

Orlicz-Morrey spaces and some integral operators

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1. INTRODUCTION

For the Hardy-Littlewood maximal operator M , Calderón-Zygmund operator T and fractional integral operator I_α , $0 < \alpha < n$, it is well known that

$$M : L^1(\mathbb{R}^n) \rightarrow L^1_{\text{weak}}(\mathbb{R}^n), \quad M : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad 1 < p \leq \infty,$$

$$T : L^1(\mathbb{R}^n) \rightarrow L^1_{\text{weak}}(\mathbb{R}^n), \quad T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad 1 < p < \infty,$$

$$I_\alpha : L^1(\mathbb{R}^n) \rightarrow L^{n/(n-\alpha)}_{\text{weak}}(\mathbb{R}^n), \quad I_\alpha : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n), \quad \begin{cases} 1 < p < q < \infty, \\ -n/p + \alpha = -n/q, \end{cases}$$

These boundedness extended to several function spaces which are generalizations of L^p -spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, etc.

For boundedness of M on Orlicz spaces $L^\Phi(\mathbb{R}^n)$, Kita [10] (1997) proved a necessary and sufficient condition on Φ and Ψ for

$$M : L^\Phi(\mathbb{R}^n) \rightarrow L^\Psi(\mathbb{R}^n).$$

Cianchi [3] (1999) proved necessary and sufficient conditions on Φ and Ψ for

$$M, T, I_\alpha : L^\Phi(\mathbb{R}^n) \rightarrow L^\Psi_{\text{weak}}(\mathbb{R}^n),$$

$$M, T, I_\alpha : L^\Phi(\mathbb{R}^n) \rightarrow L^\Psi(\mathbb{R}^n).$$

For boundedness of I_α on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, see Peetre (Spanne) [18] (1969) Adams [1], 1975 Chiarenza and Frasca [2] (1987), Nakai (1995). Chiarenza and Frasca showed

$$M, T : L^{p,\lambda}(\mathbb{R}^n) \rightarrow L^{p,\lambda}(\mathbb{R}^n),$$

$$I_\alpha : L^{p,\lambda}(\mathbb{R}^n) \rightarrow L^{q,\lambda}(\mathbb{R}^n).$$

The author studied boundedness of generalized fractional integral operators on Orlicz-Morrey spaces in [17]. Orlicz-Morrey spaces are useful to estimate generalized fractional integral operators. In this paper we investigate boundedness and

weak boundedness of the Hardy-Littlewood maximal operator, singular integral operators and generalized fractional integral operators. We refine on the results in [17].

2. DEFINITIONS

A function $\theta : (0, +\infty) \rightarrow (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \leq C\theta(s) \quad (\theta(r) \geq C\theta(s)) \quad \text{for } r \leq s.$$

A function $\theta : (0, +\infty) \rightarrow (0, +\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{for } \frac{1}{2} \leq \frac{r}{s} \leq 2.$$

For functions $\theta, \kappa : (0, +\infty) \rightarrow (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ ($\theta(r) \approx \kappa(r)$) if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \leq \kappa(r) \leq C\theta(r) \quad (\theta(C^{-1}r) \leq \kappa(r) \leq \theta(Cr)) \quad \text{for } r > 0.$$

Let $\mathcal{Y}$ be the set of all convex functions $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\Phi(r) > 0$ for $r > 0$. If $\Phi \in \mathcal{Y}$, then Φ is increasing, absolutely continuous and bijective.

For a measurable set $\Omega \subset \mathbb{R}^n$, we denote the characteristic function of Ω by χ_Ω and the Lebesgue measure of Ω by $|\Omega|$. For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f, t)$.

Definition 2.1 (Orlicz space). For $\Phi \in \mathcal{Y}$ let

$$\begin{aligned} L^\Phi(\mathbb{R}^n) &= \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^\Phi} < +\infty\}, \\ \|f\|_{L^\Phi} &= \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}, \\ L^\Phi_{\text{weak}}(\mathbb{R}^n) &= \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^\Phi_{\text{weak}}} < +\infty\}, \\ \|f\|_{L^\Phi_{\text{weak}}} &= \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right) \leq 1 \right\}. \end{aligned}$$

If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^\Phi_{\text{weak}}(\mathbb{R}^n) = L^p_{\text{weak}}(\mathbb{R}^n)$.

