

Runge-Kutta 系 7 段 7 次極限公式について

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1. はじめに

7 段陽的 Runge-Kutta 公式で達成し得る次数は高々 6 次の精度までであるが 5 段と 6 段の場合と同様に, 分点を近づけた極限で 7 次公式が得られる.

2. Runge-Kutta 系公式

常微分方程式の初期値問題

$$(2-1) \quad \frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad (y, f: \text{vector})$$

の 7 段陽的 Runge-Kutta 公式:

$$y_{n+1} = y_n + h \sum_{i=1}^7 \mu_i f_i$$

$$(2-2) \quad \begin{cases} f_1 = f(t_n, y_n) \\ f_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} f_j), \quad (i=2, \dots, 7) \end{cases}$$

2.1 7段6次公式の条件方程式

$$(2.1-0) \sum_{j=1}^{i-1} \beta_{ij} = \alpha_i$$

$$(2.1-1) \sum_{i=1}^7 \mu_i = 1$$

$$(2.1-2) \sum_{i=2}^7 \mu_i \alpha_i = 1/2$$

$$(2.1-3) \left| \begin{array}{l} \sum_{i=2}^7 \mu_i \alpha_i^2 = 1/3 \\ \sum_{i=3}^7 \mu_i c_i = 1/6 \end{array} \right.$$

$$(2.1-4) \left| \begin{array}{l} \sum_{i=2}^7 \mu_i \alpha_i^3 = 1/4 \\ \sum_{i=3}^7 \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/12 \\ \sum_{i=3}^7 \mu_i \alpha_i c_i = 1/8 \\ \sum_{i=3}^6 T_i c_i = 1/24 \end{array} \right.$$

$$(2.1-5) \left| \begin{array}{l} \sum_{i=2}^7 \mu_i \alpha_i^4 = 1/5 \\ \sum_{i=3}^7 \mu_i \alpha_i^2 c_i = 1/10 \\ \sum_{i=3}^7 \mu_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/15 \\ \sum_{i=3}^6 T_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/60 \\ \sum_{i=3}^7 \mu_i c_i^2 = 1/20 \\ \sum_{i=3}^6 T_i \alpha_i c_i = 1/40 \\ \sum_{i=3}^5 c_i \sum_{j=i+1}^6 \mu_j \alpha_j \beta_{ji} = 1/30 \\ \sum_{i=3}^7 \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/20 \\ \sum_{i=3}^5 c_i \sum_{j=i+1}^6 T_j \beta_{ji} = 1/120 \end{array} \right.$$

$$\sum_{i=2}^7 \mu_i \alpha_i^5 = 1/6$$

$$\sum_{i=3}^7 \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^4 = 1/30$$

$$\sum_{i=3}^6 T_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/120$$

$$\sum_{i=3}^5 \sigma_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/360$$

$$\sum_{i=3}^4 C_i = 1/720$$

$$\sum_{i=3}^7 \mu_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/24$$

$$\sum_{i=3}^6 T_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/90$$

$$\sum_{i=3}^6 \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 \right) \sum_{k=i+1}^7 \mu_k \alpha_k \beta_{ki} = 1/72$$

$$\sum_{i=2}^5 \sigma_i \alpha_i C_i = 1/240$$

$$(2.1-6) \quad \sum_{i=3}^5 C_i \sum_{j=i+1}^6 T_j \alpha_j \beta_{ji} = 1/180$$

$$\sum_{i=3}^5 C_i \sum_{j=i+1}^6 \beta_{ji} \sum_{k=i+1}^7 \mu_k \alpha_k \beta_{ki} = 1/144$$

$$\sum_{i=3}^7 \mu_i \alpha_i^3 C_i = 1/12$$

$$\sum_{i=3}^6 T_i \alpha_i^2 C_i = 1/60$$

$$\sum_{i=3}^6 C_i \sum_{j=i+1}^7 \mu_j \alpha_j^2 \beta_{ji} = 1/36$$

$$\sum_{i=3}^7 \mu_i \alpha_i^2 \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/18$$

$$\sum_{i=3}^7 \mu_i C_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/36$$

$$\sum_{i=3}^6 T_i C_i^2 = 1/120$$

$$\sum_{i=3}^6 C_i \sum_{j=i+1}^7 \mu_j C_j \beta_{ji} = 1/72$$

$$\sum_{i=3}^7 \mu_i \alpha_i C_i^2 = 1/24$$

$$\sum_{i=3}^6 \alpha_i C_i \sum_{j=i+1}^7 \mu_j \alpha_j \beta_{ji} = 1/48$$

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$$T_i = \sum_{j=i+1}^7 \mu_j \beta_{ji}, \quad (i=2, 3, \dots, 6)$$

$$C_i = \sum_{j=2}^{i-1} \beta_{ij} \alpha_j, \quad (i=3, 4, \dots, 7)$$

$$\sigma_i = \sum_{j=i+1}^6 T_j \beta_{ji}, \quad (i=2, 3, 4, 5)$$

7 段 6 次公式の場合分け

(I) $\alpha_7 = 1$

(II) $\alpha_7 \neq 1, p\alpha_7(\alpha_2 - \alpha_7) + q\alpha_7^2(\alpha_2 - \alpha_7) + r\alpha_7^3(\alpha_2 - \alpha_7) + sC_7 + t\alpha_7 C_7 = 0$

 $\alpha_7 = 1$ の場合の条件式 :

(2.1-7) $\sum_{i=1}^7 \mu_i = 1$

$$(2.1-8) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 0 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ 0 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 & \alpha_6^3 \\ 0 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 & \alpha_5^4 & \alpha_6^4 \end{bmatrix} \begin{bmatrix} \mu_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/12 \\ 1/20 \\ 1/30 \end{bmatrix}$$

$$(2.1-9) \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ C_3 & C_4 & C_5 & C_6 \end{bmatrix} \begin{bmatrix} C_3 T_3 \\ C_4 T_4 \\ C_5 T_5 \\ C_6 T_6 \end{bmatrix} = \begin{bmatrix} 1/24 \\ 1/40 \\ 1/60 \\ 1/120 \end{bmatrix}$$

(2.1-10) $\mu_3 C_3^2 + \mu_4 C_4^2 + \mu_5 C_5^2 + \mu_6 C_6^2 + \mu_7 C_7^2 = 1/20$

$$(2.1-11) \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 \\ 0 & C_3 & C_4 & C_5 \\ 0 & \alpha_2 C_3 & \alpha_3 C_4 & \alpha_4 C_5 \end{bmatrix} \begin{bmatrix} \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} 1/60 \\ 1/120 \\ 1/120 \\ 1/240 \end{bmatrix}$$

$$(2.1-12) \quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ 0 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} \sigma_3 \beta_{32} + \sigma_4 \beta_{42} + \sigma_5 \beta_{52} \\ \sigma_4 \beta_{43} + \sigma_5 \beta_{53} \\ \sigma_5 \beta_{54} \end{bmatrix} = \begin{bmatrix} 1/360 \\ 1/720 \end{bmatrix}$$

$$(2.1-13) \quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ 0 & c_3 & c_4 & c_5 \end{bmatrix} \begin{bmatrix} T_3 \alpha_3 \beta_{32} + T_4 \alpha_4 \beta_{42} + T_5 \alpha_5 \beta_{52} + T_6 \alpha_6 \beta_{62} \\ T_4 \alpha_4 \beta_{43} + T_5 \alpha_5 \beta_{53} + T_6 \alpha_6 \beta_{63} \\ T_5 \alpha_5 \beta_{54} + T_6 \alpha_6 \beta_{64} \\ T_6 \alpha_6 \beta_{65} \end{bmatrix} = \begin{bmatrix} 1/90 \\ 1/180 \end{bmatrix}$$

$$(2.1-14) \quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ 0 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} \mu_3 c_3 \beta_{32} + \mu_4 c_4 \beta_{42} + \mu_5 c_5 \beta_{52} + \mu_6 c_6 \beta_{62} + \mu_7 c_7 \beta_{72} \\ \mu_4 c_4 \beta_{43} + \mu_5 c_5 \beta_{53} + \mu_6 c_6 \beta_{63} + \mu_7 c_7 \beta_{73} \\ \mu_5 c_5 \beta_{54} + \mu_6 c_6 \beta_{64} + \mu_7 c_7 \beta_{74} \\ \mu_6 c_6 \beta_{65} + \mu_7 c_7 \beta_{75} \\ \mu_7 c_7 \beta_{76} \end{bmatrix} = \begin{bmatrix} 1/36 \\ 1/72 \end{bmatrix}$$

$$(2.1-15) \quad \sum_{j=i}^{i-1} \beta_{ij} = \alpha_i, \quad (i=3, 4, \dots, 7)$$

