

STEPPING-UP SEMIPROPERNESS OF NAMBA FORCINGS

KENTA TSUKUURA

ABSTRACT. We show that if both of $\text{Nm}(\kappa, \lambda)$ and $\text{Nm}(\lambda^+)$ are semiproper then $\text{Nm}(\kappa, \lambda^+)$ is semiproper.

1. INTRODUCTION

In [3], we introduced Namba forcings $\text{Nm}(\kappa, \lambda)$ over $\mathcal{P}_\kappa\lambda$ and studied these. We also showed that for regular cardinals $\aleph_2 \leq \kappa \leq \lambda$, the following holds.

- (1) $\text{Nm}(\kappa, \lambda)$ is semiproper if $\text{Nm}(\kappa, 2^{\lambda^{<\kappa}})$ is semiproper.
- (2) $\text{Nm}(\kappa, \lambda)$ forces that $\text{cf}(\delta) = \omega$ for all $\delta \in [\kappa, \lambda] \cap \text{Reg}$.

These properties and [2, Theorem 2.2 of Chapter XI] shows that $\text{Nm}(\kappa, \lambda)$ and $\text{Nm}(\lambda^+)$ are semiproper if $\text{Nm}(\kappa, \lambda^+)$ is semiproper and $2^\lambda = \lambda^+$.

In this paper, we show the converse of this as follows.

Theorem 1.1. *Suppose that $\kappa \leq \lambda$ are regular cardinals greater than $\aleph_2$. If both of $\text{Nm}(\kappa, \lambda)$ and $\text{Nm}(\lambda^+)$ are semiproper then $\text{Nm}(\kappa, \lambda^+)$ is semiproper.*

The structure of this paper is as follows. In Section 2, we recall the basic materials for Namba forcings. Section 3 is devoted to Theorem 1.1.

2. PRELIMINARIES

We use [1] as a reference for set theory in general. In this paper, κ and λ denote regular cardinals greater than $\aleph_2$ unless otherwise stated. Reg is the class of regular cardinals.

A tree $p \subseteq [X]^{<\omega}$ is a set that is closed under the initial segment, that is, $s \in p$ and $l \sqsubseteq s$ implies $l \in p$. By $l \sqsubseteq s$, we mean that s end-extends l . Note that, when we consider a tree structure or an end-extension relation $l \sqsubseteq s$, we assume that X has an ordered relation and l and s are finite increasing sequences in the sense of a fixed order. In this paper, X is λ or $\mathcal{P}_\kappa\lambda$. These have natural orders.

2.1. Namba forcings. Namba forcing $\text{Nm}(\lambda)$ is the set of all Namba trees over λ . Namba tree is a tree $p \subseteq [\lambda]^{<\omega}$ with the following conditions:

- (1) p has a trunk $\text{tr}(p)$, that is, $\text{tr}(p)$ is the maximal $t \in p$ such that $\forall s \in p (s \subseteq t \vee t \subseteq s)$.
- (2) For all $s \in p$, if $s \supseteq \text{tr}(p)$ then $\text{Suc}(s) = \{\xi < \lambda \mid s \smallfrown \langle \xi \rangle \in p\}$ is unbounded in λ .

$\text{Nm}(\lambda)$ is ordered by inclusion.

Namba forcing $\text{Nm}(\kappa, \lambda)$ is the set of all Namba trees over $\mathcal{P}_\kappa\lambda$. Then Namba tree is a tree $p \subseteq [\mathcal{P}_\kappa\lambda]^{<\omega}$ with the following conditions:

2020 *Mathematics Subject Classification.* 03E35, 03E55.

Key words and phrases. Namba forcing, semiproper forcing.

- (1) p has a trunk $\text{tr}(p)$, that is, $\text{tr}(p)$ is the maximal $t \in p$ such that $\forall s \in p (s \subseteq t \vee t \subseteq s)$.
- (2) For all $s \in p$, if $s \supseteq \text{tr}(p)$ then $\text{Suc}(s) = \{a \in \mathcal{P}_\kappa \lambda \mid s \frown \langle a \rangle \in p\}$ is unbounded in $\mathcal{P}_\kappa \lambda$.

