$. By the definition, $\kappa\text{-cl}_X A$ is always κ -closed in X .

Lemma 4. *Let X be a space. Then, $t(X) \leq \kappa$ if and only if every κ -closed set in X is closed.*

Lemma 5. *Let X and Y be spaces such that $\pi_Y : X \times Y \rightarrow Y$ is closed map. Then, $\pi_Y(A)$ is κ -closed in Y for each κ -closed set A in $X \times Y$.*

Theorem 1 will be proved by using Lemmas 4 and 5. We now proceed to corollaries. The first one follows immediately from Theorem 1:

Corollary 6. *Let X be an initially κ -compact, acc (resp. hacc) T_3 -space and Y a compact T_2 -space with $t(Y) \leq \kappa$. Assume that $\pi_Y : X \times Y \rightarrow Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

Since an acc space is countably compact (i.e., initially ω -compact), the following corollary is a special case of the preceding corollary.

Corollary 7. *Let X be an acc (resp. hacc) T_3 -space and Y a compact T_2 -space with $t(Y) \leq \omega$. Assume $\pi_Y : X \times Y \rightarrow Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

It is known (cf. [3, Theorem 3.10.7]) that if X is countably compact and Y is sequential, then $\pi_Y : X \times Y \rightarrow Y$ is closed. Hence, we have the following corollary, which is Vaughan's theorem (i) and Bonanzinga's theorem (iii) stated in the introduction:

Corollary 8. (Vaughan [12] and Bonanzinga [1]) *Let X be an acc (resp. hacc) T_3 -space and Y a sequential, compact T_2 -space. Then, $X \times Y$ is acc (resp. hacc).*

Recall that a space X is κ -bounded if $\text{cl}_X A$ is compact for each $A \in [X]^{\leq \kappa}$. It is known (cf. [10]) that all κ -bounded spaces are initially κ -compact. Further, Kombarov [6] proved that if X is κ -bounded and $t(Y) \leq \kappa$, then $\pi_Y : X \times Y \rightarrow Y$ is closed. Hence, we have the following corollary, which generalizes Vaughan's theorem (ii) and Bonanzinga's theorem (iv) stated in the introduction.

Corollary 9. *Let X be a κ -bounded, acc (resp. hacc) T_3 -space and Y a compact T_2 -space with $t(Y) \leq \kappa$, then $X \times Y$ is acc (resp. hacc).*

§ 3. PROOF OF THEOREM 2

In this section, we give a proof of Theorem 2. We omit a simple proof of the following lemma.

Lemma 10. *Let $\mathbb{R}$ be the space of real numbers with the usual topology and A a discrete subspace of $\mathbb{R}$. Then, $|A| \leq \omega$ and $\text{cl}_{\mathbb{R}} A$ is nowhere dense in $\mathbb{R}$.*

Proof of Theorem 2. Let $A = \bigcup_{n \in \mathbb{N}} A_n$, where $A_n = \mathbb{Q} \times \{1/n\}$ and let $\mathcal{A} = \{S : S \text{ is a discrete subspace of } A\}$. Then, we have:

Claim 1. $|\mathcal{A}| = \mathfrak{c}$.

Proof. Since $|A| \leq \omega$, $|\mathcal{A}| \leq \mathfrak{c}$. Let $S = \{\langle n, 1 \rangle : n \in \mathbb{N}\} \subseteq A$. Since every subset of S is discrete, $\{F : F \subseteq S\} \subseteq \mathcal{A}$. Hence, $|\mathcal{A}| \geq |\{F : F \subseteq S\}| = \mathfrak{c}$. $\square$

Since $|\mathcal{A}| = \mathfrak{c}$, we can enumerate the family $\mathcal{A}$ as $\{S_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$ and each $n \in \mathbb{N}$, put $S_{\alpha,n} = \{q \in \mathbb{Q} : \langle q, 1/n \rangle \in S_\alpha\}$.

Claim 2. For each $\alpha < \mathfrak{c}$, $|\mathbb{R} \setminus \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}| = \mathfrak{c}$.