$$(2.1-16) \quad \mu_i(1 - \alpha_i) = T_i, \quad (i=2, 3, \dots, 6)$$

2.2 誤差の主要項

$$(2.2-1) \quad E = y_{n+1} - y(t_n + h) \\ = h^7 \sum_{j,k} \delta_{\eta,j}^{(k)} D_{\eta,j}^{(k)} + h^8 \sum_{j,k} \delta_{\varepsilon,j}^{(k)} D_{\varepsilon,j}^{(k)} + O(h^9) \\ (D_{\eta,j}^{(k)}, D_{\varepsilon,j}^{(k)} : \text{derivatives})$$

$$\begin{aligned}\delta_{7,1} &= \frac{1}{6!} \left[\sum_{i=2}^7 \mu_i \alpha_i^6 - \frac{1}{7} \right] \\ &= \frac{-1}{6!} \left[T_6 \alpha_6 (\alpha_6 - \alpha_2)(\alpha_6 - \alpha_3)(\alpha_6 - \alpha_4)(\alpha_6 - \alpha_5) \right. \\ &\quad \left. - \frac{1}{420} (70 g_6(2,3,4,5) - 35 g_3(2,3,4,5) + 21 g_2(2,3,4,5) - 14 g_1(2,3,4,5) + 10) \right] = -\frac{1}{6} \delta_{7,2}\end{aligned}$$

$$\begin{aligned}\delta_{7,3} &= \frac{1}{24} \left[\sum_{i=4}^7 \mu_i \sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k^4 - \frac{1}{210} \right] \\ &= \frac{1}{24} \left[\sigma_5 \alpha_5 (\alpha_5 - \alpha_2)(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4) - \frac{1}{840} (-35 \alpha_3 \alpha_4 + 14(\alpha_2 \alpha_3 + \alpha_3 \alpha_4 + \alpha_4 \alpha_2) - 7(\alpha_2 + \alpha_3 + \alpha_4) + 4) \right] = -\delta_{7,7}\end{aligned}$$

$$\begin{aligned}\delta_{7,4} &= \frac{1}{6} \left[\sum_{i=5}^7 \mu_i \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk} \sum_{\ell=2}^{k-1} \beta_{k\ell} \alpha_\ell^3 - \frac{1}{840} \right] \\ &= \frac{1}{6} \left[\sigma_5 \beta_{54} \alpha_4 (\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3) - \frac{1}{2520} (21 \alpha_3 \alpha_4 - 7(\alpha_2 + \alpha_3) + 3) \right] = -\delta_{7,8}^{(2)}\end{aligned}$$

$$\begin{aligned}\delta_{7,5} &= \frac{1}{2} \left[\sum_{i=6}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \sum_{k=4}^{j-1} \beta_{jk} \sum_{\ell=3}^{k-1} \beta_{k\ell} \sum_{m=2}^{\ell-1} \beta_{\ell m} \alpha_m^2 - \frac{1}{2520} \right] \\ &= \frac{1}{2} \left[\sigma_5 \beta_{56} \beta_{65} \alpha_5 (\alpha_5 - \alpha_2) - \frac{1}{5040} (-7 \alpha_2 + 2) \right] = -\delta_{7,9}^{(3)}\end{aligned}$$

$$\delta_{7,6} = \mu_7 \beta_{76} \beta_{65} \beta_{54} \beta_{43} \beta_{32} \alpha_2 - \frac{1}{5040} = \sigma_5 \beta_{54} \beta_{43} C_3 - \frac{1}{5040} = -\delta_{7,10}^{(4)}$$

$$\begin{aligned}\delta_{7,7}^{(1)} &= \frac{1}{6} \left[\sum_{i=6}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \alpha_j \sum_{k=3}^{j-1} \beta_{jk} \alpha_k^3 - \frac{1}{168} \right] \\ &= \frac{1}{6} \left[T_5 \beta_{56} \alpha_6 (\alpha_6 - \alpha_2)(\alpha_6 - \alpha_3)(\alpha_6 - \alpha_4) \right. \\ &\quad \left. - \frac{1}{2520} (-105 \alpha_2 \alpha_3 \alpha_6 + 63 \alpha_2 \alpha_3 + 42(\alpha_2 \alpha_6 + \alpha_3 \alpha_6) - 21 \alpha_6 - 28(\alpha_2 + \alpha_3) + 15) \right] = -2 \delta_{7,16}\end{aligned}$$

$$\begin{aligned}\delta_{7,8}^{(1)} &= \frac{1}{2} \left[\sum_{i=5}^7 \mu_i \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk} \alpha_k \sum_{\ell=2}^{k-1} \beta_{k\ell} \alpha_\ell^2 - \frac{1}{630} \right] \\ &= \frac{-1}{2} \left[\sigma_4 \beta_{43} \alpha_3 (\alpha_3 - \alpha_2)(\alpha_4 - \alpha_5) - \frac{1}{5040} (42 \alpha_2 \alpha_5 - 14 \alpha_5 - 21 \alpha_2 + 8) \right] = -\delta_{7,21}\end{aligned}$$

$$\begin{aligned}\delta_{7,7}^{(2)} &= \frac{1}{2} \left[\sum_{i=5}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \alpha_j \sum_{k=3}^{j-1} \beta_{jk} \sum_{\ell=2}^{k-1} \beta_{k\ell} \alpha_\ell^2 - \frac{1}{504} \right] \\ &= \frac{1}{2} \left[T_5 \beta_{54} \beta_{43} \alpha_3 (\alpha_3 - \alpha_2) (\alpha_3 - \alpha_1) - \frac{1}{2520} (21\alpha_2\alpha_6 - 7\alpha_6 - 14\alpha_2 + 5) \right] = -2 \delta_{7,15}^{(2)}\end{aligned}$$

$$\begin{aligned}\delta_{7,10}^{(1)} &= \sum_{i=6}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \sum_{k=4}^{j-1} \beta_{jk} \sum_{\ell=3}^{k-1} \beta_{k\ell} \alpha_\ell \sum_{m=2}^{\ell-1} \beta_{\ell m} \alpha_m - \frac{1}{1680} \\ &= \sigma_5 \beta_{54} C_4 (\alpha_4 - \alpha_3) - \frac{1}{5040} (-7\alpha_3 + 3) = -\delta_{7,22}^{(3)}\end{aligned}$$

$$\begin{aligned}\delta_{7,10}^{(2)} &= \sum_{i=6}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \sum_{k=4}^{j-1} \beta_{jk} \alpha_k \sum_{\ell=3}^{k-1} \beta_{k\ell} \sum_{m=2}^{\ell-1} \beta_{\ell m} \alpha_m - \frac{1}{1260} \\ &= \sigma_4 \beta_{43} C_3 (\alpha_4 - \alpha_5) - \frac{1}{5040} (-7\alpha_5 + 4) = -\delta_{7,22}^{(2)}\end{aligned}$$

$$\begin{aligned}\delta_{7,10}^{(3)} &= \sum_{i=6}^7 \mu_i \sum_{j=5}^{i-1} \beta_{ij} \alpha_j \sum_{k=4}^{j-1} \beta_{jk} \sum_{\ell=3}^{k-1} \beta_{k\ell} \sum_{m=2}^{\ell-1} \beta_{\ell m} \alpha_m - \frac{1}{1008} \\ &= T_5 \beta_{54} \beta_{43} C_3 (\alpha_5 - \alpha_6) - \frac{1}{5040} (-7\alpha_6 + 5) = -2 \delta_{7,13}^{(3)}\end{aligned}$$

$$\begin{aligned}\delta_{7,11} &= \frac{1}{24} \left[\sum_{i=3}^7 \mu_i \alpha_i^4 \sum_{j=2}^{i-1} \beta_{ij} \alpha_j - \frac{1}{14} \right] \\ &= \frac{-1}{24} \left[T_6 C_6 (\alpha_6 - \alpha_3) (\alpha_6 - \alpha_4) (\alpha_6 - \alpha_5) - \frac{1}{840} (-35\alpha_3\alpha_4\alpha_5 + 21(\alpha_3\alpha_6 + \alpha_4\alpha_5 + \alpha_5\alpha_3) - 14(\alpha_3 + \alpha_4 + \alpha_5) + 10) \right] \\ &= -\frac{1}{4} \delta_{7,12}^{(1)}\end{aligned}$$

$$\begin{aligned}\delta_{7,12}^{(2)} &= \frac{1}{6} \left[\sum_{i=4}^7 \mu_i \alpha_i^3 \sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k - \frac{1}{42} \right] \\ &= \frac{-1}{6} \left[T_4 \beta_{43} C_3 (\alpha_4 - \alpha_6) (\alpha_4 - \alpha_5) - \frac{1}{2520} (21\alpha_5\alpha_6 - 14(\alpha_5 + \alpha_6) + 10) \right] = -\frac{1}{3} \delta_{7,13}^{(2)}\end{aligned}$$