If $\Phi(r) \leq \Psi(Cr)$, then $L^\Phi(\mathbb{R}^n) \supset L^\Psi(\mathbb{R}^n)$ and $\|f\|_{L^\Phi} \leq C\|f\|_{L^\Psi}$. If $\Phi \approx \Psi$, then $L^\Phi(\mathbb{R}^n) = L^\Psi(\mathbb{R}^n)$ and $\|f\|_{L^\Phi} \sim \|f\|_{L^\Psi}$.

Let $B(a, r)$ be the ball $\{x \in \mathbb{R}^n : |x - a| < r\}$ with center a and of radius $r > 0$.

Definition 2.2 (Morrey space). For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, let

$$\begin{aligned} L^{p,\lambda}(\mathbb{R}^n) &= \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < +\infty\}, \\ \|f\|_{L^{p,\lambda}} &= \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(x)|^p dx \right)^{1/p}, \\ L^{p,\lambda}_{\text{weak}}(\mathbb{R}^n) &= \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}_{\text{weak}}} < +\infty \right\}, \\ \|f\|_{L^{p,\lambda}_{\text{weak}}} &= \sup_{B=B(a,r)} \sup_{t>0} \frac{t^p m(B, f, t)}{r^\lambda}. \end{aligned}$$

If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,\lambda}_{\text{weak}}(\mathbb{R}^n) = L^p_{\text{weak}}(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Let $\mathcal{G}$ be the set of all functions $\phi : (0, +\infty) \rightarrow (0, +\infty)$ such that ϕ is almost decreasing and $\phi(r)r^n$ is almost increasing. If $\phi \in \mathcal{G}$ then ϕ satisfies doubling condition.

Definition 2.3. For $1 \leq p < \infty$ and a function $\phi \in \mathcal{G}$,

$$\begin{aligned} L^{(p,\phi)}(\mathbb{R}^n) &= \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{(p,\phi)}} < +\infty\}, \\ \|f\|_{L^{(p,\phi)}} &= \sup_{B=B(a,r)} \left(\frac{1}{|B|\phi(r)} \int_B |f(x)|^p dx \right)^{1/p}, \\ L^{(p,\phi)}_{\text{weak}}(\mathbb{R}^n) &= \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{(p,\phi)}_{\text{weak}}} < +\infty \right\}, \\ \|f\|_{L^{(p,\phi)}_{\text{weak}}} &= \sup_{B=B(a,r)} \sup_{t>0} \frac{t^p m(B, f, t)}{|B|\phi(r)}. \end{aligned}$$

If $\phi(r) = r^{-n+\lambda}$, $0 \leq \lambda \leq n$, then $L^{(p,\phi)}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ and $L^{(p,\phi)}_{\text{weak}}(\mathbb{R}^n) = L^{p,\lambda}_{\text{weak}}(\mathbb{R}^n)$.

If $p \leq q$, then $L^{(p,\phi)}(\mathbb{R}^n) \supset L^{(q,\phi^{q/p})}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \leq \|f\|_{L^{(q,\phi^{q/p})}}$. If $\phi(r) \leq C\psi(r)$, then $L^{(p,\phi)}(\mathbb{R}^n) \subset L^{(p,\psi)}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \geq C^{-1/p} \|f\|_{L^{(p,\psi)}}$. If $\phi \sim \psi$, then $L^{(p,\phi)}(\mathbb{R}^n) = L^{(p,\psi)}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \sim \|f\|_{L^{(p,\psi)}}$.

For $\Phi \in \mathcal{Y}$, a function $\phi \in \mathcal{G}$ and a ball $B = B(a, r)$, let

$$\begin{aligned} \|f\|_{\Phi,\phi,B} &= \inf \left\{ \lambda > 0 : \frac{1}{|B|\phi(r)} \int_B \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}, \\ \|f\|_{\Phi,\phi,B,\text{weak}} &= \inf \left\{ \lambda > 0 : \sup_{t>0} \frac{\Phi(t) m(B, f/\lambda, t)}{|B|\phi(r)} \leq 1 \right\}. \end{aligned}$$

We note that

$$\sup_{t>0} \Phi(t)m(\Omega, f, t) = \sup_{t>0} t m(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} t m(\Omega, \Phi(|f|), t).$$

Definition 2.4 (Orlicz-Morrey space). For $\Phi \in \mathcal{Y}$ and a function $\phi \in \mathcal{G}$, let

$$\begin{aligned} L^{(\Phi, \phi)}(\mathbb{R}^n) &= \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi, \phi)}} < +\infty\}, \\ \|f\|_{L^{(\Phi, \phi)}} &= \sup_B \|f\|_{\Phi, \phi, B}, \\ L^{(\Phi, \phi)}_{\text{weak}}(\mathbb{R}^n) &= \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi, \phi)}_{\text{weak}}} < +\infty\}, \\ \|f\|_{L^{(\Phi, \phi)}_{\text{weak}}} &= \sup_B \|f\|_{\Phi, \phi, B, \text{weak}}. \end{aligned}$$

If $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^{(\Psi, \psi)}(\mathbb{R}^n)$.

