

On puncture variation

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1. Introduction.

Let Δ be the unit disc and let $D (\neq \mathbb{C})$ be a plane domain containing the origin. Set $D_p = D \setminus \{p\}$ for $p \in D \setminus \{0\}$. Then there exists unique holomorphic universal covering $f_p: \Delta \rightarrow D_p$ satisfying

$$f_p(0) = 0, \quad f_p'(0) > 0.$$

Our aim is to derive the variation of the covering f_p by moving the puncture p in the domain D . Such a variation is called a puncture variation. Theorem 4.2. is the main result of this paper which gives explicitly the variation of f_p . As a corollary we have a puncture variation of the Poincaré metric. To obtain the formula we use quasiconformal mappings and apply a well-known representation theorem for quasiconformal maps with small dilatation.

2. Construction of $f_{p+\varepsilon}$ from f_p .

For sufficiently small $\rho \in \mathbb{R}$ let $N = \{z | 0 < |z-p| < e^\rho\}$ be a punctured disc contained in $f_p(\Delta)$ with $0 \notin N$. Let Δ_0 be a fixed component of $f_p^{-1}(N)$. Note that Δ_0 does not contain the origin. Let Γ be the covering group of f_p . The following Lemma is well known.

LEMMA 2.1. There exist a parabolic element $\beta \in \Gamma$ and a Möbius transformation A with the following properties,

- (1) A maps the upper half-plane onto Δ ,
- (2) $A(\infty) \in \partial\Delta$ is the fixed point of β and $A^{-1} \cdot \beta \cdot Az = z+1$,
- (3) Δ_0 is simply connected and contains a disc $A(U_c)$ with $U_c = \{z \in \mathbb{C} | \text{Im} z > c\}$ ($c > 0$), and
- (4) two points z_1 and z_2 of Δ_0 are equivalent under Γ if and only if $z_2 = \beta^n(z_1)$ for some integer n .

PROOF. See Kra [3, p.52] or Ahlfors [1, Lemma 1] where more general Kleinian case is considered. q.e.d

Let Γ_0 be the cyclic subgroup of Γ generated by $\beta \in \Gamma$. Expressing $f_p^{-1}(N)$ as a disjoint union of the components, we have

$$f_p^{-1}(N) = \bigcup_{\gamma \in \Gamma/\Gamma_0} \gamma \Delta_0 \quad (2.1)$$

where Γ/Γ_0 denotes the set of left cosets. Let $\Pi: L \rightarrow N$ be a universal covering given by

$$\Pi(z) = p + e^{z+\rho},$$

where L is the left half-plane $\{z | \text{Re } z < 0\}$. By the theory of covering surface we can find a conformal map $\varphi: \Delta_0 \rightarrow L$ such that

$$\varphi \circ \beta = \varphi + 2\pi i \quad (2.2)$$

and

$$f_p = \Pi \circ \varphi \quad \text{on } \Delta_0. \quad (2.3)$$

LEMMA 2.2. For $\varepsilon \in \mathbb{C}$ small, there exists a quasiconformal map $\psi_\varepsilon: L \rightarrow \mathbb{C}$ such that

$$\psi_\varepsilon(z+2\pi i) = \psi_\varepsilon(z) + 2\pi i \quad \text{on } L \quad (2.4)$$

and

$$\varepsilon + \Pi \circ \psi_\varepsilon = \Pi \quad \text{on } \partial L \quad (2.5)$$

with complex dilatation

$$-\varepsilon e^{\bar{z}-\rho} + O(\varepsilon^2) \quad (2.6)$$

for $z \in L$. The estimate is uniform for $z \in L$.

PROOF. Taking a branch of the logarithm, we set

$$\psi_\varepsilon(z) = z + \ln(1 - \varepsilon e^{\bar{z}-\rho}).$$

It is easy to see that this is a desired quasiconformal map.
q.e.d.