$$\begin{aligned}\delta_{7,12}^{(1)} &= \frac{1}{2} \left[\sum_{i=5}^7 \mu_i \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk} \alpha_k^2 \sum_{\ell=2}^{k-1} \beta_{k\ell} \alpha_\ell - \frac{1}{420} \right] \\ &= \frac{1}{2} \left[\sigma_5 C_5 (\alpha_5 - \alpha_3) (\alpha_5 - \alpha_4) - \frac{1}{1680} (14\alpha_3\alpha_4 - 7(\alpha_3 + \alpha_4) + 4) \right] = -\delta_{7,17}^{(1)}\end{aligned}$$

$$\begin{aligned}
\delta_{7,16} &= \frac{1}{12} \left[\sum_{i=3}^7 \mu_i \alpha_i^3 \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 - \frac{1}{21} \right] \\
&= \frac{-1}{12} \left[T_6 \beta_{63} \alpha_3 (\alpha_3 - \alpha_2) (\alpha_6 - \alpha_6) (\alpha_6 - \alpha_5) - \frac{1}{2520} (-105 \alpha_5 \alpha_5 \alpha_6 + 63 (\alpha_5 \alpha_6 + \alpha_5 \alpha_5) + 42 \alpha_5 \alpha_6 - 28 (\alpha_5 + \alpha_6) \right. \\
&\quad \left. - 42 \alpha_2 + 20) \right] = -\frac{1}{3} \delta_{7,15}^{(1)}
\end{aligned}$$

$$\begin{aligned}
\delta_{7,17}^{(2)} &= \frac{1}{2} \left[\sum_{i=4}^7 \mu_i \alpha_i^2 \sum_{j=3}^{i-1} \beta_{ij} \alpha_j \sum_{k=2}^{j-1} \beta_{jk} \alpha_k - \frac{1}{56} \right] \\
&= \frac{-1}{2} \left[T_5 \beta_{54} C_4 (\alpha_4 - \alpha_3) (\alpha_5 - \alpha_4) - \frac{1}{5040} (42 \alpha_3 \alpha_6 - 21 \alpha_6 - 28 \alpha_3 + 15) \right] = -\frac{1}{2} \delta_{7,22}^{(1)}
\end{aligned}$$

$$\begin{aligned}
\delta_{7,18} &= \frac{1}{6} \left[\sum_{i=3}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 \right) - \frac{1}{56} \right] \\
&= \frac{1}{6} \left[(\mu_3 C_3 \beta_{32} + \mu_4 C_4 \beta_{42} + \mu_5 C_5 \beta_{52} + \mu_6 C_6 \beta_{62} + \mu_7 C_7 \beta_{72}) \alpha_2^3 + (\mu_4 C_4 \beta_{43} + \mu_5 C_5 \beta_{53} + \mu_6 C_6 \beta_{63} + \mu_7 C_7 \beta_{73}) \alpha_3^3 \right. \\
&\quad \left. + (\mu_5 C_5 \beta_{54} + \mu_6 C_6 \beta_{64} + \mu_7 C_7 \beta_{74}) \alpha_4^3 + (\mu_6 C_6 \beta_{65} + \mu_7 C_7 \beta_{75}) \alpha_5^3 + \mu_7 C_7 \beta_{76} \alpha_6^3 - \frac{1}{56} \right]
\end{aligned}$$

$$\begin{aligned}
\delta_{7,19} &= \frac{1}{2} \left[\sum_{i=3}^7 \mu_i \alpha_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 \right) - \frac{1}{42} \right] \\
&= \frac{-1}{2} \left[T_3 C_3 d_3 + T_4 C_4 d_4 + T_5 C_5 d_5 + T_6 C_6 d_6 - \frac{1}{252} \right] = -\delta_{7,20}^{(1)}
\end{aligned}$$

$$\begin{aligned}
\delta_{7,20}^{(2)} &= \frac{1}{2} \left[\sum_{i=4}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k^2 - \frac{1}{168} \right] \\
&= \frac{1}{2} \left[(\mu_4 C_4 \beta_{43} + \mu_5 C_5 \beta_{53} + \mu_6 C_6 \beta_{63} + \mu_7 C_7 \beta_{73}) d_3 + (\mu_5 C_5 \beta_{54} + \mu_6 C_6 \beta_{64} + \mu_7 C_7 \beta_{74}) d_4 \right. \\
&\quad \left. + (\mu_6 C_6 \beta_{65} + \mu_7 C_7 \beta_{75}) d_5 + \mu_7 C_7 \beta_{76} d_6 - \frac{1}{168} \right]
\end{aligned}$$

$$\begin{aligned}
\delta_{7,20}^{(3)} &= \frac{1}{2} \left[\sum_{i=4}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 \right) \sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k - \frac{1}{126} \right] \\
&= \frac{1}{2} \left[(\mu_4 d_4 \beta_{43} + \mu_5 d_5 \beta_{53} + \mu_6 d_6 \beta_{63} + \mu_7 d_7 \beta_{73}) C_3 + (\mu_5 d_5 \beta_{54} + \mu_6 d_6 \beta_{64} + \mu_7 d_7 \beta_{74}) C_4 \right. \\
&\quad \left. + (\mu_6 d_6 \beta_{65} + \mu_7 d_7 \beta_{75}) C_5 + \mu_7 d_7 \beta_{76} C_6 - \frac{1}{126} \right]
\end{aligned}$$

$$\delta_{7,23} = \frac{1}{4} \left[\sum_{i=3}^7 \mu_i \alpha_i^2 \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right)^2 - \frac{1}{28} \right] = \frac{-1}{4} \left[T_3 \alpha_3 C_3^2 + T_4 \alpha_4 C_4^2 + T_5 \alpha_5 C_5^2 + T_6 \alpha_6 C_6^2 - \frac{1}{168} \right] = -\frac{1}{2} \delta_{7,24}^{(1)}$$

$$\begin{aligned} \delta_{7,24}^{(2)} &= \sum_{i=4}^7 \mu_i \alpha_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \left(\sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k \right) - \frac{1}{84} \\ &= - \left[(T_4 C_4 \beta_{43} + T_5 C_5 \beta_{53} + T_6 C_6 \beta_{63}) C_3 + (T_5 C_5 \beta_{54} + T_6 C_6 \beta_{64}) C_4 + T_6 C_6 \beta_{65} C_5 - \frac{1}{504} \right] = -\delta_{7,26}^{(4)} \end{aligned}$$

$$\delta_{7,25}^{(1)} = \frac{1}{2} \left[\sum_{i=4}^7 \mu_i \alpha_i \sum_{j=3}^{i-1} \beta_{ij} \left(\sum_{k=2}^{j-1} \beta_{jk} \alpha_k \right)^2 - \frac{1}{140} \right] = \frac{-1}{2} \left[\sigma_3 C_3^2 + \sigma_4 C_4^2 + \sigma_5 C_5^2 - \frac{1}{840} \right] = -\delta_{7,26}^{(3)}$$

$$\begin{aligned} \delta_{7,25}^{(2)} &= \sum_{i=4}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \left(\sum_{j=3}^{i-1} \beta_{ij} \alpha_j \sum_{k=2}^{j-1} \beta_{jk} \alpha_k \right) - \frac{1}{112} \\ &= (\mu_4 C_4 \beta_{43} + \mu_5 C_5 \beta_{53} + \mu_6 C_6 \beta_{63} + \mu_7 C_7 \beta_{73}) \alpha_3 C_3 + (\mu_5 C_5 \beta_{54} + \mu_6 C_6 \beta_{64} + \mu_7 C_7 \beta_{74}) \alpha_4 C_4 \\ &\quad + (\mu_6 C_6 \beta_{65} + \mu_7 C_7 \beta_{75}) \alpha_5 C_5 + \mu_7 C_7 \beta_{76} \alpha_6 C_6 - \frac{1}{112} \end{aligned}$$

$$\begin{aligned} \delta_{7,26}^{(1)} &= \frac{1}{2} \left[\sum_{i=4}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk} \alpha_k \right)^2 - \frac{1}{252} \right] \\ &= \frac{1}{2} \left[\mu_4 (\beta_{43} C_3)^2 + \mu_5 (\beta_{53} C_3 + \beta_{54} C_4)^2 + \mu_6 (\beta_{63} C_3 + \beta_{64} C_4 + \beta_{65} C_5)^2 \right. \\ &\quad \left. + \mu_7 (\beta_{73} C_3 + \beta_{74} C_4 + \beta_{75} C_5 + \beta_{76} C_6)^2 - \frac{1}{252} \right] \end{aligned}$$