Here, $[\mathcal{P}_\kappa \lambda]_{\subseteq}^{\leq \omega}$ is the set of all $\subseteq$ -increasing finite set of subsets of $\mathcal{P}_\kappa \lambda$. $\text{Nm}(\kappa, \lambda)$ is ordered by inclusion.

2.2. Galvin games. For an ideal I over Z and $A \in I^+$, the Galvin game $\mathcal{G}(I, A)$ is a game of length ω with two players as follows:

Player I	$F_0 : A \rightarrow \omega_1$		$\cdots$	$F_i : A \rightarrow \omega_1$		$\cdots$
Player II		$\xi_0 < \omega_1$	$\cdots$		$\xi_i < \omega_1$	$\cdots$

Let $\xi = \sup_n \xi_n$. II wins if $\bigcap_{n < \omega} F_n^{-1} \xi \in I^+$. $\Phi_{\kappa, \lambda}$ is the statement that claims Player II has a winning strategy for $\mathcal{G}(J_{\kappa \lambda}^{\text{bd}}, A)$ for all $A \in (J_{\kappa \lambda}^{\text{bd}})^+$. Φ_λ is the statement that claims Player II has a winning strategy for $\mathcal{G}(J_\lambda^{\text{bd}}, A)$ for all $A \in (J_\lambda^{\text{bd}})^+$. Here, $J_{\kappa \lambda}^{\text{bd}}$ and J_λ^{bd} are bounded ideals over $\mathcal{P}_\kappa \lambda$ and λ , respectively. The importance is as follows.

Lemma 2.1 (Shelah). *The following are equivalent.*

- (1) Φ_λ .
- (2) $\text{Nm}(\lambda)$ is semiproper.

Lemma 2.2 is an analogue of Lemma 2.1.

Lemma 2.2. *The following are equivalent.*

- (1) $\Phi_{\kappa \lambda}$.
- (2) $\text{Nm}(\kappa, \lambda)$ is semiproper.

Proof. See [3]. □

3. PROOF OF MAIN THEOREM

By Lemmas 2.1 and 2.2, for Theorem 1.1, it is enough to show

Lemma 3.1. $\Phi_{\kappa \lambda} \wedge \Phi_{\lambda^+} \rightarrow \Phi_{\kappa \lambda^+}$.

Proof. For $\lambda \leq \alpha < \lambda^+$, let $\varphi_\alpha : \alpha \rightarrow \lambda$ be a bijection. For an unbounded set $A \subseteq \mathcal{P}_\kappa \lambda^+$, $A^{\upharpoonright \alpha} = \{\varphi_\alpha \restriction (a \cap \alpha) \mid a \in A\}$ is unbounded in $\mathcal{P}_\kappa \lambda$. Let ws_α be Player II's winning strategy for $\mathcal{G}(J_{\kappa \lambda}^{\text{bd}}, A^{\upharpoonright \alpha})$ for each α . Let ws^* be Player II's winning strategy for $\mathcal{G}(J_{\kappa \lambda}^{\text{bd}}, \lambda)$.

For each $a \in A^{\upharpoonright \alpha}$, by the definition, there is a $b \in A$ such that $a = \varphi_\alpha \restriction (b \cap \alpha)$. We fix a such b as $b_{a, \alpha}$. For an $F : A \rightarrow \omega_1$, define $F^\alpha : A^{\upharpoonright \alpha} \rightarrow \omega_1$ by $F^\alpha(a) = F(b_{a, \alpha})$.

Let us define a strategy of Player II for $\mathcal{G}(J_{\kappa \lambda^+}^{\text{bd}}, A)$. Suppose that $F_0, \dots, F_n$ be Player I's partial play. Note that, for each $\alpha < \lambda^+$, Player II knows $\xi_{i, \alpha} = \text{ws}_\alpha(F_0^{\upharpoonright \alpha}, \dots, F_i^{\upharpoonright \alpha})$ for each $i \leq n$. Then $H_i(\alpha) = \xi_{i, \alpha}$ defines a mapping $H_i : \lambda^+ \rightarrow \omega_1$. Let us instruct Player II to choose $\xi_n = \text{ws}^*(H_0, \dots, H_n)$, so we define $\text{ws}(F_0, \dots, F_n) = \xi_n$.

