

## On Strict Ergodicity

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### Introduction

Having split the category of all ergodic normalized dynamical systems  $(\Omega, \mathcal{L}, m, T)$  in the measure-theoretical sense into isomorphy classes, one would next try to establish a system of nice representatives in these classes. The definition of "nice" is, of course, partly a matter of taste, but a rather natural approach is offered by topological dynamics. In that theory one considers systems  $(\Omega, T)$  where  $\Omega$  is a compact metric space and  $T$  a homeomorphism of  $\Omega$  onto itself. The natural analogon of ergodicity in topological dynamics is minimality, which means that there is no non-empty proper subset of  $\Omega$  which is closed and  $T$ -invariant. It is well known (see Oxtoby [17], Keane [14], Jacobs-Keane [11]) that minimal invariant sets, carrying always at least one  $T$ -invariant normalized measure by the Markoff-Kakutani fixed point theorem, may quite well carry several ones. If one, however, imposes the condition that there be exactly one  $T$ -invariant normalized measure  $m$  living on the minimal compact metric  $\Omega$ , one arrives at the definition of a strictly ergodic  $(\Omega, T)$ , and these give rise to well-defined measure-theoretical dynamical systems  $(\Omega, \mathcal{L}, m, T)$  which are, in

addition, ergodic, and thus candidates for the task of being "nice" representatives in the isomorphy classes mentioned above. The question "Is every ergodic  $(\Omega, \mathcal{L}, m, T)$  measure-theoretically isomorphic to some strictly ergodic system?" or, more suggestively "Can every ergodic  $(m, T)$  live in a strictly ergodic  $(\Omega, T)$ ?" is at present practically definitely answered, and it is the purpose of this article to give a survey of the answers obtained so far. The time-continuous analogon of the problem and its solutions will be included here. I am, however, not to go into a description of the combinatorial devices which have been invented in order to obtain even mechanizable constructions of strictly ergodic representatives of various kinds in shift space (see Kakutani [13], Hahn-Katznelson [7], Keane [14], Jacobs-Keane [11], Grillenberger [5], [6], compare also Jacobs [10]).

#### §1. Strictly ergodic generators.

It was R. Jewett [12] who proved first the following Theorem 1.1. Let  $(\Omega, \mathcal{L}, m, T)$  be any weakly mixing dynamical system such that  $(\Omega, \mathcal{L}, m)$  is a Lebesgue measure space with  $m(\Omega)=1$ . Then there exists an isomorphic dynamical system living in a strictly ergodic  $(\Omega', T')$  where  $\Omega'$  is a totally disconnected compact metric space.

The following improvement is due to W. Krieger [16] (see also Denker [1]).

Theorem 1.2. Let  $(\Omega, \mathcal{L}, m, T)$  be any ergodic dynamical

system such that  $(\Omega, \mathcal{L}, m)$  is a Lebesgue measure space with  $m(\Omega)=1$ .

- 1) If the entropy  $h(m, T)$  is finite, then there is a strictly ergodic invariant subset of the shift space with  $[2^{h(m, T)}]+1$  symbols, which carries a dynamical system isomorphic to the given one.
- 2) If  $h(m, T)=\infty$ , then there is a strictly ergodic compact minimal invariant subset of the shift space with countably many symbols, which carries a dynamical system isomorphic to the given one.

This theorem may equivalently be stated as a theorem on the existence of "strictly ergodic generators" in the original  $(\Omega, \mathcal{L}, m, T)$  together with a bound of the length of the generator in the case of finite entropy.

The following theorem of Denker [2] more or less brings the theory to a definite conclusion.

Theorem 1.3. Let  $(\Omega, \mathcal{L}, m, T)$  be as in theorem 1.2. Then for  $h(m, T) < \infty$  the finite strictly ergodic generators are dense (with respect to the entropy metric) in the set  $\{\alpha \mid \alpha \text{ a finite measurable partition of } \Omega, h(\alpha, m, T) = h(m, T)\}$  of all partitions with maximal mean entropy (this set is closed).

## §2. The continuous case.

In the time-continuous case one considers a one-parameter group  $(T_t)_{t \text{ real}}$  of automorphisms of a given Lebesgue space  $(\Omega, \mathcal{L}, m, T)$ . Under mild regularity assumptions this system  $(\Omega, \mathcal{L}, m, (T_t)_{t \text{ real}})$  can always be viewed as a flow under a function, and this representation is one of the most effective tools in squeezing it isomorphically into a strictly ergodic system  $(\Omega', (T'_t)_{t \text{ real}})$ .