$$\begin{aligned} \delta_{7,26}^{(2)} &= \sum_{i=5}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right) \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk} \sum_{l=2}^{k-1} \beta_{kl} \alpha_l - \frac{1}{336} \\ &= ((\mu_5 C_5 \beta_{54} + \mu_6 C_6 \beta_{64} + \mu_7 C_7 \beta_{74}) \beta_{43} + (\mu_6 C_6 \beta_{65} + \mu_7 C_7 \beta_{75}) \beta_{53} + \mu_7 C_7 \beta_{76} \beta_{63}) C_3 \\ &\quad + ((\mu_6 C_6 \beta_{65} + \mu_7 C_7 \beta_{75}) \beta_{54} + \mu_7 C_7 \beta_{76} \beta_{64}) C_4 + \mu_7 C_7 \beta_{76} \beta_{65} C_5 - \frac{1}{336} \end{aligned}$$

$$\delta_{7,27} = \frac{1}{6} \left[\sum_{i=3}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j \right)^3 - \frac{1}{56} \right] = \frac{1}{6} \left[\mu_3 C_3^3 + \mu_4 C_4^3 + \mu_5 C_5^3 + \mu_6 C_6^3 + \mu_7 C_7^3 - \frac{1}{56} \right]$$

$$\delta_{7,28} = \frac{1}{8} \left[\sum_{i=3}^7 \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 \right)^2 - \frac{1}{63} \right] = \frac{1}{8} \left[\mu_3 d_3^2 + \mu_4 d_4^2 + \mu_5 d_5^2 + \mu_6 d_6^2 + \mu_7 d_7^2 - \frac{1}{63} \right]$$

たに

$$g_1(ijkl) = \alpha_i + \alpha_j + \alpha_k + \alpha_l,$$

$$g_2(ijkl) = \alpha_i \alpha_j + \alpha_i \alpha_k + \alpha_i \alpha_l + \alpha_j \alpha_k + \alpha_j \alpha_l + \alpha_k \alpha_l,$$

$$g_3(ijkl) = \alpha_j \alpha_k \alpha_l + \alpha_i \alpha_k \alpha_l + \alpha_i \alpha_j \alpha_l + \alpha_i \alpha_j \alpha_k,$$

$$g_4(ijkl) = \alpha_i \alpha_j \alpha_k \alpha_l,$$

$$\alpha_i = \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2, \quad (i=3, 4, \dots, 7).$$

2.3 7次極限公式の条件

2.3.1 条件 $\alpha_2 \rightarrow 0$ の誘導

α_i のほか T_6 だけを含む $\delta_{7,1}$ を 0 とおくと (2.1-8) から

$$(2.3-1) \quad T_i = \frac{70g_4(jklm) - 35g_3(jklm) + 21g_2(jklm) - 14g_1(jklm) + 10}{420 \alpha_i (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_i - \alpha_l)(\alpha_i - \alpha_m)}$$

$$(i, j, k, l, m = 2, \dots, 6; j, k, l, m \neq i)$$

(2.1-7), (2.1-16) から μ_i が決る.

(2.1-9) のうち線形のはじめの3式と $\delta_{7,11} = 0$ から

$$(2.3-2) \quad C_i = \frac{-35\alpha_j \alpha_k \alpha_l + 21(\alpha_j \alpha_k + \alpha_k \alpha_l + \alpha_l \alpha_j) - 14(\alpha_j + \alpha_k + \alpha_l) + 10}{840 T_i (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_i - \alpha_l)}$$

$$(i, j, k, l = 3, 4, 5, 6; j, k, l \neq i)$$

極限公式の可能性を調べる.

$$\delta_{7,3} = 0 \quad \text{と} \quad \delta_{7,13}^{(1)} = 0 \quad \text{から}$$

$$\sigma_5 = \frac{-35\alpha_2\alpha_3\alpha_4 + 14(\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_2) - 7(\alpha_2 + \alpha_3 + \alpha_4) + 4}{840\alpha_5(\alpha_5 - \alpha_2)(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4)}$$

$$\begin{aligned} \sigma_5 &= \frac{-70g_4(2,3,4,6) + 35g_3(2,3,4,6) - 21g_2(2,3,4,6) + 14g_1(2,3,4,6) - 10}{35\alpha_3\alpha_4\alpha_6 - 21(\alpha_3\alpha_4 + \alpha_4\alpha_6 + \alpha_6\alpha_3) + 14(\alpha_3 + \alpha_4 + \alpha_6) - 10} \\ &\quad \times \frac{14\alpha_3\alpha_4 - 7(\alpha_3 + \alpha_4) + 4}{840\alpha_5(\alpha_5 - \alpha_2)(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4)} \end{aligned}$$

この二つの σ_5 が等しい条件:

$$(2.3-3) \quad \alpha_2(\alpha_6 - 1) \{ 35\alpha_3^2\alpha_4^2 - 35\alpha_3\alpha_4(\alpha_3 + \alpha_4) - 23\alpha_3\alpha_4 + 7(\alpha_3 + \alpha_4)^2 - 8(\alpha_3 + \alpha_4) + 2 \} = 0$$

$$\delta_{7,4} = 0 \quad \text{と} \quad \delta_{7,10}^{(1)} = 0 \quad \text{から}$$

$$\sigma_5\beta_{54} = \frac{21\alpha_2\alpha_3 - 7(\alpha_2 + \alpha_3) + 3}{2520\alpha_4(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3)}$$

$$\begin{aligned} \sigma_5\beta_{54} &= \frac{70g_4(2,3,5,6) - 35g_3(2,3,5,6) + 21g_2(2,3,5,6) - 14g_1(2,3,5,6) + 10}{-35\alpha_3\alpha_5\alpha_6 + 21(\alpha_3\alpha_5 + \alpha_5\alpha_6 + \alpha_6\alpha_3) - 14(\alpha_3 + \alpha_5 + \alpha_6) + 10} \\ &\quad \times \frac{-7\alpha_3 + 3}{2520\alpha_4(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3)} \end{aligned}$$

$$(2.3-4) \quad \alpha_2 \{ 7\alpha_3^2(5\alpha_5\alpha_6 - 4(\alpha_5 + \alpha_6) + 3) - \alpha_3(33\alpha_5\alpha_6 - 27(\alpha_5 + \alpha_6) + 21) + 6\alpha_5\alpha_6 - 5(\alpha_5 + \alpha_6) + 4 \} = 0$$

$$\delta_{7,8}^{(1)} = 0 \quad \text{と} \quad \delta_{7,11}^{(2)} = 0 \quad \text{から}$$

$$T_5 \beta_{54} = \frac{105\alpha_2\alpha_3\alpha_6 - 63\alpha_2\alpha_3 - 42(\alpha_3\alpha_6 + \alpha_6\alpha_2) + 28(\alpha_2 + \alpha_3) + 21\alpha_6 - 15}{2520\alpha_4(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3)(\alpha_6 - \alpha_5)}$$

$$T_5 \beta_{54} = \frac{70g_4(2,3,5,6) - 35g_3(2,3,5,6) + 21g_2(2,3,5,6) - 14g_1(2,3,5,6) + 10}{-35\alpha_3\alpha_5\alpha_6 + 21(\alpha_3\alpha_5 + \alpha_5\alpha_6 + \alpha_6\alpha_3) - 14(\alpha_3 + \alpha_5 + \alpha_6) + 10} \\ \times \frac{-42\alpha_3\alpha_6 + 28\alpha_3 + 21\alpha_6 - 15}{2520\alpha_4(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_3)(\alpha_6 - \alpha_5)}$$

$$(2.3-5) \quad \alpha_2 \left[\alpha_5 \{ (15\alpha_3\alpha_6 - 9\alpha_3 - 6\alpha_6 + 4)(35\alpha_3\alpha_6 - 21(\alpha_3 + \alpha_6) + 14) \right. \\ \left. - (42\alpha_3\alpha_6 - 28\alpha_3 - 21\alpha_6 + 15)(10\alpha_3\alpha_6 - 5(\alpha_3 + \alpha_6) + 3) \} \right. \\ \left. - \{ (15\alpha_3\alpha_6 - 9\alpha_3 - 6\alpha_6 + 4)(21\alpha_3\alpha_6 - 14(\alpha_3 + \alpha_6) + 10) \right. \\ \left. - (42\alpha_3\alpha_6 - 28\alpha_3 - 21\alpha_6 + 15)(5\alpha_3\alpha_6 - 3(\alpha_3 + \alpha_6) + 2) \} \right] = 0$$

$$\delta_{7,5} = 0 \quad \text{と} \quad \delta_{7,6} = 0 \quad \text{から}$$

$$\sigma_5 \beta_{54} \beta_{43} = \frac{-7\alpha_2 + 2}{5040\alpha_3(\alpha_3 - \alpha_2)}$$

$$\sigma_5 \beta_{54} \beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{2520\alpha_3(\alpha_3 - \alpha_2)(35\alpha_4\alpha_5\alpha_6 - 21(\alpha_4\alpha_5 + \alpha_5\alpha_6 + \alpha_6\alpha_4) + 14(\alpha_4 + \alpha_5 + \alpha_6) - 10)}$$

