

# 完備距離空間におけるシャウダーの不動点定理 と無限区間ファジィ境界値問題

## Schauder's Fixed Point Theorems in Complete Metric Spaces and Fuzzy Boundary Value Problems on an Infinite Interval

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### Abstract

Aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem. Finally we discuss a sufficient conditions for the existence of solutions of fuzzy differential equations on an infinite interval with boundary conditions.

## 1 Complete Metric Space of Fuzzy Numbers

Denote  $I = [0, 1]$ . The following definition means that a fuzzy number can be identified with a membership function.

**Definition 1** Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

$$\mathcal{F}_b^{st} = \{\mu : \mathbf{R} \rightarrow I \text{ satisfying (i)-(iv) below}\}.$$

- (i)  $\mu$  has a unique number  $m \in \mathbf{R}$  such that  $\mu(m) = 1$  (normality);
- (ii)  $\text{supp}(\mu) = \text{cl}(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$  is bounded in  $\mathbf{R}$  (bounded support);
- (iii)  $\mu$  is strictly fuzzy convex on  $\text{supp}(\mu)$  as follows:
  - (a) if  $\text{supp}(\mu) \neq \{m\}$ , then
 
$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$
 for  $\xi_1, \xi_2 \in \text{supp}(\mu)$  with  $\xi_1 \neq \xi_2$  and  $0 < \lambda < 1$ ;
  - (b) if  $\text{supp}(\mu) = \{m\}$ , then  $\mu(m) = 1$  and  $\mu(\xi) = 0$  for  $\xi \neq m$ ;
- (iv)  $\mu$  is upper semi-continuous on  $\mathbf{R}$  (upper semi-continuity).

It follows that  $\mathbf{R} \subset \mathcal{F}_b^{st}$ . Because  $m$  has a membership function as follows:

$$\mu(m) = 1; \quad \mu(\xi) = 0 \quad (\xi \neq m) \tag{1.1}$$

Then  $\mu$  satisfies the above (i)-(iv).

In usual case a fuzzy number  $x$  satisfies *fuzzy convex* on  $\mathbf{R}$ , i.e.,

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)] \quad (1.2)$$

for  $0 \leq \lambda \leq 1$  and  $\xi_1, \xi_2 \in \mathbf{R}$ . Denote  $\alpha$ -cut sets by

$$L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$$

for  $\alpha \in I$ . When the membership function is fuzzy convex, then we have the following remarks.

**Remark 1** The following statements (1) - (4) are equivalent each other, provided with (i) of Definition 1.

- (1) (1.2) holds;
- (2)  $L_\alpha(\mu)$  is convex with respect to  $\alpha \in I$ ;
- (3)  $\mu$  is non-decreasing in  $\xi \in (-\infty, m)$ , non-increasing in  $\xi \in [m, +\infty)$ , respectively;
- (4)  $L_\alpha(\mu) \subset L_\beta(\mu)$  for  $\alpha > \beta$ .

**Remark 2** The above condition (iiia) is stronger than (1.2). From (iiia) it follows that  $\mu(\xi)$  is strictly monotonously increasing in  $\xi \in [\min \text{supp}(\mu), m]$ . Suppose that  $\mu(\xi_1) \geq \mu(\xi_2)$  for  $\xi_1 < \xi_2 \leq m$ . From Remark 1(3), it follows that  $\mu(\xi_1) = \mu(\xi_2)$  for some  $\xi_1 < \xi_2$ , so we get  $\mu(\xi) = \mu(\xi_1) = \mu(\xi_2)$  for  $\xi \in [\xi_1, \xi_2]$ . This contradicts with Definition 1 (iiia). Thus  $\mu$  is strictly monotonously increasing. In the similar way  $\mu$  is strictly monotonously decreasing in  $\xi \in [m, \max \text{supp}(\mu)]$ . This condition plays an important role in Theorem 1.

We introduce the following parametric representation of  $\mu \in \mathcal{F}_b^{st}$  as

$$\begin{aligned} x_1(\alpha) &= \min L_\alpha(\mu), \\ x_2(\alpha) &= \max L_\alpha(\mu) \end{aligned}$$

for  $0 < \alpha \leq 1$  and

$$\begin{aligned} x_1(0) &= \min \text{supp}(\mu), \\ x_2(0) &= \max \text{supp}(\mu). \end{aligned}$$

In the following example we illustrate typical types of fuzzy numbers.

