

SMITH PROBLEM FOR A FINITE OLIVER GROUP WITH NON-TRIVIAL CENTER

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1. INTRODUCTION

The Smith problem is that two tangential representations are isomorphic or not for a smooth action on a homotopy sphere with exactly two fixed points. Two real G -modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $S^G = \{x, y\}$ for two points x and y at which $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as real G -modules. We will consider a subset $Sm(G)$ of the real representation ring $RO(G)$ of G consisting of the differences $U - V$ of real G -modules U and V which are Smith equivalent. We also define a subset $CSm(G)$ of $RO(G)$ consisting of the differences $U - V \in Sm(G)$ of real G -modules U and V such that for the sphere Σ appearing in the notion of Smith equivalence of U and V satisfies that Σ^P is connected for every $P \in \mathcal{P}(G)$. Moreover, we assume that $0 \in CSm(G)$ as definition.

In many groups, Smith equivalent modules are not isomorphic. In this paper we discuss the Smith problem for an Oliver group with non-trivial center. Throughout this paper we assume a group is finite.

2. TOPOLOGICAL VIEWPOINT

We denote by $\mathcal{P}(G)$ the family of subgroups of G consisting of the trivial subgroup of G and all subgroups of G of prime power order, and by $\mathcal{L}(G)$ the family of large subgroups of G . Here, by a *large subgroup* of G we mean a subgroup $H \leq G$ such that $O^p(G) \leq H$ for some prime p , where $O^p(G)$ is the smallest normal subgroup of G such that $|G/O^p(G)| = p^k$ for some integer $k \geq 0$. A real G -module V is called $\mathcal{L}(G)$ -free if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{O^p(G)} = 0$ for each prime p dividing $|G|$. Following [PSo], we denote by $LO(G)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of two real $\mathcal{L}(G)$ -free G -modules U and V such that $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for every $P \in \mathcal{P}(G)$.

For two subgroups $P < H$ of G with $P \in \mathcal{P}(G)$, and a smooth G -manifold X or a real G -module X , we consider the number

$$d_X(P, H) = \dim X^P - 2 \dim X^H$$

2000 *Mathematics Subject Classification.* 57S17, 20C15.

Key words and phrases. real representation, Smith problem, Oliver group.

where $\dim$ means the dimension of the G -CW complex. Furthermore we define by $\dim Z = \dim X - \dim Y$ for a virtual real G -module $Z = X - Y$ of $RO(G)$. A smooth G -manifold X satisfies the *gap condition* (GC) if $d_X(P, H) > 0$ for every pair (P, H) of subgroups $P < H$ of G with $P \in \mathcal{P}(G)$.

The following theorem goes back to [PSo], the Realization Theorem.

Theorem 2.1 ([PSo]). *Let G be a finite Oliver gap group. Then $LO(G) \subseteq CSm(G)$.*

We impose a number of restrictions on a smooth G -manifold, in particular, a real G -module X . The restrictions are collected in the following conditions, where we consider series $P < H \leq G$ of subgroups P and H of G *always* with $P \in \mathcal{P}(G)$. We say that a smooth G -manifold X satisfies the *weak gap condition* (WGC) if the conditions (WGC1)–(WGC4) all hold (cf. [LM], [MP]), and we say that X satisfies the *semi-weak gap condition* (SWGC) if the conditions (WGC1) and (WGC2) both hold.

(WGC1) $d_X(P, H) \geq 0$ for every $P < H \leq G$, $P \in \mathcal{P}(G)$.

(WGC2) If $d_X(P, H) = 0$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, then $[H : P] = 2$, $\dim X^H > \dim X^K + 1$ for every $H < K \leq G$, and X^H is connected.

(WGC3) If $d_X(P, H) = 0$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, and $[H : P] = 2$, then X^H can be oriented in such a way that the map $g: X^H \rightarrow X^H$ is orientation preserving for any $g \in N_G(H)$.

(WGC4) If $d_X(P, H) = d_X(P, H') = 0$ for some $P < H$, $P < H'$, $P \in \mathcal{P}(G)$, then the smallest subgroup $\langle H, H' \rangle$ of G containing H and H' is not a large subgroup of G .