$$(2.3-6) \quad \alpha_2(15\alpha_4\alpha_5\alpha_6 - 11(\alpha_4\alpha_5 + \alpha_5\alpha_6 + \alpha_6\alpha_4) + 8(\alpha_4 + \alpha_5 + \alpha_6) - 6) = 0$$

$$\delta_{7,9}^{(1)} = 0 \quad \text{と} \quad \delta_{7,10}^{(2)} = 0 \quad \text{から}$$

$$\sigma_4 \beta_{43} = \frac{-42\alpha_2\alpha_5 + 21\alpha_2 + 14\alpha_5 - 8}{5040\alpha_3(\alpha_3 - \alpha_2)(\alpha_5 - \alpha_4)}$$

$$\sigma_4 \beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{35\alpha_4\alpha_5\alpha_6 - 21(\alpha_4\alpha_5 + \alpha_5\alpha_6 + \alpha_6\alpha_4) + 14(\alpha_4 + \alpha_5 + \alpha_6) - 10} \\ \times \frac{7\alpha_5 - 4}{2520\alpha_3(\alpha_3 - \alpha_2)(\alpha_5 - \alpha_4)}$$

$$(2.3-7) \quad \alpha_2 \left[2d_5 \{ 35d_3d_5d_6 - 21(d_3d_5 + d_5d_6 + d_6d_3) + 21(d_3 + d_5 + d_6) - 16 \} \right. \\ \left. - \{ 25d_3d_5d_6 - 23(d_3d_5 + d_5d_6 + d_6d_3) + 18(d_3 + d_5 + d_6) - 14 \} \right] = 0$$

$$\delta_{7,9}^{(2)} = 0 \quad \text{と} \quad \delta_{7,10}^{(3)} = 0 \quad \text{から}$$

$$T_5 \beta_{54} \beta_{43} = \frac{-21d_2d_6 + 14d_2 + 7d_6 - 5}{2520 \alpha_3 (d_3 - d_2)(d_6 - d_5)}$$

$$T_5 \beta_{54} \beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{35d_4d_5d_6 - 21(d_4d_5 + d_5d_6 + d_6d_4) + 14(d_4 + d_5 + d_6) - 10} \\ \times \frac{7d_6 - 5}{2520 \alpha_3 (d_3 - d_2)(d_6 - d_5)}$$

$$(2.3-8) \quad \alpha_2 \left[d_6 \{ 35d_4d_5d_6 - 28(d_4d_5 + d_5d_6 + d_6d_4) + 21(d_4 + d_5 + d_6) - 16 \} \right. \\ \left. - \{ 20d_4d_5d_6 - 17(d_4d_5 + d_5d_6 + d_6d_4) + 13(d_4 + d_5 + d_6) - 10 \} \right] = 0$$

$$\delta_{7,14} = 0 \quad \text{と} \quad \delta_{7,12}^{(2)} = 0 \quad \text{から}$$

$$T_4 \beta_{43} = \frac{-105d_2d_5d_6 + 63(d_2d_5 + d_5d_6 + d_6d_2) + 42d_5d_6 - 42d_2 - 28(d_5 + d_6) + 20}{2520 \alpha_3 (d_3 - d_2)(d_5 - d_4)(d_6 - d_4)}$$

$$T_4 \beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{35d_4d_5d_6 - 21(d_4d_5 + d_5d_6 + d_6d_4) + 14(d_4 + d_5 + d_6) - 10} \\ \times \frac{21d_5d_6 - 14(d_5 + d_6) + 10}{1260 \alpha_3 (d_3 - d_2)(d_5 - d_4)(d_6 - d_4)}$$

$$(2.3-9) \quad \alpha_2 \left[(15d_5d_6 - 9(d_5 + d_6) + 6)(35d_4d_5d_6 - 21(d_4d_5 + d_5d_6 + d_6d_4) + 14(d_4 + d_5 + d_6) - 10) \right. \\ \left. - (21d_5d_6 - 14(d_5 + d_6) + 10)(20d_4d_5d_6 - 10(d_4d_5 + d_5d_6 + d_6d_4) + 6(d_4 + d_5 + d_6) - 4) \right] \\ = 0$$

(2.3-3) から (2.3-9) は $\alpha_2 = 0$ なるはずの条件が成り立つ。

2.3.2 $\alpha_2 \rightarrow 0$ のときの残りの条件方程式と $\delta_{\eta,d}^{(k)}$

(2.3-1) から

$$(2.3-10) \quad \left\{ \begin{aligned} T_2 \alpha_2 &= \frac{10g_4(3,4,5,6) - 35g_3(3,4,5,6) + 21g_2(3,4,5,6) - 14g_1(3,4,5,6) + 10}{420 \alpha_3 \alpha_4 \alpha_5 \alpha_6} \\ T_i \alpha_i &= \frac{-35\alpha_j \alpha_k \alpha_l + 21(\alpha_j \alpha_k + \alpha_k \alpha_l + \alpha_l \alpha_j) - 14(\alpha_j + \alpha_k + \alpha_l) + 10}{420 \alpha_i (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_i - \alpha_l)} \\ &\quad (i, j, k, l = 3, 4, 5, 6; j, k, l \neq i) \end{aligned} \right.$$

(2.3-2), (2.1-3) から

$$(2.3-11) \quad C_i \rightarrow \frac{\alpha_i^2}{2}, \quad (i = 3, 4, 5, 6)$$

$$(2.3-12) \quad C_7 \rightarrow \frac{1}{2} (= \frac{\alpha_7^2}{2})$$

残りの条件式:

$$\sigma_2 \alpha_2^2 \rightarrow 0$$

$$(\sigma_3 \beta_{32} + \sigma_4 \beta_{42} + \sigma_5 \beta_{52}) \alpha_2^2 \rightarrow 0$$

$$(T_3 \alpha_3 \beta_{32} + T_4 \alpha_4 \beta_{42} + T_5 \alpha_5 \beta_{52}) \alpha_2^2 \rightarrow 0$$

つまり, 残りの条件式は次の4式になる:

$$(2.3-13) \quad \sigma_3 \alpha_3^2 + \sigma_4 \alpha_4^2 + \sigma_5 \alpha_5^2 = \frac{1}{60}$$

$$(2.3-14) \quad \sigma_3 \alpha_3^3 + \sigma_4 \alpha_4^3 + \sigma_5 \alpha_5^3 = \frac{1}{120}$$

$$(2.3-15) \quad (\sigma_4 \beta_{43} + \sigma_5 \beta_{53}) \alpha_3^2 + \sigma_5 \beta_{54} \alpha_4^2 = \frac{1}{260}$$

$$(2.3-16) \quad (T_4 \alpha_4 \beta_{43} + T_5 \alpha_5 \beta_{53} + T_6 \alpha_6 \beta_{63}) \alpha_3^2 + (T_5 \alpha_5 \beta_{54} + T_6 \alpha_6 \beta_{64}) \alpha_4^2 + T_6 \alpha_6 \beta_{65} \alpha_5^2 = \frac{1}{90}$$

7 次の誤差項の係数:

$$-720\delta_{7,1}, 120\delta_{7,2}, -48\delta_{7,11}, 12\delta_{7,12}^{(1)}, -16\delta_{7,23}, 8\delta_{7,24}^{(1)}, -48\delta_{7,27}$$

$$\rightarrow T_3\alpha_3^5 + T_4\alpha_4^5 + T_5\alpha_5^5 + T_6\alpha_6^5 - 1/42$$

$$24\delta_{7,3}, -24\delta_{7,7}, 4\delta_{7,13}^{(1)}, -4\delta_{7,17}^{(1)}, -8\delta_{7,25}^{(1)}, 8\delta_{7,26}^{(3)}$$

$$\rightarrow \sigma_5\alpha_5^2(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4) - (14\alpha_3\alpha_4 - 7(\alpha_3 + \alpha_4) + 4)/840$$

$$6\delta_{7,4}, -6\delta_{7,8}^{(2)}, 2\delta_{7,10}^{(1)}, -2\delta_{7,22}^{(3)} \rightarrow \sigma_5\beta_{54}\alpha_4^2(\alpha_4 - \alpha_3) - (-7\alpha_3 + 3)/2520$$

$$2\delta_{7,5}, -2\delta_{7,9}^{(3)}, 2\delta_{7,6}, -2\delta_{7,10}^{(4)} \rightarrow \sigma_5\beta_{54}\beta_{43}\alpha_3^2 - 1/2520$$

$$-6\delta_{7,8}^{(1)}, 12\delta_{7,16}, 4\delta_{7,11}^{(2)}, -2\delta_{7,22}^{(1)}, 12\delta_{7,18}, 4\delta_{7,25}^{(2)}$$