Define a map $\tilde{f}: \Delta \rightarrow D_{p+\varepsilon}$ by

$$\tilde{f}(z) = \begin{cases} f_p(z) & , z \in f_p^{-1}(N) \\ \varepsilon + \Pi \circ \psi_\varepsilon \circ \varphi \circ \gamma^{-1}(z) & , z \in \gamma \Delta_0 \quad (\gamma \in \Gamma/\Gamma_0). \end{cases}$$

It is seen from (2.1)-(2.5) that $\tilde{f}$ is a well-defined (topological) covering of $D_{p+\varepsilon}$. Let g_μ denote the quasiconformal automorphism of Δ with complex dilatation μ which is holomorphic near the origin and satisfies $g_\mu(0) = 0$, $g'_\mu(0) > 0$.

LEMMA 2.3. We have the identity

$$f_{p+\varepsilon} = \tilde{f} \cdot g_{\varepsilon\mu}^{-1} \quad \text{on } \Delta$$

where the complex dilatation μ is given by

$$\mu(z) = \begin{cases} 0 & , z \in f_p^{-1}(N) \\ \varepsilon^{-1}(\varphi \circ \gamma^{-1})^* \mu_{\psi_\varepsilon}(z) & , z \in \gamma \Delta_0 \quad (\gamma \in \Gamma/\Gamma_0) \end{cases} \quad (2.7)$$

Here, $\varphi^* \mu$ denotes as usual the pull-back $\mu \circ \varphi \cdot \frac{\overline{\varphi'}}{\varphi'}$ of the Beltrami coefficient μ .

PROOF. Computing the complex dilatation we have

$\mu_{\tilde{f} \cdot g_{\varepsilon\mu}^{-1}} = 0$ a.e. on Δ . Hence $\tilde{f} \cdot g_{\varepsilon\mu}^{-1}$ is a holomorphic covering of

$D_{p+\varepsilon}$ such that

$$\tilde{f} \cdot g_{\varepsilon\mu}^{-1}(0) = 0, \quad (\tilde{f} \cdot g_{\varepsilon\mu}^{-1})'(0) > 0.$$

Since these conditions determine a holomorphic covering uniquely,

we conclude that $f_{p+\varepsilon} = \tilde{f} \cdot g_{\varepsilon\mu}^{-1}$. q.e.d.

3. Integral representation of the variation.

Let f_μ be the quasiconformal automorphism of Δ with complex dilatation μ which leaves 0 and 1 fixed. The following perturbation formula is well known [2, p.105].

LEMMA 3.1. For $\varepsilon \in \mathbb{C}$ small and $\zeta \in \Delta$, $f_{\varepsilon\mu}$ is given by

$$f_{\varepsilon\mu}(\zeta) = \zeta + \hat{f}(\zeta) + O(\varepsilon^2)$$

where

$$\hat{f}(\zeta) = -\frac{\varepsilon}{\pi} \iint_{\Delta} \mu(z) R(z, \zeta) dx dy + \frac{\overline{\varepsilon}}{\pi} \iint_{\Delta} \overline{\mu(z)} \zeta^2 \overline{R(z, 1/\overline{\zeta})} dx dy$$

and

$$R(z, \xi) = \frac{\xi(\xi-1)}{z(z-1)(z-\xi)}.$$

The estimate is uniform for compact subsets of Δ .

It is convenient for our purpose to have a lemma with different normalization. The next lemma is a useful perturbation formula for g_μ . Recall that μ vanishes near the origin and that $g_\mu(0)=0$ and $g'_\mu(0)>0$.

LEMMA 3.2. For $\varepsilon \in \mathbb{C}$ small and $\xi \in \Delta$, $g_{\varepsilon\mu}$ is given by

$$g_{\varepsilon\mu}(\xi) = \xi + \dot{g}(\xi) + O(\varepsilon^2)$$

where

$$\dot{g}(\xi) = -\frac{\varepsilon\xi}{2\pi} \iint_{\Delta} \mu(z) Q(z, \xi) dx dy + \frac{\bar{\varepsilon}\xi}{2\pi} \iint_{\Delta} \overline{\mu(z) Q(z, 1/\bar{\xi})} dx dy$$

and

$$Q(z, \xi) = \frac{z+\xi}{z^2(z-\xi)}.$$

The estimate is uniform for compact subsets of Δ .