In the time-discrete case we chose (two-sided) shift space as an appropriate topological system in which subsequently a strictly ergodic system was to be established. One of the most important question in the time-continuous case is what is to be the most natural time-continuous analogon of shift space. Physical considerations lead to the choice of

$$\text{Lip}_1 = \{\omega | \omega: \mathbb{R} \rightarrow \langle 0, 1 \rangle, |\omega(s) - \omega(t)| \leq |s - t|\}$$

of all unit-interval-valued functions on the line which have Lipschitz constant 1. It is obviously a compact metric space with the compact-uniform topology, and the shifts  $T_t$  ( $t$  real) defined by

$$(T_t \omega)(s) = \omega(s + t) \quad (s \text{ real})$$

form a nice one-parameter group of homeomorphisms of  $\text{Lip}_1$ . The points of  $\text{Lip}_1$  may be viewed as the possible outcomes

of a time-continuous experiment watched by instruments which have a certain inertia.

A time-continuous generator theorem by Krengel [15] (and likewise another generator theorem by Eberlein [4]) allow to embed every reasonable flow into  $Lip_1$  with its shifts. The task of making these embeddings strictly ergodic requires a formidable arsenal of techniques and has been settled in the form of the following theorem by Denker-Eberlein [3] which strongly improves my result [9].

Theorem 2.1. Let  $(\Omega, \mathcal{L}, m, (T_t)_{t \text{ real}})$  be a time-continuous ergodic dynamical system representable as a flow under a function, such that  $(\Omega, \mathcal{L}, m)$  is a Lebesgue space with  $m(\Omega)=1$ . Then there is an isomorphic strictly ergodic subflow of  $Lip_1$ , with its shifts as transformations.

It seems to be difficult to even formulate an analogon of the discrete-time density theorem 1.3, for the time-continuous case. Nothing is known about time-continuous analoga of the combinatorial constructions in [13], [14], [11], [7], [5], [6].

## Bibliography

- [1] Denker, M., On strict ergodicity, Math. Zeitschr., to appear.
- [2] Denker, M., Einige Bemerkungen zu Erzeugersätzen, Z. f. Wahrsch.-Th. und verw. Geb., to appear.
- [3] Denker, M. and E. Eberlein, Ergodic flows are strictly ergodic, Adv. Math., to appear.
- [4] Eberlein, E., Einbettung von Strömungen in Funktionenräume durch Erzeuger vom endlichen Typ, Z. f. Wahrsch., to appear.
- [5] Grillenberger, Chr., Construction of strictly ergodic systems I, Z. f. Wahrsch. 25, 323-334(1973).
- [6] Grillenberger, Chr., Construction of strictly ergodic systems II, Z. f. Wahrsch. 25, 335-342(1973).
- [7] Hahn, F. and Y. Katznelson, On the entropy of uniquely ergodic transformations, Trans. AMS 126, 335-360(1967).
- [8] Hansel, G. and J. P. Raoult; Ergodicité, uniformité et unique ergodicité, to appear.
- [9] Jacobs, K., Lipschitz functions and the prevalence of strict ergodicity for continuous-time flows, Lect. Not. Math. 160, Springer-Verlag 1970.
- [10] Jacobs, K., Systèmes dynamiques Riemanniens, Czech. Math. J. 20, 628-631(1970).
- [11] Jacobs, K. and M. Keane, 0-1-sequences of Toeplitz type, Z. f. Wahrsch. 13, 123-131(1969).
- [12] Jewett, R., The prevalence of Uniquely ergodic systems, J. Math. Mech. 19(1970), 717-729.
- [13] Kakutani, S., Ergodic theory of shift transformations, Proc. V. Berkeley Symp. Prob. Stat. II, 405-414 (1967).

- [14] Keane, M., Generalized Morse sequences, Z. f. Wahrsch.  
10, 335-353(1968).
- [15] Krengel, V., Recent results on generators in ergodic  
theory, to appear.
- [16] Krieger, W., On unique ergodicity, Proc. VI. Berkeley  
Symp. Math. Stat. Prob., June/July 1970, 327-345.
- [17] Oxtoby, J. C., Ergodic sets, Bull. AMS 58(1952), 116-  
136.