$$\rightarrow T_5\beta_{54}\alpha_4^2(\alpha_4 - \alpha_3)(\alpha_6 - \alpha_5) - (-42\alpha_3\alpha_6 + 21\alpha_6 + 28\alpha_3 - 15)/2520$$

$$2\delta_{7,9}^{(1)}, -2\delta_{7,21}, -2\delta_{7,10}^{(2)}, 2\delta_{7,22}^{(2)} \rightarrow \sigma_4\beta_{43}\alpha_3^2(\alpha_5 - \alpha_4) - (7\alpha_5 - 4)/2520$$

$$-2\delta_{7,9}^{(2)}, 4\delta_{7,15}^{(2)}, -2\delta_{7,10}^{(3)}, 4\delta_{7,13}^{(3)}, 4\delta_{7,20}^{(2)}, 4\delta_{7,26}^{(2)}$$

$$\rightarrow T_5\beta_{54}\beta_{43}\alpha_3^2(\alpha_6 - \alpha_5) - (7\alpha_6 - 5)/2520$$

$$-12\delta_{7,12}^{(2)}, 4\delta_{7,13}^{(2)}, -12\delta_{7,14}, 4\delta_{7,15}^{(1)}, -4\delta_{7,19}, 4\delta_{7,20}^{(1)}, -4\delta_{7,24}^{(2)}, 4\delta_{7,26}^{(4)}$$

$$\rightarrow T_4\beta_{43}\alpha_3^2(\alpha_6 - \alpha_4)(\alpha_5 - \alpha_4) - (21\alpha_5\alpha_6 - 14(\alpha_5 + \alpha_6) + 10)/1260$$

$$4\delta_{7,20}^{(3)} \rightarrow (\mu_4 d_4 \beta_{43} + \mu_5 d_5 \beta_{53} + \mu_6 d_6 \beta_{63} + \mu_7 d_7 \beta_{73})\alpha_3^2 + (\mu_5 d_5 \beta_{54} + \mu_6 d_6 \beta_{64} + \mu_7 d_7 \beta_{74})\alpha_4^2 \\ + (\mu_6 d_6 \beta_{65} + \mu_7 d_7 \beta_{75})\alpha_5^2 + \mu_7 d_7 \beta_{76}\alpha_6^2 - 1/63$$

$$8\delta_{7,26}^{(1)} \rightarrow \mu_4(\beta_{43}\alpha_3^2)^2 + \mu_5(\beta_{53}\alpha_3^2 + \beta_{63}\alpha_3^2)^2 + \mu_6(\beta_{63}\alpha_3^2 + \beta_{64}\alpha_4^2 + \beta_{65}\alpha_5^2)^2 \\ + \mu_7(\beta_{73}\alpha_3^2 + \beta_{74}\alpha_4^2 + \beta_{75}\alpha_5^2 + \beta_{76}\alpha_6^2)^2 - 1/63$$

$$8\delta_{7,28} = \mu_3 d_3^2 + \mu_4 d_4^2 + \mu_5 d_5^2 + \mu_6 d_6^2 + \mu_7 d_7^2 - 1/63$$

2.3.3 7次極限公式の条件

すべての項が0になるためには, はじめの7式から

$$(2.3-17) \quad \sigma_5 = \frac{14\alpha_3\alpha_4 - 7(\alpha_3 + \alpha_4) + 4}{840\alpha_5^2(\alpha_5 - \alpha_3)(\alpha_5 - \alpha_4)}$$

$$(2.3-18) \quad \sigma_5\beta_{54} = \frac{-7\alpha_3 + 3}{2520\alpha_4^2(\alpha_4 - \alpha_3)}$$

$$(2.3-19) \quad \sigma_5\beta_{54}\beta_{43} = \frac{1}{2520\alpha_3^2}$$

$$(2.3-20) \quad T_5\beta_{54} = \frac{-42\alpha_3\alpha_6 + 28\alpha_3 + 21\alpha_6 - 15}{2520\alpha_4^2(\alpha_4 - \alpha_3)(\alpha_6 - \alpha_5)}$$

$$(2.3-21) \quad \sigma_4\beta_{43} = \frac{7\alpha_5 - 4}{2520\alpha_3^2(\alpha_5 - \alpha_4)}$$

$$(2.3-22) \quad T_5\beta_{54}\beta_{43} = \frac{7\alpha_6 - 5}{2520\alpha_3^2(\alpha_6 - \alpha_5)}$$

$$(2.3-23) \quad T_4\beta_{43} = \frac{21\alpha_5\alpha_6 - 14(\alpha_5 + \alpha_6) + 10}{1260\alpha_3^2(\alpha_6 - \alpha_4)(\alpha_5 - \alpha_4)}$$

が得られる.

(2.3-17), (2.3-13), (2.3-14) から

$$(2.3-24) \quad \sigma_i = \frac{14\alpha_i\alpha_k - 7(\alpha_i + \alpha_k) + 4}{840\alpha_i^2(\alpha_i - \alpha_j)(\alpha_i - \alpha_k)}, \quad (i, j, k = 3, 4, 5; j, k \neq i)$$

(2.3-18)と(2.3-20)から得られる β_{54} が等しいための条件:

$$(2.3-25) \quad (\alpha_6 - 1)((14\alpha_3^2 - 12\alpha_3 + 3)\alpha_4 - \alpha_3) = 0$$

残りの4式から得られる β_{43} を $=$ かつ $=$ 等しいと置いて得られる条件:

$$(2.3-26) \quad \left\{ \begin{array}{l} \alpha_3(\alpha_6-1)(7\alpha_5^2-8\alpha_5+2)=0 \\ \alpha_3(\alpha_5-1)=0 \\ \alpha_3(3\alpha_6-2\alpha_5-1)=0 \\ \alpha_3(6\alpha_5\alpha_6-5(\alpha_5+\alpha_6)+4)=0 \\ \alpha_3\{(21\alpha_6^2-30\alpha_6+11)\alpha_5-(21\alpha_6^2-29\alpha_6+10)\}=0 \\ \alpha_3(\alpha_6-1)=0 \end{array} \right.$$

(2.3-25) と (2.3-26) を共に満足する場合分け

i) $\alpha_6=1$ かつ $\alpha_3=0$ (α_4, α_5 : 任意)

ii) $\alpha_6=1$ かつ $\alpha_5=1$ (α_3, α_4 : 任意)

iii) $\alpha_3=0$ かつ $\alpha_4=0$ (α_5, α_6 : 任意)

i) の場合に限って考察する.

$$T_i(1-\alpha_i)\alpha_i^2 \rightarrow 2\sigma_i\alpha_i^2, \quad (i=3, 4, 5)$$

残りの条件式:

$$(2.3-16)' \quad (T_4\alpha_4\beta_{43}+T_5\alpha_5\beta_{53}+T_6\alpha_6\beta_{63})\alpha_3^2 + (T_5\alpha_5\beta_{54}+T_6\alpha_6\beta_{64})\alpha_4^2 + T_6\alpha_6\beta_{65}\alpha_5^2 \\ = T_3\alpha_3d_3 + T_4\alpha_4d_4 + T_5\alpha_5d_5 + T_6\alpha_6d_6 = \frac{1}{90}$$

残りの7次の誤差項の係数:

(2.3-13), (2.3-16)', $\delta_{7,14}=0$ と (2.1-6) の15番目の式から

$$(2.3-28) \quad d_i \rightarrow \frac{\alpha_i^3}{3}, \quad (i=4, 5, 6)$$

$$(2.3-29) \quad d_7 \rightarrow \frac{1}{3} (= \frac{\alpha_7^3}{3})$$

$$d_4 = \beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2 \rightarrow \frac{\alpha_4^3}{3}, \quad \beta_{43}\alpha_3^2 \rightarrow \frac{\alpha_4^3}{3}, \quad \alpha_2\alpha_2^2 \rightarrow 0, \quad \sum_{i=3}^5 \sigma_i \beta_{i2}\alpha_2^2 \rightarrow 0, \quad \sum_{i=3}^5 T_i \alpha_i \beta_{i2}\alpha_2^2 \rightarrow 0$$

から

$$\beta_{i2}\alpha_2^2 \rightarrow 0, \quad (i=4, 5, 6)$$

従って

$$(2.3-30) \quad \sum_{j=3}^{i-1} \beta_{ij}\alpha_j^2 \rightarrow \sum_{j=2}^{i-1} \beta_{ij}\alpha_j^2 = d_i, \quad (i=4, 5, 6, 7)$$