If $\Phi(r) = r^p$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^{(p, \phi)}(\mathbb{R}^n)$ and $L^{(\Phi, \phi)}_{\text{weak}}(\mathbb{R}^n) = L^{(p, \phi)}_{\text{weak}}(\mathbb{R}^n)$. If $\phi(r) = r^{-n}$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^\Phi(\mathbb{R}^n)$ and $L^{(\Phi, \phi)}_{\text{weak}}(\mathbb{R}^n) = L^\Phi_{\text{weak}}(\mathbb{R}^n)$. If $\phi(r) \equiv 1$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\Phi(r) = r^p$ and $\phi(r) = r^{\lambda-n}$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^{p, \lambda}(\mathbb{R}^n)$.

A function $\Phi \in \mathcal{Y}$ is said to satisfy the Δ_2 -condition, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad r \geq 0,$$

for some $C > 0$.

A function $\Phi \in \mathcal{Y}$ is said to satisfy the ∇_2 -condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k} \Phi(kr), \quad r \geq 0,$$

for some $k > 1$.

Let $\mathcal{Y}^+$ be the set of all $\Phi \in \mathcal{Y}$ with $\int_0^1 \Phi(t)t^{-2} dt < +\infty$. For $\Phi \in \mathcal{Y}^+$, let

$$\Phi^+(r) = r \int_0^r \frac{\Phi(t)}{t^2} dt, \quad r \geq 0.$$

Then $\Phi^+ \in \mathcal{Y}$ and $\Phi(r) \leq \Phi^+(2r)$. If Φ satisfies the ∇_2 -condition, then $\Phi \in \mathcal{Y}^+$ and $\Phi^+ \approx \Phi$.

Example 2.1. For $\epsilon > 0$ and $\delta \geq 0$, let $\Phi \in \mathcal{Y}$ with

$$\Phi(r) = \begin{cases} r(\log(1/r))^{-\epsilon-1} & \text{for small } r > 0, \\ r(\log r)^\delta & \text{for large } r > 0. \end{cases}$$

Then $\Phi \in \mathcal{Y}^+$ and

$$\Phi^+(r) \approx \begin{cases} r(\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases}$$

Example 2.2. For $1 < p < \infty$, $\epsilon \in \mathbb{R}$ and $\delta \in \mathbb{R}$, let $\Phi \in \mathcal{Y}$ with

$$\Phi(r) = \begin{cases} r^p (\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r^p (\log r)^\delta & \text{for large } r > 0. \end{cases}$$

Then $\Phi \in \nabla_2$ and $\Phi^+ \approx \Phi$.

For a function $\theta : (0, +\infty) \rightarrow (0, +\infty)$, let

$$\theta^*(r) = \int_0^r \frac{\theta(t)}{t} dt, \quad \theta_*(r) = \int_r^{+\infty} \frac{\theta(t)}{t} dt.$$

If θ satisfies the doubling condition, then $\theta(r) \leq C\theta^*(r)$ and $\theta(r) \leq C\theta_*(r)$. If $\theta(r)r^{-\epsilon}$ is almost increasing for some $\epsilon > 0$, then $\theta^*(r) \leq C\theta(r)$. If $\theta(r)r^\epsilon$ is almost decreasing for some $\epsilon > 0$, then $\theta_*(r) \leq C\theta(r)$.

If $\phi \in \mathcal{G}$ and $\int_0^1 \phi(t) dt < +\infty$, then $\phi^* \in \mathcal{G}$. If $\phi \in \mathcal{G}$ and $\int_1^{+\infty} \phi(t) dt < +\infty$, then $\phi_* \in \mathcal{G}$.

3. MAIN RESULTS

The Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x .

The following are our main results.

