

SOME RESULTS IN THE EXTENSION WITH A COHERENT SUSLIN TREE

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ABSTRACT. We show that under $\text{PFA}(S)$, the coherent Suslin tree S (which is a witness of the axiom $\text{PFA}(S)$) forces that there are no ω_2 -Aronszajn trees. We also determine the values of cardinal invariants of the continuum in this extension.

1. INTRODUCTION

In [20], Stevo Todorćević introduced the forcing axiom $\text{PFA}(S)$, which says that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing which preserves S to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves S to be Suslin and $\aleph_1$ -many dense subsets D_α , $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all the D_α . $\text{PFA}(S)[S]$ denotes the forcing extension with the coherent Suslin tree S which is a witness of $\text{PFA}(S)$. Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to Tadatashi Miyamoto [15]), it is consistent relative to some large cardinal assumption that $\text{PFA}(S)$ holds.

The first appearance of such a forcing axiom is in the paper [13] due to Paul B. Larson and Todorćević. In this paper, they introduced the weak version of $\text{PFA}(S)$, called Souslin's Axiom (in which the properness is replaced by the cccness), and under this axiom, the coherent Suslin tree S , which is a witness of the axiom, forces a weak fragment of Martin's Axiom. In [20], it is also proved that under $\text{PFA}(S)$, S forces the open graph dichotomy ⁽¹⁾ and the P -ideal dichotomy. Namely, many consequences of PFA are satisfied in the extension with S under

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¹This is the so called open coloring axiom [18, §8].

$\text{PFA}(S)$. On the other hand, many people proved that some consequences from $\diamond$ are satisfied in the extension with a Suslin tree (e.g. [16, Theorem 6.15.]). In particular, the pseudo-intersection number $\mathfrak{p}$ is $\aleph_1$ in the extension with a Suslin tree. In fact, the extension with S under $\text{PFA}(S)$ is designed as a universe which satisfied some consequences of $\diamond$ and PFA simultaneously. By the use of this model, Larson and Todorćević proved that the affirmative answer to Katětov's problem is consistent [13].

In this note, we point out the values of cardinal invariants of the continuum (e.g. in [2, 6]) in the extension with S under $\text{PFA}(S)$. And we show that under $\text{PFA}(S)$, S forces that there are no ω_2 -Aronszajn trees. In [19], Todorćević demonstrated that many consequences of PFA are deduced from PID plus $\mathfrak{p} > \aleph_1$. In [17], the first author proved that PID plus $\mathfrak{p} > \aleph_1$ implies the failure of $\square_{\kappa, \omega_1}$ whenever $\text{cf}(\kappa) > \omega_1$. It is not yet known whether PID plus $\mathfrak{p} > \aleph_1$ implies the failure of $\square_{\omega_1, \omega_1}$. Since $\square_{\omega_1, \omega_1}$ is equivalent to the existence of a special ω_2 -Aronszajn tree, our result concludes that it is consistent that PID holds, $\mathfrak{p} = \aleph_1$ and $\square_{\omega_1, \omega_1}$ fails.

At last in the introduction, we introduce a coherent Suslin tree. A coherent Suslin tree S consists of functions in $\omega^{<\omega_1}$ and is closed under finite modifications. That is,

- for any s and t in S , $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S , the set

$$\{\alpha \in \min\{\text{lv}(s), \text{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$$

is finite (here, $\text{lv}(s)$ is the length of s , that is, the size of s), and

- for any $s \in S$ and $t \in \omega^{\text{lv}(s)}$, if the set $\{\alpha \in \text{lv}(s); s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$ also.

For a countable ordinal α , let S_α be the set of the α -th level nodes, that is, the set of all members of S of domain α , and let $S_{\leq \alpha} := \bigcup_{\beta \leq \alpha} S_\beta$. For $s \in S$, we let

$$S \restriction s := \{u \in S; s \leq_S u\}.$$

We note that $\diamond$, or adding a Cohen real, builds a coherent Suslin tree. A coherent Suslin tree has canonical commutative isomorphisms. Let s and t be nodes in S with the same level. Then we define a function $\psi_{s,t}$ from $S \restriction s$ into $S \restriction t$ such that for each $v \in S \restriction s$,

$$\psi_{s,t}(v) := t \cup (v \restriction [\text{lv}(s), \text{lv}(v)))$$

(here, $v \restriction [\text{lv}(s), \text{lv}(v))$ is the function v restricted to the domain $[\text{lv}(s), \text{lv}(v))$).

We note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S of

the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commute. (On a coherent Suslin tree, see e.g. [10, 12].)