(2.3-11), (2.3-12), (2.3-28), (2.3-29), (2.3-30) から,

$$\begin{aligned} \delta_{7,20}^{(3)} &\rightarrow (\mu_4 d_4 \beta_{43} + \mu_5 d_5 \beta_{53} + \mu_6 d_6 \beta_{63} + \mu_7 d_7 \beta_{73}) \alpha_3^2 \\ &\quad + (\mu_5 d_5 \beta_{54} + \mu_6 d_6 \beta_{64} + \mu_7 d_7 \beta_{74}) \alpha_4^2 \\ &\quad + (\mu_6 d_6 \beta_{65} + \mu_7 d_7 \beta_{75}) \alpha_5^2 \\ &\quad + \mu_7 d_7 \beta_{76} \alpha_6^2 - \frac{1}{63} \\ &\rightarrow \frac{1}{3} \left[\mu_4 \alpha_4^3 d_4 + \mu_5 \alpha_5^3 d_5 + \mu_6 \alpha_6^3 d_6 + \mu_7 d_7 - \frac{1}{21} \right] \rightarrow 4\delta_{7,14} \rightarrow 0 \end{aligned}$$

$$\begin{aligned} \delta_{7,26}^{(1)} &\rightarrow \frac{1}{8} \left[\mu_4 (\beta_{43}\alpha_3^2)^2 + \mu_5 \left(\sum_{j=3}^4 \beta_{5j}\alpha_j^2 \right)^2 + \mu_6 \left(\sum_{j=3}^5 \beta_{6j}\alpha_j^2 \right)^2 + \mu_7 \left(\sum_{j=3}^6 \beta_{7j}\alpha_j^2 \right)^2 - \frac{1}{63} \right] \\ &\rightarrow \frac{1}{24} \left[\mu_4 \alpha_4^3 d_4 + \mu_5 \alpha_5^3 d_5 + \mu_6 \alpha_6^3 d_6 + \mu_7 d_7 - \frac{1}{21} \right] \rightarrow \frac{1}{2} \delta_{7,14} \rightarrow 0 \end{aligned}$$

$$\begin{aligned} \delta_{7,28} &= \frac{1}{8} \left[\mu_3 d_3^2 + \mu_4 d_4^2 + \mu_5 d_5^2 + \mu_6 d_6^2 + \mu_7 d_7^2 - \frac{1}{63} \right] \\ &\rightarrow \frac{1}{72} \left[\mu_3 \alpha_3^6 + \mu_4 \alpha_4^6 + \mu_5 \alpha_5^6 + \mu_7 - \frac{1}{7} \right] \rightarrow 10\delta_{7,1} = 0 \end{aligned}$$

3. 7次極限公式

3.1 極限公式

$$\begin{aligned}
 (3.1-1) \quad & f_1 = f(t_n, y_n) \\
 & F_2 = \frac{\partial}{\partial t} f(t_n, y_n) + f_1 \frac{\partial}{\partial y} f(t_n, y_n) \\
 & F_3 = \frac{\partial^2}{\partial t^2} f(t_n, y_n) + 2f_1 \frac{\partial^2}{\partial t \partial y} f(t_n, y_n) + f_1^2 \frac{\partial^2}{\partial y^2} f(t_n, y_n) + F_2 \frac{\partial}{\partial y} f(t_n, y_n) \\
 & f_4 = f(t_n + \alpha_4 h, y_n + h(b_{41}f_1 + h b_{42}F_2 + \frac{h^2}{2} b_{43}F_3)) \\
 & f_5 = f(t_n + \alpha_5 h, y_n + h(b_{51}f_1 + h b_{52}F_2 + \frac{h^2}{2} b_{53}F_3 + b_{54}f_4)) \\
 & y_p = y_n + h(b_{61}f_1 + h b_{62}F_2 + \frac{h^2}{2} b_{63}F_3 + b_{64}f_4 + b_{65}f_5) \\
 & f_7 = f(t_n + h, y_p) \\
 & F_6 = \frac{\partial}{\partial t} f(t_n + h, y_p) \\
 & \quad + \frac{\partial}{\partial y} f(t_n + h, y_p) \cdot \{b_{671}f_1 + h b_{672}F_2 + \frac{h^2}{2} b_{673}F_3 + b_{674}f_4 + b_{675}f_5 + b_{676}f_7\} \\
 & y_{n+1} = y_n + h(m_1f_1 + h m_2F_2 + \frac{h^2}{2} m_3F_3 + m_4f_4 + m_5f_5 + h m_6F_6 + m_7f_7)
 \end{aligned}$$

$\Gamma = \Gamma_n \cup$

$$b_{i1} = \lim(\beta_{i1} + \beta_{i2} + \beta_{i3}), \quad b_{i2} = \lim(\beta_{i2}\alpha_2 + \beta_{i3}\alpha_3), \quad b_{i3} = \lim \beta_{i3}\alpha_3^2, \quad (i=4, 5, 6, 7);$$

$$b_{ij} = \lim \beta_{ij}, \quad (i=5, j=4; i=7, j=4, 5);$$

$$b_{6j} = \lim \frac{b_{7j} - b_{6j}}{1 - \alpha_6}, \quad (j=1, 2, \dots, 5); \quad b_{676} = \lim \frac{\beta_{76}}{1 - \alpha_6};$$

$$m_1 = \lim(\mu_1 + \mu_2 + \mu_3), \quad m_2 = \lim(\mu_2\alpha_2 + \mu_3\alpha_3), \quad m_3 = \lim \mu_3\alpha_3^2,$$

$$m_i = \lim \mu_i, \quad (i=4, 5); \quad m_6 = -\lim \mu_6(1 - \alpha_6), \quad m_7 = \lim(\mu_6 + \mu_7)$$

係数は α_4, α_5 を自由パラメータとして次のように表わされる:

$$m_2 = \frac{\alpha_4 \alpha_5 (63 \alpha_4 \alpha_5 - 28(\alpha_4 + \alpha_5) + 15) + (\alpha_4 + \alpha_5)(14 \alpha_4 \alpha_5 - 7(\alpha_4 + \alpha_5) + 4)}{420 \alpha_4^2 \alpha_5^2}$$

$$m_3 = \frac{14 \alpha_4 \alpha_5 - 7(\alpha_4 + \alpha_5) + 4}{420 \alpha_4^3 \alpha_5^2}, \quad m_4 = \frac{7 \alpha_5 - 4}{420 \alpha_4^3 (\alpha_5 - \alpha_4)(1 - \alpha_4)^2}, \quad m_5 = \frac{-7 \alpha_4 + 4}{420 \alpha_5^3 (\alpha_5 - \alpha_4)(1 - \alpha_5)^2}$$

$$m_6 = \frac{-21 \alpha_4 \alpha_5 + 14(\alpha_4 + \alpha_5) - 10}{420(1 - \alpha_4)(1 - \alpha_5)}$$

$$m_7 = \frac{2 \alpha_4 \alpha_5 (84 \alpha_4 \alpha_5 - 63(\alpha_4 + \alpha_5) + 50) - (\alpha_4 + \alpha_5)(189 \alpha_4 \alpha_5 - 140(\alpha_4 + \alpha_5) + 110) + 210 \alpha_4 \alpha_5 - 154(\alpha_4 + \alpha_5) + 120}{420(1 - \alpha_4)^2(1 - \alpha_5)^2}$$

$$m_1 = 1 - m_4 - m_5 - m_7$$

$$b_{41} = \alpha_4, \quad b_{42} = \frac{\alpha_4^2}{2}, \quad b_{43} = \frac{\alpha_4^3}{3}$$

$$b_{54} = \frac{\alpha_5^3(\alpha_5 - \alpha_4)}{\alpha_4^3(4 - 7\alpha_4)}, \quad b_{51} = \alpha_5 - b_{54}, \quad b_{52} = \frac{\alpha_5^2}{2} - b_{54}\alpha_4, \quad b_{53} = \frac{\alpha_5^3}{3} - b_{54}\alpha_4^2$$

$$b_{75} = \frac{-m_5(1 - \alpha_5)^2}{2m_6}, \quad b_{74} = \frac{1}{m_6} \left(m_5(1 - \alpha_5)b_{54} - \frac{1}{2}m_4(1 - \alpha_4)^2 \right)$$

$$b_{71} = 1 - b_{74} - b_{75}, \quad b_{72} = \frac{1}{2} - b_{74}\alpha_4 - b_{75}\alpha_5, \quad b_{73} = \frac{1}{3} - b_{74}\alpha_4^2 - b_{75}\alpha_5^2$$

$$b_{675} = \frac{1}{m_6} (m_5(1 - \alpha_5) - m_7 b_{75})$$

$$b_{674} = \frac{1}{m_6} (m_4(1 - \alpha_4) - m_5 b_{54} - m_7 b_{74})$$

$$b_{673} = 2 - b_{674}\alpha_4^2 - b_{675}\alpha_5^2, \quad b_{672} = 2 - b_{674}\alpha_4 - b_{675}\alpha_5$$

$$b_{671} = 2 - b_{674} - b_{675}, \quad b_{676} = -1$$

3.2 自由パラメタ α_4, α_5 の決め方

$O(R^8)$ の誤差項の係数 $\sigma_{ij}^{(k)}$ 115項の分類:

$$K = \frac{1}{6720}, \quad \xi = \frac{14\alpha_4\alpha_5 - 8(\alpha_4 + \alpha_5) + 5}{3360}, \quad \eta = \frac{8\alpha_5 - 5}{6720}, \quad \zeta = \frac{2\alpha_4 - 1}{1680} \text{ とおく}$$