Theorem 3.1. Let $\phi \in \mathcal{G}$ and $\Phi \in \mathcal{Y}$. Then

$$M : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Phi, \phi)}_{\text{weak}}(\mathbb{R}^n).$$

If $\Phi \in \mathcal{Y}^+$, then

$$M : L^{(\Phi^+, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Phi, \phi)}(\mathbb{R}^n).$$

If $\Phi \in \nabla_2$, then

$$M : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Phi, \phi)}(\mathbb{R}^n).$$

Next let T be a singular integral operator with a kernel $K(x, y)$ which satisfies

$$(3.1) \quad |K(x, y)| \leq \frac{C}{|x - y|^n}, \quad x, y \in \mathbb{R}^n, x \neq y,$$

and, for $f \in C^\infty_{\text{comp}}(\mathbb{R}^n)$,

$$(3.2) \quad Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad \text{a.e. } x \notin \text{supp } f.$$

Theorem 3.2. Let $\Phi \in \Delta_2$, $\phi, \psi \in \mathcal{G}$ and $\int_1^{+\infty} \Phi^{-1}(\phi(t))/t dt < +\infty$. Assume that there exists a constant $A > 0$ such that

$$\Phi((\Phi^{-1} \circ \phi)_*(r)) \leq A\psi(r), \quad r > 0.$$

(i) If $T : L^\Phi(\mathbb{R}^n) \rightarrow L_{\text{weak}}^\Phi(\mathbb{R}^n)$, then

$$T : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L_{\text{weak}}^{(\Phi, \psi)}(\mathbb{R}^n).$$

(ii) If $T : L^\Phi(\mathbb{R}^n) \rightarrow L^\Phi(\mathbb{R}^n)$, then

$$T : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Phi, \psi)}(\mathbb{R}^n).$$

Remark 3.1. Since $\Phi^{-1} \circ \phi$ satisfies doubling condition, $\phi(r) = \Phi(\Phi^{-1} \circ \phi(r)) \leq \Phi(C(\Phi^{-1} \circ \phi)_*(r)) \leq C'\psi(r)$ for all $r > 0$.

For a function $\rho : (0, +\infty) \rightarrow (0, +\infty)$, let

$$I_\rho f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy.$$

We consider the following conditions on ρ :

$$(3.3) \quad \int_0^1 \frac{\rho(t)}{t} dt < +\infty,$$

$$(3.4) \quad \frac{1}{A_1} \leq \frac{\rho(s)}{\rho(r)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

$$(3.5) \quad \frac{\rho(r)}{r^n} \leq A_2 \frac{\rho(s)}{s^n} \quad \text{for} \quad s \leq r.$$

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the fractional integral denoted by I_α .

Theorem 3.3. Let $\phi \in \mathcal{G}$, $\Phi, \Psi \in \mathcal{Y}$. Assume that ϕ is bijective. If there exists a constant $A > 0$ such that

$$\Psi \left(\frac{\Phi^{-1} \circ \phi(r) \rho^*(r) + ((\Phi^{-1} \circ \phi)\rho)_*(r)}{A} \right) \leq \phi(r), \quad r > 0,$$

then

$$I_\rho : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L_{\text{weak}}^{(\Psi, \phi)}(\mathbb{R}^n).$$

If $\Phi \in \mathcal{Y}^+$ and there exists a constant $A > 0$ such that

$$\Psi \left(\frac{(\Phi^+)^{-1} \circ \phi(r) \rho^*(r) + (((\Phi^+)^{-1} \circ \phi)\rho)_*(r)}{A} \right) \leq \phi(r), \quad r > 0,$$

then

$$I_\rho : L^{(\Phi^+, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Psi, \phi)}(\mathbb{R}^n).$$

Moreover, if $\Phi \in \nabla_2$, then

$$I_\rho : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Psi, \phi)}(\mathbb{R}^n).$$

Example 3.1. Let $0 < \alpha < n$, $1 < p < q < \infty$, $-1/p + \alpha/(n - \lambda) = -1/q$,

$$\begin{aligned}\rho(r) &= r^\alpha, \\ \Phi(r) &= r^p, \quad \Psi(r) = r^q, \\ \phi(r) &= r^{\lambda-n}.\end{aligned}$$

Then $\Phi \in \nabla_2$ and

$$\Phi^{-1} \circ \phi(r) \rho^*(r) + ((\Phi^{-1} \circ \phi) \rho)_*(r) \sim r^{(\lambda-n)/p+\alpha} = r^{(\lambda-n)/q}.$$

We have

$$I_\alpha : L^{p, \lambda}(\mathbb{R}^n) \rightarrow L^{q, \lambda}(\mathbb{R}^n).$$

This is the result of Adams [1] (1975).