2. CARDINAL INVARIANTS

Proposition 2.1 ([20, 4.3 Theorem]). *PFA(S) implies that $\mathfrak{p} = \text{add}(\mathcal{N}) = \mathfrak{c} = \aleph_2$ holds.*

Proof. A forcing with property K doesn't destroy a Suslin tree ([14, Theorem 11.]). So, since a σ -centered forcing satisfies property K and $\mathfrak{p} = \mathfrak{m}(\sigma\text{-centered})$ (due to Bell, see e.g. in [6, 7.12 Theorem]), PFA(S) implies $\mathfrak{p} > \aleph_1$.

To see that PFA(S) implies $\text{add}(\mathcal{N}) > \aleph_1$, here we consider the characterization of the additivity of the null ideal by the amoeba forcing $\mathbb{A}$ as follows (see [2, 6.1 Theorem] or [3, Theorem 3.4.17]).

$$\text{add}(\mathcal{N}) = \min \left\{ |\mathcal{D}| : \mathcal{D} \text{ is a set of dense subsets of } \mathbb{A} \text{ such that} \right. \\ \left. \text{there are no filters of } \mathbb{A} \text{ which meet every member of } \mathcal{D} \right\}.$$

Since the amoeba forcing is σ -linked (so satisfies property K), PFA(S) implies $\text{add}(\mathcal{N}) > \aleph_1$.

A proof that PFA(S) implies $\mathfrak{c} = \aleph_2$ is same to one for PFA due to Todorćević [5, 3.16 Theorem] (see also [9, Theorem 31.25]). We note that PFA(S) implies OCA ([8, Lemma 4]), so $\mathfrak{b} = \aleph_2$ holds ([18, 8.6 Theorem], also [9, Theorem 29.8]). In a proof that $\mathfrak{b} = \mathfrak{c}$ holds under PFA, an iteration of a σ -closed forcing and a ccc forcing which is defined by an unbounded family in ω^ω is used. A σ -closed forcing doesn't destroy a Suslin tree (see e.g. [15]). Since the cccness of the second iterand comes from the unboundedness of a family in ω^ω , this preserves a Suslin tree because a Suslin tree doesn't add new reals. So this iteration doesn't destroy a Suslin tree. Therefore $\mathfrak{b} = \mathfrak{c}$ holds under PFA(S). $\square$

Proposition 2.2 ([8, Lemma 2.]). *$\mathfrak{t} = \aleph_1$ holds in the extension with a Suslin tree.*

Proof. Suppose that T is a Suslin tree, and let π be an order preserving function from T into the order structure $([\omega]^{\aleph_0}, \supseteq^*)$ such that if members s and t of T are incomparable in T , then $\pi(s) \cap \pi(t)$ is finite. Then for a generic branch G through T , the set $\{\pi(s) : s \in G\}$ is a $\supseteq^*$ -decreasing sequence which doesn't have its lower bound in $[\omega]^{\aleph_0}$ (because T doesn't add new reals). $\square$

Proposition 2.3. *Under PFA(S), S forces that $\text{add}(\mathcal{N}) = \mathfrak{c} = \aleph_2$.*

Proof. Since S doesn't add new reals and preserves all cardinals, by Proposition 2.1, S forces that $\mathfrak{c} = \aleph_2$ ([20, 4.4 Corollary.]).

To see that S forces $\text{add}(\mathcal{N}) > \aleph_1$, here we consider another characterization of the additivity of the null ideal (see [1], also [2, 3]). A function in the set $\prod_{n \in \omega} ([\omega]^{\leq n+1} \setminus \{\emptyset\})$ is called a slalom, and for a function f in ω^ω and a slalom φ , we say that φ captures f (denoted by $f \sqsubseteq \varphi$) if for all but finitely many $n \in \omega$, $f(n) \in \varphi(n)$. Then

$$\text{add}(\mathcal{N}) = \min \left\{ |F| : F \subseteq \omega^\omega \right. \\ \left. \& \forall \varphi \in \prod_{n \in \omega} ([\omega]^{\leq n+1} \setminus \{\emptyset\}) \exists f \in F (f \not\sqsubseteq \varphi) \right\}.$$

Let $\dot{X}$ be an S -name for a set of $\aleph_1$ -many functions in ω^ω . For each $s \in S$, let

$$Y_s := \left\{ f \in \omega^\omega : s \Vdash_S "f \in \dot{X}" \right\}.$$

Since $\dot{X}$ is an S -name for a set of size $\aleph_1$, Y_s is of size at most $\aleph_1$ for each $s \in S$, so is the set $\bigcup_{s \in S} Y_s$. And we note that

$$\Vdash_S "\dot{X} \subseteq \bigcup_{s \in S} Y_s".$$

By $\text{add}(\mathcal{N}) > \aleph_1$ (Proposition 2.1), there exists a slalom φ which captures all functions in the set $\bigcup_{s \in S} Y_s$. Then

$$\Vdash_S "\varphi \text{ captures all functions in } \dot{X}",$$

which finishes the proof. $\square$

Proposition 2.4. *Under $\text{PFA}(S)$, S forces that $\mathfrak{h} = \aleph_2$.*

Proof. By Proposition 2.1, $\mathfrak{h} = \aleph_2$ holds in the ground model because of the inequality $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{c}$ (see e.g. [6, §6]).