PROOF. Observe that

$$\dot{g}(\xi) = \dot{f}(\xi) + \frac{1}{2}\xi(\dot{f}'(0) - \overline{\dot{f}'(0)}).$$

LEMMA 3.1. yields

$$\dot{f}'(0) = \frac{\varepsilon}{\pi} \iint_{\Delta} \frac{\mu(z)}{z^2(z-1)} dx dy - \frac{\bar{\varepsilon}}{\pi} \iint_{\Delta} \frac{\overline{\mu(z)}}{\bar{z}(\bar{z}-1)} dx dy.$$

Combining these identities, we obtain the Lemma. q.e.d.

From Lemmas 2.3 and 3.2 we have, for $\xi \in f_p^{-1}(N)$,

$$f_{p+\varepsilon} = f_p(\zeta) + \zeta f_p'(\zeta)(\varepsilon I(\zeta) + \bar{\varepsilon} J(\zeta)) + O(\varepsilon^2) \quad (3.1)$$

where

$$I(\zeta) = \frac{1}{2\pi} \iint_{f_p^{-1}(N)} \mu(z) Q(z, \zeta) dx dy$$

and

$$J(\zeta) = \overline{-I(1/\bar{\zeta})}. \quad (3.2)$$

Since (2.1) is a disjoint union, $I(\zeta)$ is expressed as a series of the form

$$I(\zeta) = \sum_{\gamma \in \Gamma/\Gamma_0} I_\gamma(\zeta) \quad (3.3)$$

where

$$I_\gamma(\zeta) = \frac{1}{2\pi} \iint_{\gamma \Delta_0} \mu(z) Q(z, \zeta) dx dy.$$

4. Evaluation of $I_\gamma(\zeta)$.

By (2.6) and (2.7) we have

$$\begin{aligned} I_\gamma(\zeta) &= \frac{1}{2\pi} \iint_{\Delta_0} \gamma^* \mu(z) \gamma^* Q(z, \zeta) dx dy \\ &= \frac{1}{2\pi} \iint_{\Delta_0} \varepsilon^{-1} \varphi^* \mu_{\psi_\varepsilon}(z) \gamma^* Q(z, \zeta) dx dy \\ &= \frac{1}{2\pi} \iint_L \varepsilon^{-1} \mu_{\psi_\varepsilon}(z) (\gamma \circ \varphi^{-1})^* Q(z, \zeta) dx dy \\ &= -\frac{1}{2\pi} \iint_{x \leq 0} e^{\bar{z} - \rho} (\gamma \circ \varphi^{-1})^* Q(z, \zeta) dx dy + O(\varepsilon), \end{aligned}$$

where $\gamma^* Q(z, \zeta) = Q(\gamma(z), \zeta) \left(\frac{d\gamma}{dz}\right)^2$ is the pull-back of Q considered

as a quadratic differential of z . Therefore,

$$I_\gamma(\xi) = I + O(\varepsilon) \quad (4.1)$$

where

$$I = -\frac{1}{2\pi} \iint_{x \leq 0} e^{\bar{z}-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \xi) dx dy.$$

Our task is to evaluate the double integral I by using the calculus of residues. For convenience we introduce the functions u and w with the following properties,

$$(1) \varphi = u \circ w,$$

(2) $w: \Delta \rightarrow L$ is a Möbius transformation onto L such that $w \circ \beta = w + 2\pi i$, and

(3) $u: w(\Delta_0) \rightarrow L$ is a conformal surjection such that $u(z + 2\pi i) = u(z) + 2\pi i$.

Obviously, such u and w exist but not uniquely. We fix once and for all a choice of w .