Example 3.2. Let $\ell : (0, +\infty) \rightarrow (0, +\infty)$ satisfy the doubling condition and

$$\ell(r) = \begin{cases} (\log 1/r)^{-1} & \text{for small } r > 0, \\ \log r & \text{for large } r > 0. \end{cases}$$

For $\beta > 0$, let

$$\rho(r) = \begin{cases} (\log 1/r)^{-\beta-1} & \text{for small } r > 0, \\ (\log r)^{\beta-1} & \text{for large } r > 0. \end{cases}$$

Then ρ satisfies (3.3)–(3.5) and

$$\rho^*(r) = \int_0^r \frac{\rho(t)}{t} dt \sim \ell^\beta(r).$$

Let

$$\Phi(r) = r^p \quad (1 \leq p < \infty), \quad \Psi(r) = r^p \ell^{p\beta}(r),$$

and $\phi(r) = r^{\lambda-n}$ ($0 \leq \lambda < n$). Then we have

$$\begin{aligned}I_\rho : L^{1, \lambda}(\mathbb{R}^n) &= L^{(1, \phi)}(\mathbb{R}^n) \rightarrow L_{\text{weak}}^{(\Psi, \phi)}(\mathbb{R}^n), \\ I_\rho : L^{p, \lambda}(\mathbb{R}^n) &= L^{(p, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Psi, \phi)}(\mathbb{R}^n), \\ 1 &< p < \infty.\end{aligned}$$

Example 3.3. Let ℓ and ρ be as in Example 3.2. For $p > 0$, let

$$e_p(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r > 0, \\ \exp(r^p) & \text{for large } r > 0. \end{cases}$$

Let

$$\begin{aligned}\Phi(r) &= e_p(r), \quad \Psi(r) = e_q(r), \\ -1/p + \beta &= -1/q < 0, \\ \phi(r) &= r^{-\mu}, \quad 0 < \mu \leq n.\end{aligned}$$

Then we have

$$I_\rho : L^{(\Phi, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Psi, \phi)}(\mathbb{R}^n).$$

Example 3.4. Let ℓ and ρ be as in Example 3.2. For $\epsilon > 0$, $\delta \geq 0$ and $\beta > 0$, let

$$\begin{aligned}\Phi(r) &= \begin{cases} r(\log(1/r))^{-\epsilon-1} & \text{for small } r > 0, \\ r(\log r)^\delta & \text{for large } r > 0, \end{cases} \\ \Psi(r) &= \begin{cases} r(\log(1/r))^{-\epsilon-\beta} & \text{for small } r > 0, \\ r(\log r)^{\delta+1+\beta} & \text{for large } r > 0, \end{cases} \\ \phi(r) &= r^{-\mu}, \quad 0 < \mu \leq n.\end{aligned}$$

Then

$$\Phi^+(r) \approx \begin{cases} r(\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases}$$

Then we have

$$I_\rho : L^{(\Phi^+, \phi)}(\mathbb{R}^n) \rightarrow L^{(\Psi, \phi)}(\mathbb{R}^n).$$

4. PROOFS

Lemma 4.1. For $B = B(a, r)$,

$$\int_B f(x)g(x) dx \leq 2|B|\phi(r)\|f\|_{\Phi, B, \phi}\|g\|_{\tilde{\Phi}, B, \phi}.$$

Proof. By Hölder's inequality for the Orlicz spaces $L^\Phi(B, dx/(|B|\phi(r)))$ and $L^{\tilde{\Phi}}(B, dx/(|B|\phi(r)))$, we have

$$\begin{aligned}\int_B f(x)g(x) \frac{dx}{|B|\phi(r)} &\leq 2\|f\|_{L^\Phi(B, dx/(|B|\phi(r)))}\|g\|_{L^{\tilde{\Phi}}(B, dx/(|B|\phi(r)))} \\ &= 2\|f\|_{\Phi, B, \phi}\|g\|_{\tilde{\Phi}, B, \phi}. \quad \square\end{aligned}$$

Lemma 4.2. If a Young function Φ is bijective and $B = B(a, r)$, then $\|1\|_{\tilde{\Phi}, B, \phi} \leq \Phi^{-1}(\phi(r))/\phi(r)$.