**Example 1** Consider the following  $L-R$  fuzzy number  $x \in \mathcal{F}_b^{st}$  with a membership function as follows:

$$\mu(\xi) = \begin{cases} L(\frac{|m-\xi|}{\ell})_+ & (\xi \leq m) \\ R(\frac{|\xi-m|}{r})_+ & (\xi > m) \end{cases}$$

Here it is said that  $m \in \mathbf{R}$  is a center and  $\ell > 0, r > 0$  are spreads.  $L, R$  are  $I$ -valued functions. Let  $L(\xi)_+ = \max(L(|\xi|), 0)$  etc. We identify  $\mu$  with  $x = (x_1, x_2)$ . As long as there exist  $L^{-1}$  and  $R^{-1}$ , we have  $x_1(\alpha) = m - L^{-1}(\alpha)\ell$  and  $x_2(\alpha) = m + R^{-1}(\alpha)r$ .

Let  $L(\xi) = -c_1\xi + 1$ , where  $c_1 > 0$  and  $|x_1 - m| \leq \ell$ . We illustrate the following cases (i)-(iv).

- (i) Let  $R(\xi) = -c_2\xi + 1$ , where  $c_2 > 0$ . Then  $c_2\ell(x_2 - m) = c_1r(m - x_1)$ .
- (ii) Let  $R(\xi) = -c_2\sqrt{\xi} + 1$ , where  $c_2 > 0$ . Then  $c_2\ell(x_2 - m)^2 = c_1r^2(m - x_1)$ .
- (iii) Let  $R(\xi) = -c_2\xi^2 + 1$ , where  $c_2 > 0$ . Then  $c_2^2\ell^2(x_2 - m) = c_1^2r(x_1 - m)^2$ .

(iv) Let  $c$  be a real number such that  $0 < c < 1$ . Denote

$$L(\xi) = \begin{cases} 1 & (\xi = 0) \\ -c\xi + c & (0 < \xi \leq 1) \end{cases}$$

and let  $R(\xi) = L(\xi)$ . Then we have  $\ell(x_2 - m) = r(m - x_1)$  for  $|x_1 - m| \leq \ell$ . The representation of  $x = (x_1, x_2)$  is as follows:

$$\begin{aligned} x_1(\alpha) &= m - \left(1 - \frac{\alpha}{c}\right)\ell, \\ x_2(\alpha) &= m + \left(1 - \frac{\alpha}{c}\right)r \quad (0 \leq \alpha < c) \\ x_1(\alpha) &= x_2(\alpha) = m \quad (c \leq \alpha \leq 1) \end{aligned}$$

The membership function is given by as follows:

$$\mu(\xi) = \begin{cases} 0 & (\xi < x_1(0), \xi > x_2(0)) \\ x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\ 1 & (\xi = m) \\ x_2^{-1}(\xi) & (m < \xi \leq x_2(0)) \end{cases}$$

Denote by  $C(I)$  the set of all the continuous functions on  $I$  to  $\mathbf{R}$ . The following theorem shows a membership function is characterized by  $x_1, x_2$ .

**Theorem 1** Denote the left-, right-end points of the  $\alpha$ -cut set of  $\mu \in \mathcal{F}_b^{st}$  by  $x_1(\alpha), x_2(\alpha)$ , respectively. Here  $x_1, x_2 : I \rightarrow \mathbf{R}$ . The following properties (i)-(iii) hold.

- (i)  $x_1, x_2 \in C(I)$ ;
- (ii)  $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$ ;
- (iii)  $x_1, x_2$  are non-decreasing, non-increasing on  $I$ , respectively, as follows :
  - (a) there exists a positive number  $c \leq 1$  such that  $x_1(\alpha) < x_2(\alpha)$  for  $\alpha \in [0, c)$  and that  $x_1(\alpha) = m = x_2(\alpha)$  for  $\alpha \in [c, 1]$ ;
  - (b)  $x_1(\alpha) = x_2(\alpha) = m$  for  $\alpha \in I$ ;

Conversely, under the above conditions (i) -(iii), if we denote

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \quad (1.3)$$

for  $\xi \in \mathbf{R}$ , then  $\mu \in \mathcal{F}_b^{st}$ .