Let $\dot{X}_\alpha$, for each $\alpha \in \omega_1$, be an S -name for a dense open subset of $[\omega]^{\aleph_0}$. For $\alpha \in \omega$ and $s \in S$, let

$$Y_{\alpha,s} := \left\{ x \in [\omega]^{\aleph_0} : \exists t \in S (s \leq_S t \& t \Vdash_S "x \in \dot{X}_\alpha") \right\}.$$

Then we note that each $Y_{\alpha,s}$ is a dense open subset of $[\omega]^{\aleph_0}$, and

$$\Vdash_S "\bigcap_{s \in S} Y_{\alpha,s} \subseteq \dot{X}_\alpha".$$

Since $\mathfrak{h} > \aleph_1$, for each $\alpha \in \omega_1$, the set $\bigcap_{\alpha \in \omega_1} \bigcap_{s \in S} Y_{\alpha, s}$ is a dense open subset of $[\omega]^{\aleph_0}$, in particular, it is nonempty. Therefore

$$\Vdash_S \text{ “ } \bigcap_{\alpha \in \omega_1} \dot{X}_\alpha \neq \emptyset \text{ ” ,}$$

which finishes the proof. $\square$

We note that $\mathfrak{h}$ is less than or equal to many standard cardinal invariants, like $\mathfrak{a}$, $\mathfrak{s}$, etc. See e.g. [3, 6, 7].

3. ω_2 -ARONSZAJN TREES

Theorem 3.1. *Under $\text{PFA}(S)$, S forces that there are no ω_2 -Aronszajn trees.*

Proof. An outline of the proof is same to the proof due to Baumgartner in [4] (see also [9, Theorem 31.32.]). So this theorem follows from the following two claims.

Claim 3.2. *Let $\mathbb{P}$ be a σ -closed forcing notion, and let $\dot{T}$ be an S -name for an ω_2 -Aronszajn tree. Then $\mathbb{P}$ adds no S -names for cofinal chains through $\dot{T}$ whenever $\mathfrak{c} > \aleph_1$ holds.*

Proof of Claim 3.2. At first, we see an easy proof by the result of product forcing ([9, Lemma 15.9] or [11, Ch.VIII, 1.4.Theorem]). We note that the two step iteration $\mathbb{P} * S$ is equal to the two step iteration $S * \mathbb{P}^V$ ⁽²⁾. In the extension with S , since $\mathfrak{c} > \aleph_1$, a σ -closed forcing $\mathbb{P}^V$ doesn't add a cofinal branch through the value of $\dot{T}$ by the generic of S , which is an ω_2 -Aronszajn tree (this can be proved as in [9, Lemma 27.10]). Therefore $\mathbb{P}$ doesn't add an S -name for a cofinal chain through $\dot{T}$.

At last, we see a direct proof. In fact, we show that if $\mathbb{P}$ is σ -closed and $\dot{T}$ is an S -name for an ω_2 -tree, then $\mathbb{P}$ adds no new S -names for cofinal chains through $\dot{T}$ whenever $\mathfrak{c} > \aleph_1$ holds.

Suppose that $\mathbb{P}$ adds a new S -name for a cofinal chain through $\dot{T}$, that is, there exists a sequence $\langle \dot{z}_\alpha; \alpha \in \omega_2 \rangle$ of $\mathbb{P}$ -names for S -names for members of $\dot{T}$ such that

$$\Vdash_{\mathbb{P}} \text{ “ } \Vdash_S \text{ “ } \forall \alpha < \beta < \omega_2, \dot{z}_\alpha <_{\dot{T}} \dot{z}_\beta \text{ ” ” }$$

and for every S -name $\dot{B}$ for a subset of $\dot{T}$ (in the ground model),

$$\Vdash_{\mathbb{P}} \text{ “ } \Vdash_S \text{ “ } \dot{B} \neq \{\dot{z}_\alpha; \alpha \in \omega_2\} \text{ ” ” }.$$

²In fact, in the first argument, we use a σ -forcing $\text{Fn}(\omega_1, \omega_2, \aleph_1)$, which collapses ω_2 to ω_1 by countable approximations. S doesn't add new countable sets, so $\text{Fn}(\omega_1, \omega_2, \aleph_1)$ doesn't change in the extension with S .