Now, for a finite group G , we define subgroups $VLO(G)$, $WLO(G)$ and $MLO(G)$ of the free abelian group $LO(G)$ as follows.

$$VLO(G) = \{U - V \in LO(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the gap condition for some real } \mathcal{L}(G)\text{-free } G\text{-module } W\}$$

$$WLO(G) = \{U - V \in LO(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the weak gap condition for some real } \mathcal{L}(G)\text{-free } G\text{-module } W\}$$

$$MLO(G) = \{U - V \in LO(G) \mid U \oplus W \text{ and } V \oplus W \text{ both satisfy the semi-weak gap condition for some real } \mathcal{L}(G)\text{-free } G\text{-module } W\}$$

Note that if $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ then for an $\mathcal{L}(G)$ -free real G -modules U and V there is a real $\mathcal{L}(G)$ -free G -module W such that both $U \oplus W$ and $V \oplus W$ satisfy (WGC2), and if G is an Oliver group then for an $\mathcal{L}(G)$ -free real G -modules U and V there is a real $\mathcal{L}(G)$ -free G -module W such that both $U \oplus W$ and $V \oplus W$ satisfy (WGC2) and (WGC4).

In general, $VLO(G) \subseteq WLO(G) \subseteq MLO(G) \subseteq LO(G)$ by definitions. But if G is a gap group, then for every $U - V \in LO(G)$, there exists a real $\mathcal{L}(G)$ -free G -module W satisfying the gap condition, such that $U \oplus W$ and $V \oplus W$ also satisfy the gap condition, and thus $U - V \in VLO(G)$, and hence

$$VLO(G) = WLO(G) = MLO(G) = LO(G).$$

Therefore, the following theorem extends the result in Theorem 2.1 by using Theorem in [MP].

Theorem 2.2. *Let G be a finite Oliver group. Then $WLO(G) \subseteq CSm(G)$.*

3. ALGEBRAIC VIEWPOINT

We denote by $PO(G)$ the subgroup of $RO(G)$ of G consisting of the differences $U - V$ of representations U and V such that $\dim U^G = \dim V^G$ and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup P of G of prime power order. We note that in [PSo], $PO(G)$ is denoted by $IO(G, G)$. Similarly, we denote by $\overline{PO}(G)$ the subgroup of $RO(G)$ of G consisting of the differences $U - V$ of representations U and V such that $\dim U^G = \dim V^G$ and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup P of G of odd prime power order and order 2, 4. By a theorem of Sanchez [Sa], the difference of two Smith equivalent representations lies in $\overline{PO}(G)$ and the difference of two $\mathcal{P}$ -matched Smith equivalent representations lies in $PO(G)$.

We define the Laitinen number a_G as the number of real conjugacy classes in G represented by elements of G not of prime power order. The rank of $PO(G)$ is equal to the maximum of 0 and $a_G - 1$. Moreover the rank of $\overline{PO}(G)$ is equal to the rank of $PO(G)$ plus the number of all real conjugacy classes represented by 2-elements of order ≥ 8 . Now, let H be a normal subgroup of G . We denote by $PO(G, H)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of representations U and V such that $U^H \cong V^H$ as representations over G/H , and $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for any subgroup P of prime power order. Again, we note that in [PSo], $PO(G, H)$ is denoted by $IO(G, H)$. It holds that $PO(G) = PO(G, G)$. Let $b_{G/H}$ be the number of all real conjugacy classes in G/H which are images from real conjugacy classes of G represented by elements not of prime power order by the surjection $G \rightarrow G/H$. Then the rank of $PO(G, H)$ is equal to $a_G - b_{G/H}$ (see [PSo]).

Proposition 3.1 (cf. [PSo]). *It holds that*

$$PO(G, G^{nil}) \leq LO(G) \leq PO(G) \leq \overline{PO}(G) \leq RO(G).$$

Note that $G^{nil} = \cap_p O^p(G)$. Also it is known that

$$LO(G) \subseteq CSm(G) \subseteq Sm(G)$$

if G is an Oliver gap group.