LEMMA 4.1. For fixed $\xi \in f_p^{-1}(N)$,

$$(\gamma \circ \varphi^{-1})^* Q(z, \xi) = O(z^{-4}) \quad \text{as } z \rightarrow \infty, \quad z \in L.$$

PROOF. Setting $\gamma_1 = \gamma \circ w^{-1}$ and $u_1 = u^{-1}$, we have

$$\gamma \circ \varphi^{-1} = \gamma_1 \circ u_1$$

where $\gamma_1: L \rightarrow \Delta$ is a Möbius transformation and $u_1: L \rightarrow L$ is holomorphic. Clearly,

$$\gamma_1'(z) = O(z^{-2}) \quad (z \rightarrow \infty). \quad (4.2)$$

On the other hand, by expanding the function $u_1(z) - z$, which is periodic with period $2\pi i$, in a Fourier series, it is not hard to see that

$$u_1(z) = z + O(1) \text{ and } u_1'(z) = O(1) \text{ (} z \rightarrow \infty \text{)}, \quad (4.3)$$

since $u_1(z)$ is analytic on ∂L and u_1 maps L into itself. (4.2)

and (4.3) show that $(\gamma \circ \varphi^{-1})'(z) = O(z^{-2}) \text{ (} z \rightarrow \infty \text{)}$. This immediately gives the Lemma. q.e.d.

Cauchy's integral theorem and Lemma 4.1. imply that the integral

$$\int_{-\infty}^{\infty} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \xi) dy$$

is independent of $x = \operatorname{Re} z$. Thus

$$\begin{aligned} I &= - \frac{1}{2\pi} \int_{-\infty}^0 e^{2x} dx \int_{-\infty}^{\infty} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q dy \\ &= - \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-iy-\rho} (\gamma \circ \varphi^{-1})^* Q(iy, \xi) dy \\ &= - \frac{1}{4\pi i} \int_{\partial L} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \xi) dz \\ &= - \frac{1}{4\pi i} \int_{\ell} \frac{(\gamma \circ w^{-1})^* Q(z, \xi)}{u'(z) e^{u(z)+\rho}} dz \end{aligned}$$

where ℓ is a vertical line contained in $w(\Delta_0)$. Since the function $f_p \circ w^{-1}(z)$ is periodic with period $2\pi i$ on L , it is of the form $f_p \circ w^{-1}(z) = F(e^z)$ where $F(z)$ is regular in Δ with $F(0) = p$. Differentiating both sides of the identity $F(e^z) = p + e^{u(z)+\rho}$, we have

$$u'(z) e^{u(z)+\rho} = F'(e^z) e^z.$$

Hence

$$I = - \frac{1}{4\pi i} \int_{\ell} \frac{(\gamma \circ w^{-1})^* Q}{F'(e^z) e^z} dz.$$

By noting the estimates

$$(\gamma \cdot w^{-1})^* Q = O(z^{-4}), \quad (z \rightarrow \infty)$$

and

$$\frac{1}{Ce^z} - \frac{1}{F'(e^z)e^z} = O(1), \quad (z \rightarrow \infty)$$

with $C=F'(0)$, a standard application of Cauchy's integral theorem yields

$$I = - \frac{1}{4\pi i C} \int_{\ell} e^{-z} (\gamma \cdot w^{-1})^* Q(z, \xi) dz.$$

Although this integral can be evaluated by computing the residues in the right half-plane determined by ℓ , it is easier to evaluate the integral by changing the variable z to $w \cdot \gamma^{-1}(z)$. Thus

$$I = - \frac{1}{4\pi i C} \int_{\gamma(h)} \frac{e^{-w \cdot \gamma^{-1}(z)}}{(w \cdot \gamma^{-1})'(z)} Q(z, \xi) dz$$

where h is a circle in Δ which is tangent to $\partial\Delta$ at the fixed point of β . Denoting the residue of the integrand at z by $\text{Res}(z)$, we have

$$I = \frac{1}{2C} [\text{Res}(\xi) + \text{Res}(0) + \text{Res}(\infty)].$$

Observe that $w \cdot \gamma^{-1}(z)$ is of the form

$$w \cdot \gamma^{-1}(z) = \frac{t\alpha + \bar{t}z}{\alpha - z}, \quad |\alpha|=1, \quad \text{Re } t < 0.$$