K の倍数 ----- 20項

ξ の倍数 ----- 38項

η の倍数 ----- 32項

ζ の倍数 ----- 25項

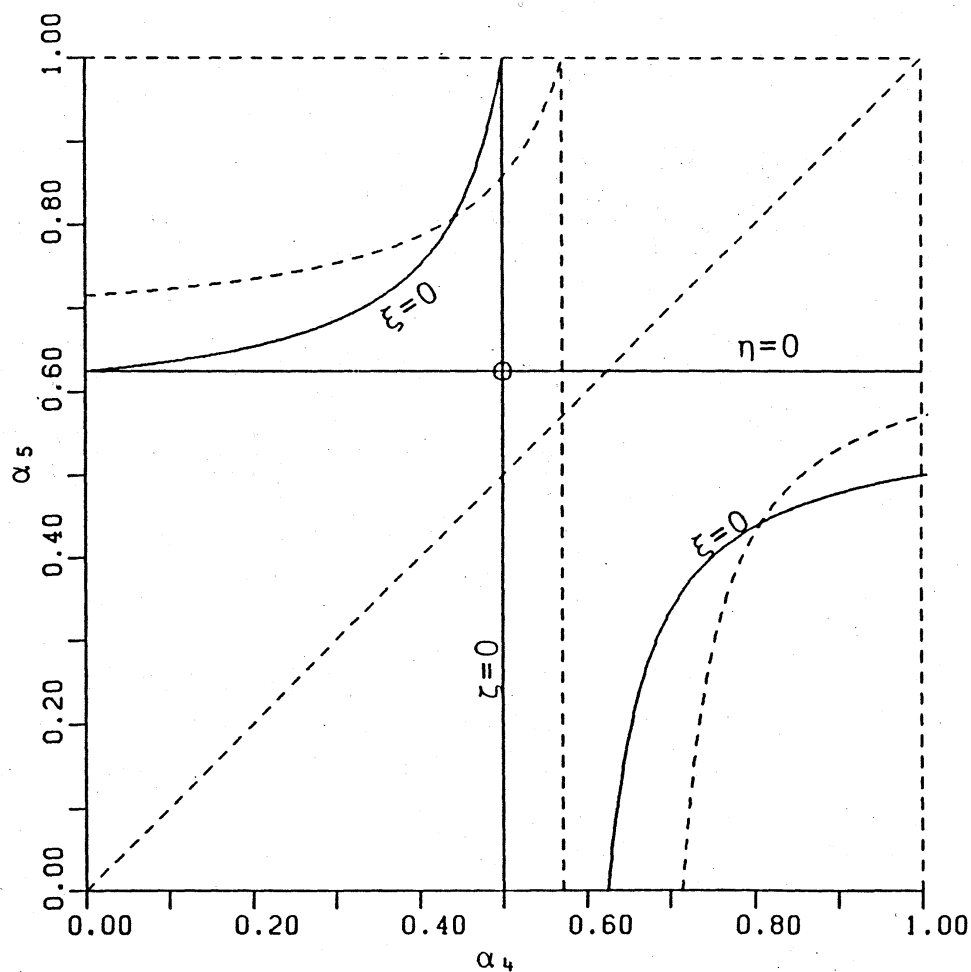


図1 $\xi, \eta, \zeta = 0$ の曲線 ----- は禁止線, $\circ \cdots (\frac{1}{2}, \frac{5}{8})$

3.3 7次極限公式 [RKD7]

$$f_1 = f(t_n, y_n)$$

$$F_2 = \frac{\partial}{\partial t} f(t_n, y_n) + f_1 \frac{\partial}{\partial y} f(t_n, y_n)$$

$$F_3 = \frac{\partial^2}{\partial t^2} f(t_n, y_n) + 2f_1 \frac{\partial^2}{\partial t \partial y} f(t_n, y_n) + f_1^2 \frac{\partial^2}{\partial y^2} f(t_n, y_n) + F_2 \frac{\partial}{\partial y} f(t_n, y_n)$$

$$f_4 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf_1 + \frac{1}{8}h^2F_2 + \frac{1}{48}h^3F_3)$$

$$f_5 = f(t_n + \frac{5}{8}h, y_n + \frac{35}{256}hf_1 - \frac{25}{512}h^2F_2 - \frac{125}{6144}h^3F_3 + \frac{125}{256}hf_4)$$

[RKD7]

$$y_p = y_n + \frac{2053}{1625}hf_1 + \frac{257}{650}h^2F_2 + \frac{1}{15}h^3F_3 - \frac{28}{13}hf_4 + \frac{3072}{1625}hf_5$$

$$f_7 = f(t_n + h, y_p)$$

$$F_6 = \frac{\partial}{\partial t} f(t_n + h, y_p)$$

$$+ \frac{\partial}{\partial y} f(t_n + h, y_p)$$

$$\times \left(\frac{62298}{4225}f_1 + \frac{4566}{845}hF_2 + \frac{12}{13}h^2F_3 - \frac{6168}{169}f_4 + \frac{100352}{4225}f_5 - f_7 \right)$$

$$y_{n+1} = y_n + \frac{2707}{8750}hf_1 + \frac{19}{500}h^2F_2 + \frac{1}{525}h^3F_3 + \frac{8}{35}hf_4 + \frac{32768}{118125}hf_5$$

$$- \frac{13}{1260}h^2F_6 + \frac{349}{1890}hf_7$$

4. 数値例

$$\frac{dy}{dt} = \frac{e^t(y^3(t+1)+1)}{3y^2(6-te^t)}, \quad y(0)=1$$

$t=1$ における累積誤差. 16進14桁15切捨て計算

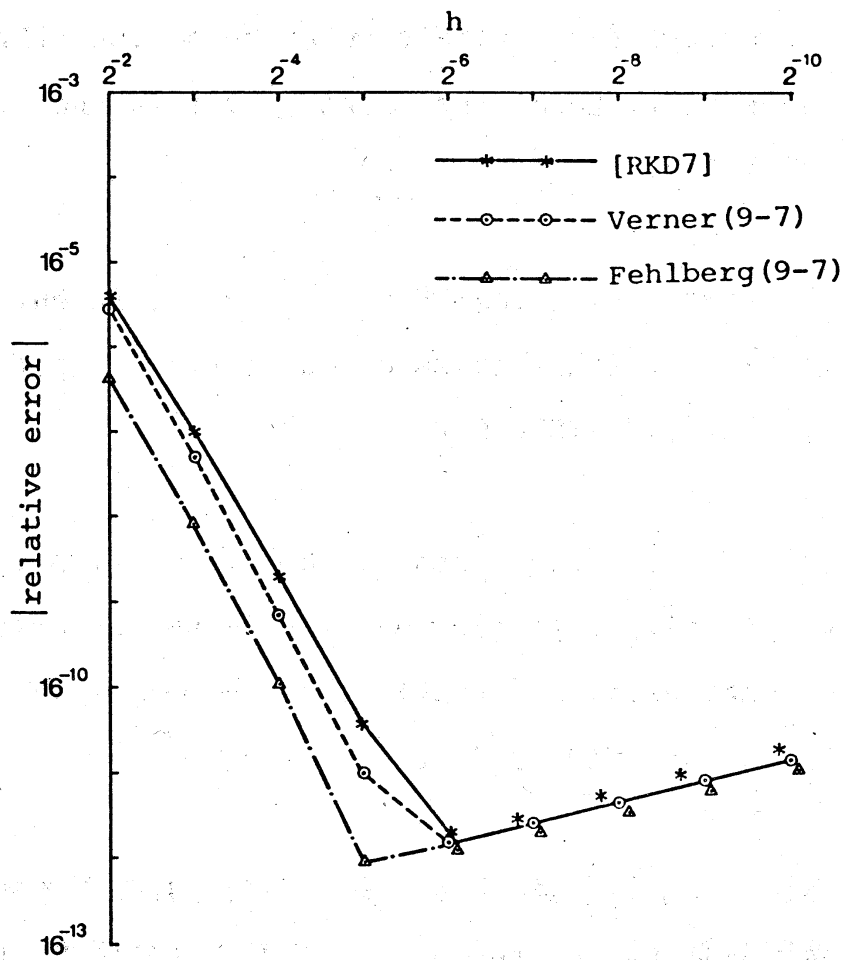


図 2. $t=1$ における累積誤差の比較

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