Proof. For each $\alpha < \mathfrak{c}$, let $X_\alpha = \mathbb{R} \setminus \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}$. Since X_α is a G_δ -set in $\mathbb{R}$, X_α is a complete metric space. To show that X_α is dense in itself, suppose that X_α has an isolated point x . Then, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap X_\alpha = \{x\}$. Let $I = (x, x + \varepsilon)$. Then, $I \subset \mathbb{R} \setminus X_\alpha \subset \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}$. Moreover, since I is open in $\mathbb{R}$, $\text{cl}_{\mathbb{R}} S_{\alpha,n} \cap I \subseteq \text{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$. Hence,

$$(6) \quad I = \left(\bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n} \right) \cap I = \bigcup_{n \in N} (\text{cl}_{\mathbb{R}} S_{\alpha,n} \cap I) \subseteq \bigcup_{n \in N} \text{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I).$$

By Lemma 10, each $\text{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$ is nowhere dense in $\mathbb{R}$. Thus, (6) contradicts the Baire Category Theorem. Hence, X_α is dense in itself. It is known ([3, 4.5.5]) that every dense in itself complete metric space includes a Cantor set. Hence, $|X_\alpha| = \mathfrak{c}$. $\square$

Claim 3. There exists a sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ satisfying the following conditions:

- (1) For each $\alpha < \mathfrak{c}$, $p_\alpha \in \mathbb{P}$.
- (2) For any $\alpha, \beta < \mathfrak{c}$, if $\alpha \neq \beta$, then $p_\alpha \neq p_\beta$.
- (3) For each $\alpha < \mathfrak{c}$, $p_\alpha \notin \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}$.

Proof. By transfinite induction, we define a sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ as follows: There is $p_0 \in \mathbb{P}$ such that $p_0 \notin \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{0,n}$ by Claim 2. Let $0 < \alpha < \mathfrak{c}$ and assume that p_β has been defined for all $\beta < \alpha$. By Claim 2, $|\mathbb{R} \setminus \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}| = \mathfrak{c}$. Hence, we can choose a point $p_\alpha \in (\mathbb{P} \setminus \bigcup_{n \in N} \text{cl}_{\mathbb{R}} S_{\alpha,n}) \setminus \{p_\beta : \beta < \alpha\}$. Now, we have completed the induction. Then, the sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ satisfies the conditions (1) (2) and (3). $\square$

Claim 4. For each $\alpha < \mathfrak{c}$, there exists a sequence $\{\varepsilon_{\alpha,n} : n \in N\}$ in $\mathbb{Q}$ satisfying the following conditions:

- (1) For each $n \in N$, $(p_\alpha - \varepsilon_{\alpha,n}, p_\alpha + \varepsilon_{\alpha,n}) \cap S_{\alpha,n} = \emptyset$.
- (2) For each $n \in N$, $\varepsilon_{\alpha,n} \geq \varepsilon_{\alpha,n+1}$.
- (3) $\lim_{n \rightarrow \infty} \varepsilon_{\alpha,n} = 0$.

Proof. Let $\alpha < \mathfrak{c}$. For $n = 1$, since $p_\alpha \notin \text{cl}_{\mathbb{R}} S_{\alpha,1}$, there exists a rational $\varepsilon_{\alpha,1} > 0$ such that $(p_\alpha - \varepsilon_{\alpha,1}, p_\alpha + \varepsilon_{\alpha,1}) \cap S_{\alpha,1} = \emptyset$. Let $n > 1$ and assume that we have defined $\{\varepsilon_{\alpha,m} : m < n\}$ satisfying that $\varepsilon_{\alpha,1} > \varepsilon_{\alpha,2} > \cdots > \varepsilon_{\alpha,n-1}$. Since $p_\alpha \notin \text{cl}_{\mathbb{R}} S_{\alpha,n}$, there exists a rational $\varepsilon'_{\alpha,n}$ such that $(p_\alpha - \varepsilon'_{\alpha,n}, p_\alpha + \varepsilon'_{\alpha,n}) \cap S_{\alpha,n} = \emptyset$. Put $\varepsilon_{\alpha,n} = n^{-1} \min\{\varepsilon_{\alpha,n-1}, \varepsilon'_{\alpha,n}\}$. Now, we have completed the induction. Then, the sequence $\{\varepsilon_{\alpha,n} : n \in N\}$ satisfies (1) (2) and (3). $\square$

Define $X = A \cup B$, where $B = \{\langle p_\alpha, 0 \rangle : \alpha < \mathfrak{c}\}$. Topologize X as follows: A basic neighborhood of a point in A is a neighborhood induced from the usual topology on the plane. For each $\alpha < \mathfrak{c}$, a basic neighborhood base $\{U_n \langle p_\alpha, 0 \rangle : n \in \omega\}$ of $\langle p_\alpha, 0 \rangle \in B$ is defined by

$$U_n \langle p_\alpha, 0 \rangle = \{\langle p_\alpha, 0 \rangle\} \cup \left(\bigcup_{i \geq n} \{((p_\alpha - \varepsilon_{\alpha,i}, p_\alpha + \varepsilon_{\alpha,i}) \cap \mathbb{Q}) \times \{1/i\}\} \right).$$

for each $n \in N$. Then, X is a first countable T_2 -space. For each $\alpha < \mathfrak{c}$ and each $n \in N$, $U_n \langle p_\alpha, 0 \rangle$ is open and closed in X , because $p_\alpha \pm \varepsilon_{\alpha,i} \notin \mathbb{Q}$ for each $i \in \omega$. It follows that X is 0-dimensional, and hence, a Tychonoff space.