Suppose that Player I choosed $F_0, F_1, \dots$ and Player II used ws . We claim that Player II wins. Let $\xi_\infty = \sup_i \xi_i$, here $\xi_n = \text{ws}(F_0, \dots, F_n)$. The goal is showing that $\bigcap_i F_i^{-1} \xi_\infty$ is unbounded.

For a $c \subseteq \mathcal{P}_\kappa \lambda^+$, there is an α such that $c \subseteq \alpha$ and $\alpha \in \bigcap_i H_i^{-1} \xi_\infty$. Note that $\bigcap_i H_i^{-1} \xi_\infty$ is unbounded subset of λ^+ since ws^* is a winning strategy. So we have

$$\bigcap_i (F_i^{\uparrow\alpha})^{-1}(\sup_i \xi_{\alpha,i}) \subseteq \bigcap_i (F_i^{\uparrow\alpha})^{-1}\xi_\infty.$$

Since ws_α is a winning strategy, these are unbounded subset of $\mathcal{P}_\kappa\lambda$. Note that $\bigcap_i (F_i^{\uparrow\alpha})^{-1}\xi_\infty \subseteq A^{\uparrow\alpha}$. Then there is an $a \in \bigcap_i (F_i^{\uparrow\alpha})^{-1}\xi_\infty$ such that $\varphi_\alpha "c \subseteq a$. Then $\xi_\infty > F_i^{\uparrow\alpha}(a) = F_i(b_{a,\alpha})$. By the choice of $b_{a,\alpha}$, we have $b_{a,\alpha} \in A$ and $a = \varphi_\alpha "(b_{a,\alpha} \cap \alpha)$. Therefore

$$\varphi_\alpha "c = \varphi_\alpha "(c \cap \alpha) \subseteq a = \varphi_\alpha "(b_{a,\alpha} \cap \alpha).$$

In particular, $c \subseteq b_{a,\alpha} \in \bigcap_i F_i^{-1}\xi_\infty$, as desired. $\square$

In [3], we show that the “local” semiproperness of Namba forcings can be characterized in terms of semistationary reflection principles.

Theorem 3.2 (Tsukuura [3]). *If $\alpha^\omega < \kappa$ for all $\alpha < \kappa$ then the following are equivalent.*

- (1) $\text{Nm}(\kappa, \lambda)$ preserves the semistationarity of any subset of $[\lambda]^\omega$.
- (2) $\text{SSR}([\lambda]^\omega, <\kappa)$.

$\text{SSR}([\lambda]^\omega, <\kappa)$ is the statement that claims, for every semistationary subset $S \subseteq [\lambda]^\omega$, there is an $R \in \mathcal{P}_\kappa\lambda$ with the following properties:

- (1) $\omega_1 \subseteq R \cap \kappa \in \kappa$
- (2) $S \cap [R]^\omega$ is semistationary.

We call the local semiproperness of $\text{Nm}(\kappa, \lambda)$ (1) of Theorem 3.2. Using this, we can obtain an analogue of Theorem 1.1 for the local semiproperness.

Theorem 3.3. *Suppose $\alpha^\omega < \kappa$ for all $\alpha < \kappa$. If $\text{Nm}(\kappa, \lambda)$ and $\text{Nm}(\lambda^+)$ are locally semiproper then $\text{Nm}(\kappa, \lambda^+)$ is locally semiproper.*

Proof. By Theorem 3.2, $\text{SSR}([\lambda]^\omega, <\kappa)$ and $\text{SSR}([\lambda]^+, <\lambda^+)$ holds. It is easy to see that $\text{SSR}([\lambda^+]^\omega, <\kappa)$ holds. Again, by Theorem 3.2, $\text{Nm}(\kappa, \lambda^+)$ is locally semiproper. $\square$

REFERENCES

- [1] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.
- [2] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
- [3] Kenta Tsukuura. On semiproperness of Namba forcings and ideals in Prikry extensions. *arXiv preprint arXiv:2311.11498*, 2023.

DEPARTMENT OF INDUSTRIAL & SYSTEMS ENGINEERING, FACULTY OF SCIENCE AND ENGINEERING, HOSEI UNIVERSITY, 184-8584, JAPAN

Email address: kenta.tsukura.85@hosei.ac.jp