Proof. By $\Phi(t)s/t - \Phi(s) \leq \Phi(t)$ for $s < t$ and $\Phi(t)s/t - \Phi(s) \leq 0$ for $s \geq t$, we have $\tilde{\Phi}(\Phi(t)/t) \leq \Phi(t)$. Let $\Phi(t) = \phi(r)$ and $\lambda = t/\Phi(t) = \Phi^{-1}(\phi(r))/\phi(r)$. Then

$$\frac{1}{|B|} \int_B \tilde{\Phi} \left(\frac{1}{\lambda} \right) dx = \tilde{\Phi} \left(\frac{1}{\lambda} \right) \leq \Phi(t) = \phi(r). \quad \square$$

Lemma 4.3. *If $f \in L^{(\Phi, \phi)}(\mathbb{R}^n)$ and $\text{supp } f \cap B(a, 2r) = \emptyset$. Then*

$$Mf(x) \leq C\Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi, \phi)}} \quad \text{for } x \in B(a, r).$$

Proof. For all balls $B = B(b, s)$, if $s \leq r/2$ then $\int_B |f(x)| dx = 0$, and, if $s > r/2$, then

$$\begin{aligned} \int_B |f(x)| dx &\leq 2\phi(s)\|f\|_{\Phi, \phi, B}\|1\|_{\tilde{\Phi}, \phi, B} \\ &\leq 2\phi(s)\|f\|_{L^{(\Phi, \phi)}}\Phi^{-1}(\phi(s))/\phi(s) \\ &\leq 2\Phi^{-1}(\phi(s))\|f\|_{L^{(\Phi, \phi)}} \\ &\leq C\Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi, \phi)}}. \quad \square \end{aligned}$$

Theorem 4.4 (see [3, 10, 12]). *Let $\Phi \in \mathcal{Y}$. Then*

$$\sup_{t>0} \Phi(t)m(f, t) \leq c_1 \int_{\mathbb{R}^n} \Phi(c_1|f(x)|) dx.$$

If $\Phi \in \mathcal{Y}^+$, then

$$\int_{\mathbb{R}^n} \Phi(Mf(x)) dx \leq c_1 \int_{\mathbb{R}^n} \Phi^+(c_1|f(x)|) dx.$$

If $\Phi \in \nabla_2$, then

$$\int_{\mathbb{R}^n} \Phi(Mf(x)) dx \leq c_1 \int_{\mathbb{R}^n} \Phi(c_1|f(x)|) dx.$$

Proof of Theorem 3.1. For all balls $B = B(a, r)$, let $f = f_1 + f_2$, $f_1 = f\chi_{2B}$. For f_1 , applying Theorem 4.4, we have

$$\begin{aligned} \sup_{t>0} \Phi(t)m(B, f_1/\lambda, t) &\leq c_1 \int_{\mathbb{R}^n} \Phi(c_1|f_1(x)|/\lambda) dx, \quad \text{if } \Phi \in \mathcal{Y}, \\ \int_B \Phi(Mf_1(x)/\lambda) dx &\leq c_1 \int_{\mathbb{R}^n} \Phi^+(c_1|f_1(x)|/\lambda) dx, \quad \text{if } \Phi \in \mathcal{Y}^+. \end{aligned}$$

We may assume that $c_1 \geq 1$. Let $c_* \geq 1$ be a constant such that $\phi(r) \leq c_*\phi(s)$ for $r \geq s$. Let $\lambda = 2^n c_* c_1^2 \|f\|_{L^{(\Phi, \phi)}}$ in the cases $\Phi \in \mathcal{Y}$. Then we have

$$\begin{aligned} c_1 \int_{\mathbb{R}^n} \Phi(c_1|f_1(x)|/\lambda) dx &= c_1 \int_{2B} \Phi(c_1|f(x)|/\lambda) dx \\ &\leq 2^{-n} c_*^{-1} \int_{2B} \Phi(|f(x)|/\|f\|_{L^{(\Phi, \phi)}}) dx \leq 2^{-n} c_*^{-1} |2B| \phi(2r) \leq |B| \phi(r). \end{aligned}$$

Let $\lambda = 2^n c_* c_1^2 \|f\|_{L(\Phi^+, \phi)}$ in the case $\Phi \in \mathcal{Y}^+$. Then we have in the same way

$$c_1 \int_{\mathbb{R}^n} \Phi^+(c_1 |f_1(x)|/\lambda) dx \leq |B| \phi(r).$$

For f_2 , applying Lemma 4.3, we have, for $x \in B$,

$$\begin{aligned} Mf_2(x) &\leq c_3 \Phi^{-1}(\phi(r)) \|f\|_{L(\Phi, \phi)}, \quad \text{if } \Phi \in \mathcal{Y}, \\ Mf_2(x) &\leq c_3 \Phi^{-1}(\phi(r)) \|f\|_{L(\Phi^+, \phi)}, \quad \text{if } \Phi \in \mathcal{Y}^+. \end{aligned}$$