4. UPPER RESTRICTION

Let S be a set of primes dividing $|G|$ and 1, and let denote by $G^{\cap S}$ the normal subgroup of G defined as

$$G^{\cap S} = \bigcap_{L \trianglelefteq G; [G:L] \in S} L.$$

Theorem 4.1 ([M07a, KMK]). *Let G be a finite Oliver group. We set $S = \{2, 3\}$ if a Sylow 2-subgroup of G is normal and set $S = \{2\}$ otherwise. Then it holds that*

$$CSm(G) \subseteq PO(G, G^{\cap S}) \quad \text{and} \quad Sm(G) \subseteq \overline{PO}(G, G^{\cap S}).$$

In addition if G is a gap group and $G^{nil} = G^{\cap S}$, then it holds that

$$LO(G) = CSm(G) = PO(G, G^{nil}).$$

Here G^{nil} is the minimal subgroup among normal subgroups N of G such that G/N is nilpotent.

In particular, $a_G = b_{G/G^{\cap S}}$ yields that $CSm(G) = 0$.

Proposition 4.2 (cf. [PSu08]). *$G/G^{\cap S}$ is an elementary abelian group.*

5. KNOWN RESULTS

In this section we summarize several known results ([Ju, M07a, M07b, PSo, PSu07, Su]). First we treat a non-solvable group. Pawałowski and Solomon [PSo] showed that $0 \neq PO(G, G^{nil}) \subseteq CSm(G)$ if G is a non-solvable gap group with $a_G \geq 2$, Pawałowski and Sumi [PSu07] showed that $0 \neq LO(G) \cap CSm(G)$ if G is a non-solvable group with $a_G \geq 2$, except $Aut(A_6)$, $P\Sigma L(2, 27)$, and Morimoto [M07a, M07b] showed that $Sm(Aut(A_6)) = 0$ and $CSm(P\Sigma L(2, 27)) \neq 0$. Combining these results we can state that

Theorem 5.1. *For a finite non-solvable group G , $Sm(G) \neq 0$ if and only if $a_G \leq 1$ or $G \cong Aut(A_6)$.*

We say that an element not of prime power order is an NPP element. Morimoto showed the following theorem to get $CSm(P\Sigma L(2, 27)) \neq 0$.

Theorem 5.2 (Morimoto). *Let G be an Oliver gap group. Suppose that $O^2(G)$ has a dihedral subgroup D_{2pq} of order $2pq$ with distinct primes p and q and G has two real conjugacy classes of NPP elements contained in $O^2(G)$. Then $CSm(G) \neq 0$.*

To show $LO(G) \cap CSm(G) \neq 0$ for a non-solvable group with $LO(G) \neq 0$, Pawałowski and Sumi introduced a basic pair (cf. [PSu07, Su]). Let $f: G \rightarrow G/G^{nil}$ be a natural homomorphism. For two NPP elements x and y of an finite Oliver group G , we call (x, y) a basic pair, if $f(x) = f(y)$, x is not real conjugate to y , and one of the following claims is satisfied:

- (1) x and y are elements of some gap subgroup of G .
- (2) $|x|$ is even and the involution of $\langle x \rangle$ is conjugate to the involution of $\langle y \rangle$ in G .

We denote by $\pi(G)$ the set of all primes dividing the order of G . Note that $\langle x \rangle G^{nil} = \langle y \rangle G^{nil}$ as $f(x) = f(y)$. Recall that if $|x|$ is even, then for the involution c of $\langle x \rangle$, $c \in O^2(G)$ or $|\pi(O^2(C_G(c)))| \geq 2$, then $\langle x \rangle O^2(G)$ is a gap group.

Theorem 5.3 ([PSu07]). *If an Oliver group has a basic pair, it holds $LO(G) \cap CSm(G) \neq 0$.*

Recall that $LO(G/G^{nil}) \subseteq LO(G)$. Furthermore we have

Proposition 5.4. $2LO(G/G^{nil}) \subseteq WLO(G)$ and in particular $LO(G/G^{nil}) \neq 0$ implies $CSm(G) \neq 0$.

Then $LO(G) \cap CSm(G) = 0$ implies $LO(G/G^{nil}) = 0$. Thus the following proposition is important.