**Remark 3** From the above Condition (i) a fuzzy number  $x = (x_1, x_2)$  means a bounded continuous curve over  $\mathbf{R}^2$  and  $x_1(\alpha) \leq x_2(\alpha)$  for  $\alpha \in I$ .

In what follows we denote  $\mu = (x_1, x_2)$  for  $\mu \in \mathcal{F}_b^{st}$ . The parametric representation of  $\mu$  is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let  $g : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  be an  $\mathbf{R}$ -valued function. The corresponding binary operation of two fuzzy numbers  $x, y \in \mathcal{F}_b^{st}$  to  $g(x, y) : \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \rightarrow \mathcal{F}_b^{st}$  is calculated by the extension principle of Zadeh. The membership function  $\mu_{g(x,y)}$  of  $g$  is as follows:

$$\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))$$

Here  $\xi, \xi_1, \xi_2 \in \mathbf{R}$  and  $\mu_x, \mu_y$  are membership functions of  $x, y$ , respectively. From the extension principle, it follows that, in case where  $g(x, y) = x + y$ ,

$$\begin{aligned}\mu_{x+y}(\xi) &= \max_{\xi=\xi_1+\xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\ &= \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i), i = 1, 2\} \\ &= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.\end{aligned}$$

Thus we get  $x + y = (x_1 + y_1, x_2 + y_2)$ . In the similar way  $x - y = (x_1 - y_2, x_2 - y_1)$ .

Denote a metric by

$$d_\infty(x, y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)$$

for  $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_b^{st}$ .

**Theorem 2**  $\mathcal{F}_b^{st}$  is a complete metric space in  $C(I)^2$ .

## 2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions  $\mu_x, \mu_y$  of  $x, y \in \mathcal{F}_b^{st}$  and  $\lambda \in \mathbf{R}$ , the following addition and a scalar product are given as follows :

$$\begin{aligned}\mu_{x+y}(\xi) &= \sup\{\alpha \in [0, 1] : \\ &\quad \xi = \xi_1 + \xi_2, \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}; \\ \mu_{\lambda x}(\xi) &= \begin{cases} \mu_x(\xi/\lambda) & (\lambda \neq 0) \\ 0 & (\lambda = 0, \xi \neq 0) \\ \sup_{\eta \in \mathbf{R}} \mu_x(\eta) & (\lambda = 0, \xi = 0) \end{cases}\end{aligned}$$

In [5] they introduced the following equivalence relation  $(x, y) \sim (u, v)$  for  $(x, y), (u, v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st}$ , i.e.,

$$(x, y) \sim (u, v) \iff x + v = u + y. \quad (2.4)$$

Putting  $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$  by the parametric representation, the relation (2.4) means that the following equations hold.

$$x_i + v_i = u_i + y_i \quad (i = 1, 2)$$

Denote an equivalence class by  $[x, y] = \{(u, v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} : (u, v) \sim (x, y)\}$  for  $x, y \in \mathcal{F}_b^{st}$  and the set of equivalence classes by

$$\mathcal{F}_b^{st} / \sim = \{[x, y] : x, y \in \mathcal{F}_b^{st}\}$$

such that one of the following cases (i) and (ii) hold:

- (i) if  $(x, y) \sim (u, v)$ , then  $[x, y] = [u, v]$ ;
- (ii) if  $(x, y) \not\sim (u, v)$ , then  $[x, y] \cap [u, v] = \emptyset$ .

Then  $\mathcal{F}_b^{st} / \sim$  is a linear space with the following addition and scalar product

$$[x, y] + [u, v] = [x + u, y + v] \quad (2.5)$$

$$\lambda[x, y] = \begin{cases} [(\lambda x, \lambda y)] & (\lambda \geq 0) \\ [((-\lambda)y, (-\lambda)x)] & (\lambda < 0) \end{cases} \quad (2.6)$$

for  $\lambda \in \mathbf{R}$  and  $[x, y], [u, v] \in \mathcal{F}_b^{st} / \sim$ . They denote a norm in  $\mathcal{F}_b^{st} / \sim$  by

$$\| [x, y] \| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here  $d_H$  is the Hausdorff metric is as follows:

$$\begin{aligned} & d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) \\ &= \max \left( \sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \right. \\ & \quad \left. \sup_{\eta \in L_\alpha(\mu_y)} \inf_{\xi \in L_\alpha(\mu_x)} |\xi - \eta| \right) \end{aligned}$$

It can be easily seen that  $\| [x, y] \| = d_\infty(x, y)$ .

