

Time periodic flows of an incompressible viscous fluid in perturbed channels

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1 The time periodic Poiseuille flow

In this section, for a straight channel in $\mathbb{R}^n (n = 2, 3)$, which is parallel to the x_1 -axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the x_1 -axis.

In the case $n = 2$, for $a > 0$ we suppose $\Sigma := (-a, a)$. In the case $n = 3$, we suppose that Σ is a bounded smooth simply connected domain in $\mathbb{R}^2$. We write

$$\omega = \mathbb{R} \times \Sigma.$$

Σ is a cross section of the channel ω .

In ω , we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \mathbb{R} \times \omega, \quad (1.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \mathbb{R} \times \omega, \quad (1.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \mathbb{R} \times \partial\omega \quad (1.3)$$

with the time periodic condition and the flux condition

$$\mathbf{u}(t) = \mathbf{u}(t + T) \quad \text{in } \omega \quad (1.4)$$

$$\int_{\Sigma} \mathbf{u}(t) \cdot \mathbf{n} dS = \alpha(t) \quad (t \in \mathbb{R}), \quad (1.5)$$

where $\mathbf{u} = \mathbf{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of the fluid motion in ω , respectively, ν is the given viscosity constant, $T (> 0)$ is a given constant, $\mathbf{n}$ is the unit parallel vector to the x_1 -axis and $\alpha(t)$ is a given T -periodic real function.

Since we look for a solution parallel to the x_1 -axis, we may assume that

$$\mathbf{u}(t, x) = (v(t, x), 0) \quad (n = 2),$$

$$\mathbf{u}(t, x) = (v(t, x), 0, 0) \quad (n = 3).$$

Then it follows that v does not depend on x_1 from (1.2), $(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{0}$ and p depends only on t and x_1 from (1.1). Therefore we obtain the equation

$$\frac{\partial v}{\partial t} - \nu \Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in } \mathbb{R} \times \Sigma, \quad (1.6)$$

where $\Delta = \partial^2/\partial x_2^2$ ($n = 2$), $\Delta = \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ ($n = 3$). It is easy to see that v does not depend on x_1 and p depends only on t and x_1 . Therefore it follows from the equation (1.6) that $\partial v/\partial t - \nu \Delta v$ and $\partial p/\partial x_1$ depends only on t . Integrating (1.6) on Σ , we obtain

$$p(t, x_1) = -\frac{1}{|\Sigma|} \left(\alpha'(t) - \nu \int_{\Sigma} \Delta v(t) dS \right),$$

where $|\Sigma|$ is the Lebesgue measure of Σ . Therefore there exists a time periodic solution $\mathbf{u}$ of the Navier-Stokes equations (1.1)–(1.5) in ω , with the form $\mathbf{u} = (v, 0)$ or $\mathbf{u} = (v, 0, 0)$, if and only if v is a solution of the problem

$$v' + \nu A v - \frac{\nu}{|\Sigma|} (A v, e) e = \frac{\alpha'}{|\Sigma|} e \quad (1.7)$$

with the time periodic condition and the flux condition

$$v(t) = v(t + T) \quad (t \in \mathbb{R}), \quad (1.8)$$

$$(v(t), e) = \alpha(t) \quad (t \in \mathbb{R}), \quad (1.9)$$

where $e(y) = 1$ ($y \in \Sigma$), $A = -\Delta$ with the domain $D(A) = H^2(\Sigma) \cap H_0^1(\Sigma)$, $(v, e) = \int_{\Sigma} v e dS$.

Before stating the time periodic result, we introduce the function space. Let X be a Banach space. We set

$$\begin{aligned} H_{\pi}^1(\mathbb{R}) &= \{\varphi \in H_{\text{loc}}^1(\mathbb{R}); \varphi(t) = \varphi(t + T) \text{ a.e. } t \in \mathbb{R}\}, \\ L_{\pi}^2(\mathbb{R}; X) &= \{\varphi \in L_{\text{loc}}^2(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for a.e. } t \in \mathbb{R}\}, \\ C_{\pi}(\mathbb{R}; X) &= \{\varphi \in C(\mathbb{R}; X); \varphi(t) = \varphi(t + T) \text{ in } X \text{ for } t \in \mathbb{R}\}. \end{aligned}$$

Beirão da Veiga [4] proved that for $n \geq 2$ if a flux $\alpha \in H_{\pi}^1(\mathbb{R})$ is given, then there exists a unique time periodic solution v^{α} of this problem (1.7)–(1.9) satisfying

$$\begin{aligned} v^{\alpha} &\in L_{\pi}^2(\mathbb{R}; H_0^1(\Sigma) \cap H^2(\Sigma)) \cap C_{\pi}(\mathbb{R}; H_0^1(\Sigma)), \\ (v^{\alpha})' &\in L_{\pi}^2(\mathbb{R}; L^2(\Sigma)). \end{aligned}$$

Set

$$\begin{aligned} \mathbf{V}^{\alpha}(t, x) &= (v^{\alpha}(t, x), 0) \quad (n = 2), \\ \mathbf{V}^{\alpha}(t, x) &= (v^{\alpha}(t, x), 0, 0) \quad (n = 3). \end{aligned}$$

Let us call $\mathbf{V}^{\alpha}$ “the time periodic Poiseuille flow”.