Proposition 5.5 ([PSu07]). *Let H be a nilpotent group with $LO(H) = 0$. Then H is isomorphic to one of the following groups:*

- (1) *a p -group for a prime p ,*
- (2) *$C_2 \times P$ for an odd prime p and a p -group P , or*
- (3) *$P \times C_3$ for a 2-group P such that any element is self-conjugate.*

Lemma 5.6. *If $a_G \geq 2$ and $LO(G) = 0$ it holds $|\pi(G/G^{nil})| = 1, 2$.*

Proof. If $|\pi(G/G^{nil})| \geq 3$, then G/G^{nil} is a gap group with $LO(G/G^{nil}) \neq 0$, a contrary. If $|\pi(G/G^{nil})| = 0$, then G is perfect and thus $\text{rank } LO(G) = a_G - 1 > 0$, a contrary. $\square$

Theorem 5.7. *If $LO(G) \cap CSm(G) = 0$, then G has no element x with $|\pi(\langle x \rangle)| \geq 3$.*

Proof. We assume that x is an element of G of order pqr such that p, q, r are distinct primes. It is clear that $a_G \geq 4$. We may assume that $x^{pq} \in G^{nil}$ by Lemma 5.6. Then $(x^{pq}x^{qr}x^{pr}, x^{qr}x^{pr})$ is a basic pair, a contrary. $\square$

Thus $|\pi(\langle c \rangle)| \leq 2$ for each non-trivial element $c \in Z(G)$.

6. INDUCED MODULES AND $PO(G)$

Let G be a finite group and $NPP(G)$ be the set of all elements of G not of prime power order. Note that $NPP(G)$ does not contain the identity element. For the real representation ring $RO(G)$, the real vector space $RO(G) \otimes \mathbb{R}$ is identified with the vector space consisting of all maps from the set of real conjugacy classes of G to the real number field $\mathbb{R}$. We denote by $1_{(g)_{\pm}^G}$ the map defined by $1_{(g)_{\pm}^G}((g)_{\pm}^G) = 1$ and $1_{(g)_{\pm}^G}((a)_{\pm}^G) = 0$ if a is not real conjugate to g . Then

$$RO(G) \otimes \mathbb{R} \cong \langle 1_{(g)_{\pm}^G} \mid (g)_{\pm}^G \subseteq G \rangle$$

and

$$RO(G)_{\mathcal{P}(G)} \otimes \mathbb{R} \cong \langle 1_{(g)_{\pm}^G} \mid g \in NPP(G) \rangle.$$

Let K be a subgroup of G . The induced map $\text{Ind}_K^G 1_{(k)_{\pm}^K}$ has a non-zero value at $(g)_{\pm}^G$ only if g is real conjugate to k in G , i.e. $(g)_{\pm}^G = (k)_{\pm}^G$, since

$$\text{Ind}_K^G 1_{(k)_{\pm}^K}((a)_{\pm}^G) = \sum_{\substack{bKeG/K \\ b^{-1}ab \in K}} 1_{(k)_{\pm}^K}((b^{-1}ab)_{\pm}^K).$$

We denote by $RO(G)_{\mathcal{P}(G)}$ the subset of $RO(G)$ consisting the differences $U - V$ of real representations U and V such that $\text{Res}_P^G(U) \cong \text{Res}_P^G(V)$ for $P \in \mathcal{P}(G)$. It is clear that

$$PO(G) = \text{Ker}(\text{Fix}^G : RO(G)_{\mathcal{P}(G)} \rightarrow \mathbb{R}).$$

We have the following commutative diagram.

$$\begin{array}{ccccc} RO(K)_{\mathcal{P}(K)} \otimes \mathbb{R} & \longrightarrow & (\text{Ind}_K^G RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{R} & \xrightarrow{\subseteq} & RO(G)_{\mathcal{P}(G)} \otimes \mathbb{R} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \langle 1_{(k)_\pm^K}^K \mid k \in \text{NPP}(K) \rangle & \longrightarrow & \langle 1_{(k)_\pm^G}^G \mid k \in \text{NPP}(K) \rangle & \xrightarrow{\subseteq} & \langle 1_{(g)_\pm^G}^G \mid g \in \text{NPP}(G) \rangle \end{array}$$