Note that  $\| [x, y] \| = 0$  in  $\mathcal{F}_b^{st} / \sim$  if and only if  $x = y$  in  $\mathcal{F}_b^{st}$ .

### 3 Schauder's Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space  $\mathcal{F}_b^{st}$  has an induced Banach space.

**Theorem 3** *Let  $S$  be a bounded closed subset in  $\mathcal{F}_b^{st}$ . Assume that  $S$  contains any segments of  $x, y \in S$ , i.e.,  $\lambda x + (1 - \lambda)y \in S$  for  $\lambda \in I$ . Let  $V$  be an into continuous mapping on  $S$ . Assume that the closure  $cl(V(S))$  is compact in  $\mathcal{F}_b^{st}$ . Then  $V$  has at least one fixed point  $x$  in  $S$ , i.e.,  $V(x) = x$ .*

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

**Theorem 4** *Let  $\mathcal{F}$  be a complete metric space with a metric  $d$ . Assume that  $\mathcal{F}$  is closed under addition and scalar product, and that  $d(\lambda x, 0) = |\lambda|d(x, 0)$  for the scalar product  $\lambda x$  and  $\lambda \in \mathbf{R}, x \in \mathcal{F}$ . Denote  $X = \{[x, 0] : x, 0 \in \mathcal{F}\}$ . Here  $[x, y]$  for  $x, y \in \mathcal{F}$  are equivalence classes of (2.4) and 0 is the origin. Then  $X$  is a Banach space concerning addition (2.5), scalar product (2.6) and norm  $\| [x, 0] \| = d(x, 0)$  for  $[x, 0] \in X$ .*

Moreover let  $S$  be a bounded closed subset in  $\mathcal{F}$ . Assume that  $S$  contains any segments of  $x, y \in S$  in the same meaning of Theorem 3. Let  $V$  be an into continuous mapping on  $S$ . Assume that the closure  $cl(V(S))$  is compact in  $\mathcal{F}$ . Then  $V$  has at least one fixed point in  $S$ .

### 4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:

$$\frac{dx}{dt} = p(t)x + f(t, x), \quad x(\infty) = c \quad (4.7)$$

Here  $p : \mathbf{R}_+ \rightarrow \mathcal{F}_b^{st}$ ,  $f : \mathbf{R}_+ \times \mathcal{F}_b^{st} \rightarrow \mathcal{F}_b^{st}$  are continuous functions. Let denote  $\mathbf{R}_+ = [0, \infty)$  and  $c \in \mathcal{F}_b^{st}$ . The following assumptions play important roles in considering the existence of solutions of (4.7).

**Assupmtion.**

(A1) Assume that

$$\int_0^\infty d(p(s), 0)ds = K < \infty.$$

(A2) There exist positive real numbers  $a, r, R$  and integrable function  $m : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  such that

$$\begin{aligned} & d(f(t, x), 0) \leq m(t) \text{ for } (t, x) \in \mathbf{R}_+ \times S_1; \\ & \int_0^\infty m(s)ds \leq rR; \\ & [R + N_p(a + \|L\| R)]K < 1. \end{aligned}$$

Here

$$S_1 = \{x \in \mathcal{F}_b^{st} : d(x, 0) \leq \min(ar, r)\}$$

and  $N_p$  is independent on the function  $p$ .

$L : C_r^{\text{lim}} \rightarrow \mathcal{F}_b^{st}$  is a linear operator as  $L(x) = x(\infty)$  and

$$C_r^{\text{lim}} = \{x \in C(\mathbf{R}_+ : \mathcal{F}_b^{st}) : \exists x(\infty), d(x, 0) \leq r\}.$$

(A3) There exists no solution of

$$\frac{dx}{dt} = p(t)x, L(x) = 0$$

except for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.

Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in  $C_r^{\text{lim}}$  for any  $c \in S_1$  by applying the Schauder's fixed point theorem in  $C_r^{\text{lim}}$ .

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