Let $\lambda = c_3 \|f\|_{L(\Phi, \phi)}$ in the case $\Phi \in \mathcal{Y}$, or $\lambda = c_3 \|f\|_{L(\Phi^+, \phi)}$ in the case $\Phi \in \mathcal{Y}^+$. Then we have

$$\sup_{t>0} \Phi(t) m(B, f_2/\lambda, t) \leq \int_B \Phi(Mf_2(x)/\lambda) dx \leq |B| \phi(r).$$

This shows the norm inequations. $\square$

Let $\Phi \in \mathcal{Y}$ and T is a operator on $L^\Phi(\mathbb{R}^n)$ which satisfies $|T(cf)| = |cTf|$. If there exists a constant $c > 0$ such that

$$(4.1) \quad \int_{\mathbb{R}^n} \Phi(|Tf(x)|) dx \leq c \int_{\mathbb{R}^n} \Phi(c|f(x)|) dx, \quad f \in L^\Phi(\mathbb{R}^n),$$

then there exists a constant $C > 0$ such that

$$(4.2) \quad \|Tf\|_{L^\Phi} \leq C \|f\|_{L^\Phi}, \quad f \in L^\Phi(\mathbb{R}^n), \quad \text{i.e. } T : L^\Phi(\mathbb{R}^n) \rightarrow L^\Phi(\mathbb{R}^n).$$

If there exists a constant $c > 0$ such that

$$(4.3) \quad \sup_{t>0} \Phi(t) m(Tf, t) \leq c \int_{\mathbb{R}^n} \Phi(c|f(x)|) dx, \quad f \in L^\Phi(\mathbb{R}^n),$$

then there exists a constant $C > 0$ such that

$$(4.4) \quad \|Tf\|_{L_{\text{weak}}^\Phi} \leq C \|f\|_{L^\Phi}, \quad f \in L^\Phi(\mathbb{R}^n), \quad \text{i.e. } T : L^\Phi(\mathbb{R}^n) \rightarrow L_{\text{weak}}^\Phi(\mathbb{R}^n).$$

If $\Phi \in \Delta_2$, then (4.1) and (4.3) are equivalent to (4.2) and (4.4), respectively. If $T : L^1(\mathbb{R}^n) \rightarrow L_{\text{weak}}^1(\mathbb{R}^n)$ and $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $1 < p < \infty$, then (4.1) holds for $\Phi \in \Delta_2 \cap \nabla_2$ and (4.3) holds for $\Phi \in \Delta_2$.

Proof of Theorem 3.2. For $f \in L^{(\Phi, \phi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we choose a ball B such that $x \in B$, and let

$$Tf(x) = Tf_1(x) + \int_{\mathbb{R}^n} K(x, y) f_2(y) dy, \quad f = f_1 + f_2, \quad f_1 = f \chi_{2B}.$$

We show that $Tf(x)$ is finite a.e. x and independent of the choice of B . Since $f_1 \in L^\Phi(\mathbb{R}^n)$, Tf_1 is in $L^\Phi(\mathbb{R}^n)$ or $L_{\text{weak}}^\Phi(\mathbb{R}^n)$. Let $B = B(a, r)$ and $B_k = B(a, 2^k r)$,

$k = 1, 2, \dots$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} |K(x, y) f_2(y)| dy &= \sum_{k=2}^{\infty} \int_{B_k \setminus B_{k-1}} |K(x, y) f_2(y)| dy \\ &\leq \sum_{k=2}^{\infty} \frac{C}{|B_k|} \int_{B_k} |f_2(y)| dy. \end{aligned}$$

By Lemmas 4.1 and 4.2 we have

$$\begin{aligned} \frac{1}{|B_k|} \int_{B_k} |f_2(y)| dy &\leq 2\phi(2^k r) \|f\|_{\Phi, \phi, B_k} \|1\|_{\tilde{\Phi}, \phi, B_k} \\ &\leq 2\phi(2^k r) \|f\|_{L(\Phi, \phi)} \Phi^{-1}(\phi(2^k r)) / \phi(2^k r) \\ &= 2\Phi^{-1}(\phi(2^k r)) \|f\|_{L(\Phi, \phi)}. \end{aligned}$$