It holds that

$$(\text{Ind}_K^G RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{R} = (\text{Ind}_K^G RO(K))_{\mathcal{P}(G)} \otimes \mathbb{R}$$

and then that

$$(\text{Ind}_K^G RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{Q} = (\text{Ind}_K^G RO(K))_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

Since an element of $RO(G)$ is a linear combination with rational coefficients of induced modules of $RO(C)$ for cyclic subgroups C of G , we obtain that

$$\sum_{\substack{(C)^G \\ C \leq G}} (\text{Ind}_C^G RO(C)_{\mathcal{P}(C)}) \otimes \mathbb{Q} = RO(G)_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

Furthermore, noting $\text{Ind}_C^G RO(C)_{\mathcal{P}(C)} = 0$ for $C \in \mathcal{P}(G)$, it holds that

$$\sum_{\substack{((g))^G \\ g \in \text{NPP}(G)}} (\text{Ind}_{\langle g \rangle}^G RO(\langle g \rangle)_{\mathcal{P}(\langle g \rangle)}) \otimes \mathbb{Q} = RO(G)_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

If g has order $2p$ for an odd prime p , then $RO(\langle g \rangle)_{\mathcal{P}(\langle g \rangle)} \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[\langle x^p \rangle]) \otimes (2\mathbb{R} - \eta)$$

for all real irreducible modules η over $\langle g^2 \rangle$ and $PO(\langle g \rangle) \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[\langle x^p \rangle]) \otimes (\eta - \eta')$$

for all non-trivial real irreducible modules η, η' over $\langle g^2 \rangle$. Hence we can investigate $LO(G)$ for a finite non-gap group G with $G/O^2(G)$ an elementary abelian 2-group. Letting C_2^n be an elementary abelian 2-group of order 2^n , we obtain the following results.

Theorem 6.1. *Let $G := K \times C_2^n$, $n \geq 2$ be an Oliver group such that $K/O^2(K)$ is an elementary abelian 2-group. Then it holds $MLO(G) \subseteq CSm(G) \subseteq LO(G)$. Furthermore if G is a gap group, it holds the equality $CSm(G) = LO(G)$.*

We will discuss in the case when G is a non-gap group in Theorem 6.1.

Proposition 6.2. *Let G be an Oliver non-gap group such that $[G : O^2(G)] = 2$. The following two claims are equivalent.*

- (1) $MLO(G) = LO(G)$.

- (2) If two involutions x and y of G outside of $O^2(G)$ are not conjugate then $C_G(x)$ or $C_G(y)$ is a 2-group.

The author does not know a group G with $MLO(G) \neq LO(G)$.

7. NON-TRIVIAL CENTRAL

In this section we consider whether $CSm(G) = 0$ or not for an Oliver group G with $a_G \geq 2$. In the section 5 we know completely it for a non-solvable group G . From now on we assume that G is an Oliver solvable group with $LO(G) \cap CSm(G) = 0$ and $a_G \geq 2$. Recall that $PO(G, G^{nil}) \neq 0$ implies $a_G \geq 2$.

Lemma 7.1. *If $Z(G) \neq \{1\}$ then $|\pi(G^{nil})| = 2$.*

Proof. Since $LO(G/G^{nil}) = 0$, G/G^{nil} is isomorphic to P , $C_2 \times P$, or $C_3 \times P_2$, where P is a p -group and P_2 is a 2-group. Then for some subgroup K of G , the sequence $G^{nil} \trianglelefteq K \trianglelefteq G$ such that $|\pi(G/K)| = 1$ and K/G^{nil} is cyclic. Thus $|\pi(G^{nil})| \geq 2$. We assume that $|\pi(G^{nil})| \geq 3$. Take distinct primes p, q, r in $\pi(G^{nil})$. Let $c \in Z(G)$ be an element of prime order. We may assume that $|c| \neq q, r$. Take elements x_q and x_r of G^{nil} of order q and r respectively. Then cx_q and cx_r are NPP elements of distinct order. Therefore (cx_q, cx_r) is a basic pair. $\square$