ABSOLUTELY COUNTABLE COMPACTNESS

Claim 5. *The space X has not the property (wa).*

Proof. Let $\mathcal{U} = \{A\} \cup \{U_1\langle p_\alpha, 0 \rangle : \alpha < \mathfrak{c}\}$. Then, $\mathcal{U}$ is an open cover of X and A is a dense subspace of X . For each discrete subset F of A , there exists $\alpha < \mathfrak{c}$ such that $F = S_\alpha$. Since $U_1\langle p_\alpha, 0 \rangle \cap S_\alpha = \emptyset$, $\langle p_\alpha, 0 \rangle \notin \text{St}(F, \mathcal{U})$. This shows that X does not have the property (wa). $\square$

§4. PROOF OF THEOREM 3

We omit the proofs of the following lemmas and only show how Theorem 3 can be deduced from the lemmas.

Lemma 11. *Let X be a space and Y a space having at least one pair of disjoint nonempty closed subsets. Assume that $X \times Y^\kappa$ is acc for an infinite cardinal κ . Then, X is initially κ -compact.*

We consider $2 = \{0, 1\}$ the discrete group of integers modulo 2. Then, 2^κ is a topological group under pairwise addition. The following lemma seems to be well known (see [10, 3.5] for the first statement), but we include it here for the sake of completeness.

Lemma 12. *There exists a separable, countably compact, non-compact subgroup G_1 of $2^\mathfrak{c}$. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact, non-compact subgroup G_2 of 2^{ω_1} .*

Proof of Theorem 3. Let G_1 be the group in Lemma 12. Then, $G_1 \times 2^\mathfrak{c}$ is a separable, countably compact T_2 -group. Since G_1 is not compact and $w(G_1) \leq \mathfrak{c}$, G_1 is not initially $\mathfrak{c}$ -compact. Hence, it follows from Lemma 11 that $G_1 \times 2^\mathfrak{c}$ is not acc. Next, assume that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, and let G_2 be the group in Lemma 12. Since $\omega_1 < \mathfrak{s}$, 2^{ω_1} is sequentially compact. Hence, $G_2 \times 2^{\omega_1}$ is a separable, sequentially compact T_2 -group which is not compact. Since $w(G_2) = \omega_1$, $G_2 \times 2^\omega$ is not acc by Lemma 11. $\square$

YAN-KUI SONG

REFERENCES

1. M. Bonanzinga, *On the product of a compact space with an hereditarily absolutely countably compact space*, Comment. Math. Univ. Carolinae **38** (1997), 557-562.
2. E. K. van Douwen, *The integer and Topology*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111-167.
3. R. Engelking, *General Topology*, Revised and completed edition, Heldermann, Berlin, 1989.
4. W. M. Fleishman, *A new extension of countable compactness*, Fund. Math **67** (1970), 1-9.
5. W. Just, M. V. Matveev and P. J. Szeptycki, *Some results on property (a)*, preprint.
6. A. P. Kombarov, *On the product of normal spaces*, Sovite. Math. Dokl. **13** (4) (1972), 1068-1071.
7. M.V. Matveev, *Absolutely countably compact spaces*, Topology and its Appl. **58** (1994), 81-92.
8. ———, *Some questions on property (a), Q and A in General Topology* **15** (1997), 103-111.
9. ———, *A countably compact topological group which is not absolutely countably compact*, Q and A in General Topology **11** (1993), 173-176.
10. R. M. Stephenson Jr, *Initially κ -Compact and Related Space*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 603-632.
11. J. E. Vaughan, *A countably compact, separable space which is not absolutely countably compact*, Comment. Math. Univ. Carolinae **34** (1) (1995), 197-200.
12. ———, *On the product of a compact space with an absolutely countably compact space*, Annals of New York Acad. Soc. **788** (1996), 203-208.
13. ———, *Countably Compact and Sequentially Compact Spaces*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 569-602.

FACULTY OF SCIENCE, SHIZUOKA UNIVERSITY, OHYA, SHIZUOKA, 422 JAPAN