Hence

$$\begin{aligned} (4.5) \quad \int_{\mathbb{R}^n} |K(x, y) f_2(y)| dy &\leq C \sum_{k=2}^{\infty} \Phi^{-1}(\phi(2^k r)) \|f\|_{L(\Phi, \phi)} \\ &\leq C \int_r^{\infty} \Phi^{-1}(\phi(t)) t^{-1} dt \|f\|_{L(\Phi, \phi)}. \end{aligned}$$

If $x \in B \subset B'$ and $f = f_1 + f_2 = f'_1 + f'_2$, $f_1 = f\chi_{2B}$, $f'_1 = f\chi_{2B'}$, then

$$\begin{aligned} Tf(x) &= Tf'_1(x) + \int_{\mathbb{R}^n} K(x, y) f'_2(y) dy \\ &= T(f_1 + f'_1 - f_1)(x) + \int_{\mathbb{R}^n} K(x, y) f'_2(y) dy \\ &= T(f_1)(x) + T(f'_1 - f_1)(x) + \int_{\mathbb{R}^n} K(x, y) f'_2(y) dy \\ &= T(f_1)(x) + \int_{\mathbb{R}^n} K(x, y) (f'_1(y) - f_1(y)) dy + \int_{\mathbb{R}^n} K(x, y) f'_2(y) dy \\ &= Tf_1(x) + \int_{\mathbb{R}^n} K(x, y) f_2(y) dy. \end{aligned}$$

Now we show the boundedness. For every ball $B = B(a, r)$, let $f = f_1 + f_2$, $f_1 = f\chi_{2B}$. For $x \in B$ we write

$$Tf(x) = Tf_1(x) + Tf_2(x), \quad Tf_2(x) = \int_{\mathbb{R}^n} K(x, y) f_2(y) dy.$$

Then, in the case (i), by (4.3) we have

$$\Phi(t)m\left(B, \frac{Tf_1}{C\|f\|_{L(\Phi, \phi)}}, t\right) \leq \Phi(t)m\left(\frac{Tf_1}{C\|f\|_{L(\Phi, \phi)}}, t\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\|f\|_{L(\Phi, \phi)}}\right) dx,$$

and, in the case (ii), by (4.1) we have

$$\int_B \Phi \left(\frac{|Tf_1(x)|}{C\|f\|_{L(\Phi, \phi)}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left(\frac{|Tf_1(x)|}{C\|f\|_{L(\Phi, \phi)}} \right) dx \leq C \int_{\mathbb{R}^n} \Phi \left(\frac{|f_1(x)|}{\|f\|_{L(\Phi, \phi)}} \right) dx.$$

By the almost decreasingness of ϕ and Remark 3.1 we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi \left(\frac{|f_1(x)|}{\|f\|_{L(\Phi, \phi)}} \right) dx &= \int_{2B} \Phi \left(\frac{|f(x)|}{\|f\|_{L(\Phi, \phi)}} \right) dx \\ &\leq |2B|\phi(2r) \leq C|B|\phi(r) \leq C|B|\psi(r), \end{aligned}$$

Hence

$$\|Tf_1\|_{\Phi, \psi, B, \text{weak}} \leq C\|f\|_{L(\Phi, \phi)}, \quad \text{in the case (i)}$$

$$\|Tf_1\|_{\Phi, \psi, B} \leq C\|f\|_{L(\Phi, \phi)}, \quad \text{in the case (ii)}.$$

By (4.5) we have

$$\int_B \Phi \left(\frac{|Tf_2(x)|}{C\|f\|_{L(\Phi, \phi)}} \right) dx \leq \int_B \Phi \left(\int_r^\infty \Phi^{-1}(\phi(t))t^{-1}dt \right) dx \leq |B|\psi(r),$$

i.e.

$$\|Tf_2\|_{\Phi, \psi, B, \text{weak}} \leq \|Tf_2\|_{\Phi, \psi, B} \leq C\|f\|_{L(\Phi, \phi)}. \quad \square$$

Proof of Theorem 3.3. Using the pointwise estimate in the proof of Theorem 2.2 in [17];

$$\Psi \left(\frac{|I_\rho f(x)|}{C_1\|f\|_{L(\Phi, \phi)}} \right) \leq \Phi \left(\frac{Mf(x)}{C_0\|f\|_{L(\Phi, \phi)}} \right),$$

and Theorem 3.1 we have the norm inequalities. $\square$

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