2 Problem in a perturbed channel

Let Ω be a smooth and unbounded domain in $\mathbb{R}^n$ ($n = 2, 3$) and $\partial\Omega$ be the boundary of the domain Ω . A domain Ω is called a perturbed channel if Ω satisfies

$$\Omega \setminus B(0, R) = \omega \setminus B(0, R) (=:\omega_0),$$

where $B(0, R) = \{x \in \mathbb{R}^n; |x| < R\}$. ω_0 is a perturbed and bounded part, ω_L is channel parts. The boundary $\partial\Omega$ of Ω has connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_J$ of C^∞ -surface such that $\Gamma_1, \dots, \Gamma_J$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and such that $\partial\Omega = \bigcup_{j=0}^J \Gamma_j$. Let us call the domain Ω “a perturbed channel”.

In the domain Ω , we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } (0, T) \times \Omega, \quad (2.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega \quad (2.2)$$

with the boundary condition

$$\mathbf{u} = \boldsymbol{\beta} \quad \text{on } (0, T) \times \partial\Omega, \quad (2.3)$$

$$\mathbf{u} \rightarrow \mathbf{V}^\alpha \quad \text{as } |x| \rightarrow \infty \quad \text{in } \omega_L \quad (2.4)$$

and the time periodic condition

$$\mathbf{u}(0) = \mathbf{u}(T) \quad \text{in } \Omega, \quad (2.5)$$

where $\mathbf{u} = \mathbf{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in Ω respectively, while $\nu > 0$ is the kinematic viscosity, $\mathbf{f} = \mathbf{f}(t, x)$ is the given external force and $\boldsymbol{\beta} = \boldsymbol{\beta}(t, x)$ is the given function on $(0, T) \times \partial\Omega$ with compact support. Since the solution $\mathbf{u}(t)$ satisfies $\operatorname{div} \mathbf{u}(t) = 0$ in Ω for a fixed $t \in (0, T)$, the given boundary data $\boldsymbol{\beta}(t)$ on $\partial\Omega$ is required to fulfill the compatibility condition which is called “General Outflow Condition” (*GOC*)

$$\int_{\partial\Omega} \boldsymbol{\beta}(t) \cdot \mathbf{n} d\sigma = 0, \quad (2.6)$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. The purpose is that if the given boundary data $\boldsymbol{\beta}$ satisfies (*GOC*), we will seek a solution of (2.1)-(2.5).

We introduce some function spaces. $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ is the set of all real smooth vector functions with compact support in Ω and $\operatorname{div} \boldsymbol{\varphi} = 0$. $\mathbb{L}_\sigma^2(\Omega)$ is the closure of $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual $\mathbb{L}^2(\Omega)$ norm. The $\mathbb{L}^2$ inner product and norm on Ω are denoted as $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_{2,\Omega}$ respectively. $\mathbb{H}_0^1(\Omega)$ and $\mathbb{H}_{0,\sigma}^1(\Omega)$ are the closures of $\mathbb{C}_0^\infty(\Omega)$ and $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual Dirichlet norm $\|\nabla \cdot\|_{2,\Omega}$, respectively. $\mathbb{H}_\sigma^1(\Omega)$ is the set of all $\mathbb{H}^1(\Omega)$ functions with $\operatorname{div} \boldsymbol{\varphi} = 0$. Let X be a Banach space. $C_\pi([0, T]; X)$ and $H_\pi^1((0, T); X)$ are the set of all the $C([0, T]; X)$ and $H^1((0, T); X)$ functions satisfying the time periodic condition $\mathbf{u}(0) = \mathbf{u}(T)$ in X .

3 Result

Our definition of a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) is as follows.

Definition 3.1 A measurable function $\mathbf{u} = \mathbf{u}(t, x)$ on $(0, T) \times \Omega$ is called a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) if $\mathbf{u}$ satisfies the following condition.

- (1) $\mathbf{v} := \mathbf{u} - \hat{\mathbf{V}}^\alpha - \mathbf{b} \in L^2((0, T); \mathbb{H}_{0,\sigma}^1(\Omega)) \cap L^\infty((0, T); \mathbb{L}_\sigma^2(\Omega))$.
 - (2) $\mathbf{u}$ satisfies $\frac{d}{dt}(\mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \boldsymbol{\varphi}) = {}_{(\mathbb{H}_{0,\sigma}^1(\Omega))'} \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\mathbb{H}_{0,\sigma}^1(\Omega)}$ ($\boldsymbol{\varphi} \in \mathbb{H}_{0,\sigma}^1(\Omega)$).
 - (3) $\mathbf{v}(0) = \mathbf{v}(T) \in \mathbb{L}^2(\Omega)$,
- where the function $\hat{\mathbf{V}}^\alpha$ and $\mathbf{b}$ are to be such that

$$\begin{aligned} \operatorname{div} \hat{\mathbf{V}}^\alpha &= 0 & \text{in } \Omega \\ \hat{\mathbf{V}}^\alpha &= \mathbf{0} & \text{on } \partial\Omega, \\ \hat{\mathbf{V}}^\alpha &= \mathbf{V}^\alpha & \text{in } \omega_L, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} \mathbf{b} &= 0 & \text{in } \Omega, \\ \mathbf{b} &= \boldsymbol{\beta} & \text{on } \partial\Omega. \end{aligned}$$

$\mathbf{V}^\alpha$ is “the extended time periodic Poiseuille flow” and $\mathbf{b}$ is “the boundary extension”.