Lemma 7.2. *$Z(G)$ has no NPP element.*

Proof. We suppose that $Z(G)$ has an NPP element c of order pq where p and q are primes. Then $|\pi(G)| = 2$ and $\pi(G) = \pi(\langle c \rangle) = \{p, q\}$ by Theorem 5.7. First we show that G^{nil} is not a subgroup of $\langle c \rangle$. Suppose $G^{nil} \leq \langle c \rangle$. Let $f: G \rightarrow G/\langle c \rangle$ be a canonical epimorphism. Note that $\pi(G/\langle c \rangle) = \{p, q\}$. Since $f(G)$ is nilpotent, $O^q(f(G))$ is a Sylow p -subgroup of $f(G)$ and a Sylow p -subgroup $O^q(G)_p$ of $O^q(G)$ is normal and its quotient $O^q(G)/O^q(G)_p$ is cyclic. This is a contrary against G is Oliver.

$$\begin{array}{ccccc}
 \langle c \rangle & \longrightarrow & G & \xrightarrow{f} & G/\langle c \rangle \\
 \uparrow & & \uparrow & & \uparrow \\
 \langle c \rangle \cap O^q(G) & \longrightarrow & O^q(G) & \longrightarrow & O^q(G/\langle c \rangle) \\
 \uparrow & & \uparrow & & \uparrow = \\
 \langle c \rangle \cap O^q(G)_p & \longrightarrow & O^q(G)_p & \longrightarrow & O^q(G/\langle c \rangle)
 \end{array}$$

Thus we can take an element x of G^{nil} which is not in $\langle c \rangle$. Since f sends two NPP elements xc and c to elements of distinct order, xc and c are not real conjugate. It is clear that they are sent to the same element by $G \rightarrow G/G^{nil}$. Then (xc, c) is a basic pair, which is a contrary. Thus $Z(G)$ has no NPP element. $\square$

The following can be straightforward checked.

Lemma 7.3. *Let $c \in Z(G)$ be an element of order a prime p . If G^{nil} has an element x of order q^2 for some prime $q \neq p$, then G has a basic pair (cx, cx^q) .*

We define the $\text{DressLength}(G)$ as the minimal length n of sequences

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \{1\}$$

such that $O^{p_j}(G_{j-1}) = G_j$ with some prime p_j for each j . In convenient, we assume $\text{DressLength}(G) = \infty$ if there is no sequence as above. For example, $\text{DressLength}(G) = \infty$ for a non-solvable group. It is easy to see that $\text{DressLength}(G) \geq 3$ if G is an Oliver group and that $\text{DressLength}(G) \geq 3$ if G is a gap group.

Now we recall classical results. A finite group is called a CP group if it has no NPP elements.

Lemma 7.4 (Higman, cf. [PSo, Lemma 2.5]). *Let H be a finite solvable CP group. Then one of the following conclusions holds:*

- (1) H is a p -group for some prime p ; or
- (2) $H = K \rtimes C$ is a Frobenius group with kernel K and complement C , where K is a p -group and C is a q -group of q -rank 1 for two distinct primes p and q ; or
- (3) $H = K \rtimes C \rtimes A$ is a 3-step group, in the sense that $K \rtimes C$ is a Frobenius group as in the conclusion (2) with C cyclic, and $C \rtimes A$ is a Frobenius group with kernel C and complement A , a cyclic p -group.

Proposition 7.5 ([Hu, Proposition 22.3 and Remark on p.193]). $\text{Aut}(C_{2^a}) = C_2 \times C_{2^{a-2}}$ where $x \mapsto x^5$ is a generator of $C_{2^{a-2}}$ and $x \mapsto x^{-1}$ is a generator of C_2 . $\text{Aut}(C_{p^a}) = C_{p^{a-1}(p-1)}$ for an odd prime p .

With these results we use a Frattini subgroup and a Fitting subgroup and then we obtain the following results.

Theorem 7.6. *Let G be an Oliver solvable group with $a_G \geq 2$ and $Z(G) \neq \{1\}$. If $\text{CSm}(G) = 0$, then it holds the following.*

- (1) $Z(G)$ has no NPP element.
- (2) If $Z(G)$ is a p -group, an element of G^{nil} not of p power order has prime order.
- (3) $|\pi(G)| = 2$.
- (4) $\text{DressLength}(G) = 3, 4$.

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