After elementary calculations, we obtain

$$I = \frac{1}{C\xi} \left[\frac{e^{-iu_r}}{(w \cdot \gamma^{-1})'(\xi)} \right] \left[e^{-w \cdot \gamma^{-1}(\xi) + iu_r} + (w \cdot \gamma^{-1}(\xi) - iu_r) \frac{\sinh t_r}{t_r} \right]$$

$$-\cosh t_\gamma] - \xi e^{-iu_\gamma} \sinh t_\gamma \} \quad (4.4)$$

with $w \cdot \gamma^{-1}(0) = t_\gamma + iu_\gamma$ ($t_\gamma, u_\gamma \in \mathbb{R}$). Since

$$w \cdot \gamma^{-1}(1/\bar{\xi}) = -\overline{w \cdot \gamma^{-1}(\xi)}, \quad \xi \in \Delta,$$

identities (3.1)-(3.3), (4.1) and (4.4) give us the following final form of the variation of f_p .

THEOREM 4.2. For sufficiently small $\varepsilon \in \mathbb{C}$, the universal covering $f_{p+\varepsilon}$ of $D_{p+\varepsilon}$ is given by

$$f_{p+\varepsilon}(z) = f_p(z) + f_p'(z) \left[\frac{\varepsilon}{C} I_1 - \frac{\bar{\varepsilon}}{C} I_2 \right] + O(\varepsilon^2), \quad z \in \Delta$$

where

$$I_1 = \sum_{\gamma \in \Gamma/\Gamma_0} \left\{ \frac{e^{-iu_\gamma}}{(w \cdot \gamma^{-1})'(z)} \left[e^{-w \cdot \gamma^{-1}(z) + iu_\gamma} + (w \cdot \gamma^{-1}(z) - iu_\gamma) \frac{\sinh t_\gamma}{t_\gamma} - \cosh t_\gamma \right] - z e^{-iu_\gamma} \sinh t_\gamma \right\},$$

and

$$I_2 = \sum_{\gamma \in \Gamma/\Gamma_0} \left\{ \frac{e^{iu_\gamma}}{(w \cdot \gamma^{-1})'(z)} \left[e^{w \cdot \gamma^{-1}(z) - iu_\gamma} - (w \cdot \gamma^{-1}(z) - iu_\gamma) \frac{\sinh t_\gamma}{t_\gamma} - \cosh t_\gamma \right] - z e^{iu_\gamma} \sinh t_\gamma \right\}$$

with $w \cdot \gamma^{-1}(0) = t_\gamma + iu_\gamma$ ($t_\gamma, u_\gamma \in \mathbb{R}$). The constant C denotes the derivative $F'(0)$ of the function F satisfying the identity $f_p \circ w^{-1}(z) = F(e^z)$. The estimate is uniform as long as z stays in compact subsets of Δ .

Let $\lambda_p(z)|dz|$ be the Poincaré metric of the domain D_p . By definition $\lambda_p(z)$ satisfies

$$\lambda_p(f_p(z))|f_p'(z)| = \frac{1}{1-|z|^2}, \quad z \in \Delta.$$

In particular, we have $\lambda_p(0) = 1/f_p'(0)$. Theorem 4.2. easily gives the following

COROLLARY. For sufficiently small $\varepsilon \in \mathbb{C}$, $\lambda_{p+\varepsilon}(0)$ is given by

$$\ln \lambda_{p+\varepsilon}(0) = \ln \lambda_p(0) + 2\operatorname{Re} \left\{ \frac{\varepsilon}{\bar{C}} \sum_{r \in \Gamma/\Gamma_0} e^{-iu_r} \left(\cosh t_r - \frac{\sinh t_r}{t_r} \right) \right\} + O(\varepsilon^2).$$

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