Before stating our result, we define a constant concerning the time periodic Poiseuille flow.

Definition 3.2 We set

$$\gamma^\alpha(t) = \sup_{\boldsymbol{\varphi} \in \mathbb{H}_{0,\sigma}^1(\omega)} \frac{((\boldsymbol{\varphi} \cdot \nabla) \boldsymbol{\varphi}, \mathbf{V}^\alpha(t))_\omega}{\|\nabla \boldsymbol{\varphi}\|_{2,\omega}^2} \quad (t \in [0, T]), \quad (3.1)$$

$$\hat{\gamma}^\alpha := \sup_{t \in [0, T]} \gamma^\alpha(t). \quad (3.2)$$

We have the following result.

Theorem 3.1 (T. Kobayashi[13])

Suppose that $\hat{\gamma}^\alpha < \nu$, $\mathbf{f} \in L^2((0, T); (\mathbb{H}_{0,\sigma}^1(\Omega))')$ and $\boldsymbol{\beta} = \mathbf{0}$. Then there exists a time periodic weak solution.

This result is not the problem of (GOC) because $\boldsymbol{\beta} = \mathbf{0}$. We need the following assumption.

Assumption 3.1 Ω is a two dimensional symmetric domain with respect to the x_1 -axis and all the inner boundaries Γ_j ($1 \leq j \leq J$) intersect the x_1 -axis.

Theorem 3.2 (T. Kobayashi[14])

We assume that the domain Ω satisfies Assumption 3.1. We suppose that $\hat{\gamma}^\alpha < \nu$, $\mathbf{f} \in L^2((0, T); (\mathbb{H}_{0,\sigma}^1(\Omega))')$, $\boldsymbol{\beta} \in H_\pi^{\frac{1}{2},S}((0, T); \mathbb{H}_{\frac{1}{2},S}^{\frac{1}{2}}(\partial\Omega))$ with compact support, (GOC) and

$$\int_{\Gamma_0^+} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = 0 \quad \text{on } [0, T].$$

Then there exists a time periodic weak solution of the Navier-Stokes equations.

We need an appropriate extension of the given boundary data β .

Proposition 3.1 *We assume that a domain Ω satisfies Assumption 3.1. Suppose that $\beta \in H_\pi^1((0, T); \mathbb{H}^{\frac{1}{2}, S}(\partial\Omega))$ satisfies (GOC), the support of β is compact and*

$$\int_{\Gamma_0^+} \beta \cdot \mathbf{n} d\sigma = \int_{\Gamma_0^-} \beta \cdot \mathbf{n} d\sigma = 0 \quad \text{on } [0, T].$$

Then for any $\varepsilon > 0$ there exists an extension $\mathbf{b}_\varepsilon \in H_\pi^1((0, T); \mathbb{H}_\sigma^{1, S}(\Omega))$ of β such that $\mathbf{b}_\varepsilon$ has compact support and the inequality

$$|((\mathbf{v} \cdot \nabla) \mathbf{v}, \mathbf{b}_\varepsilon(t))| < \varepsilon \|\nabla \mathbf{v}\|_{2, \Omega}^2 \quad (\mathbf{v} \in \mathbb{H}_{0, \sigma}^{1, S}(\Omega), t \in [0, T]) \quad (3.3)$$

holds true.

The estimate (3.3) is “Leray’s inequality”. The estimate (3.3) is its symmetric version in an unbounded perturbed channel.

Remark 3.1 *In this paper, the domain Ω has two outlets. We can solve K ($K \geq 3$) outlets problem. We consider a straight channel ω_i ($i = 1, \dots, K$), where Σ_i is a cross section of ω_i as Section 1 and the center line of ω_i may not be parallel to the x_1 -axis. We assume that a given flux function $\alpha_i \in H_\pi^1(\mathbb{R})$ ($i = 1, \dots, K$) satisfies $\sum_{i=1}^K \alpha_i(t) = 0$ ($t \in \mathbb{R}$). For each α_i , we have the time periodic Poiseuille flow $\mathbf{V}_i^\alpha$ in ω_i . We assume that Ω has K outlets ω_{0i} ($i = 1, \dots, K$) where ω_{0i} is a semi-infinite channel with the cross section Σ_i . In the domain Ω , we consider a time periodic problem with the time periodic Poiseuille flow $\mathbf{V}_i^\alpha$. We define constant $\hat{\gamma} = \max_{1 \leq i \leq K} \{\hat{\gamma}_i^\alpha\}$ as Definition 3.2. Suppose that $\hat{\gamma} < \nu$. Then there exists a time periodic weak solution in Ω with K outlets.*

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