We note that we look at $\dot{T}$ as an object in the ground model even in the extension with $\mathbb{P}$. So for any $\mathbb{P}$ -name $\dot{t}$ for an S -name for a member of $\dot{T}$ and $p \in \mathbb{P}$, densely many extensions of p in $\mathbb{P}$ decides the value of $\dot{t}$ as an S -name for a member of $\dot{T}$. By induction on $\sigma \in 2^{<\omega}$, we choose a condition p_σ in $\mathbb{P}$, an S -name $\dot{x}_\sigma$ for a member of $\dot{T}$ and countable ordinals $\alpha_{|\sigma|}$ and $\beta_{|\sigma|}$ such that

- for σ and τ in $2^{<\omega}$ with $\sigma \subseteq \tau$, $p_\tau \leq_{\mathbb{P}} p_\sigma$,
- $\Vdash_{\mathbb{P}} \text{ “ } \Vdash_S \dot{x}_\sigma \in \{\dot{z}_\alpha; \alpha \in \omega_2\} \text{ ” }$ for each $\sigma \in 2^{<\omega}$,
- $\Vdash_S \text{ “ both } \dot{x}_{\sigma \smallfrown \langle 0 \rangle} \text{ and } \dot{x}_{\sigma \smallfrown \langle 1 \rangle} \text{ are above } \dot{x}_\sigma \text{ in } \dot{T} \text{ ” }$ for each $\sigma \in 2^{<\omega}$,
- $\Vdash_S \text{ “ } \dot{x}_{\sigma \smallfrown \langle 0 \rangle} \text{ and } \dot{x}_{\sigma \smallfrown \langle 1 \rangle} \text{ are incomparable in } \dot{T} \text{ ” }$ for each $\sigma \in 2^{<\omega}$,
- for each $n \in \omega$ and $\sigma \in 2^n$, every α_n -th level node of S decides the value of $\dot{x}_\sigma$ which is of level less than β_n in $\dot{T}$.

This can be done because of the property of the sequence $\langle \dot{z}_\alpha; \alpha \in \omega_2 \rangle$ and the cccness of S as a forcing notion.

Since $\mathbb{P}$ is σ -closed, for any $f \in 2^\omega$, there is $p_f \in \mathbb{P}$ such that $p_f \leq_{\mathbb{P}} p_{f \restriction n}$ holds for every $n \in \omega$. Since it is forced with $\mathbb{P}$ that $\langle \dot{z}_\alpha; \alpha \in \omega_2 \rangle$ is a cofinal chain through $\dot{T}$, there exists an S -name $\dot{x}_f$ for a member of $\dot{T}$ which is of level $\sup_{n \in \omega} \beta_n$ such that

$$p_f \Vdash_{\mathbb{P}} \text{ “ } \Vdash_S \dot{x}_f \in \{\dot{z}_\alpha; \alpha \in \omega_2\} \text{ ” }.$$

Then it holds that

$$p_f \Vdash_{\mathbb{P}} \text{ “ } \Vdash_S \dot{x}_f \text{ is above } \dot{x}_{f \restriction n} \text{ in } \dot{T} \text{ for every } n \in \omega \text{ ” }.$$

We note that the phrase $\Vdash_S \dot{x}_f \text{ is above } \dot{x}_{f \restriction n} \text{ in } \dot{T} \text{ for every } n \in \omega$ is also true in the ground model, so we conclude that

$$\begin{aligned} \Vdash_S \text{ “ } \{\dot{x}_f : f \in 2^\omega\} \text{ is a subset of the set of the members of } \dot{T} \\ \text{ whose levels are } \sup_{n \in \omega} \beta_n, \text{ and is of size } \mathfrak{c} > \aleph_1 \text{ ” }, \end{aligned}$$

which contradicts to the assumption that $\dot{T}$ is an S -name for an ω_2 -tree.

⊢ **Claim 3.2**

Claim 3.3. *Let $\dot{T}$ be an S -name for a tree of size $\aleph_1$ and of height ω_1 which doesn't have uncountable (i.e. cofinal) chains through $\dot{T}$. Then there exists a ccc forcing notion which preserves S to be Suslin and forces $\dot{T}$ to be special (i.e. to be a union of countably many antichains through $\dot{T}$).*

We note that this claim has been known if $\dot{T}$ is an S -name for an ω_1 -Aronszajn tree.

Proof of Claim 3.3. For simplicity, we assume that $\dot{T}$ is an S -name for an order structure on ω_1 , that is, $\dot{<}_T$ is an S -name such that

$$\Vdash_S "\dot{T} = \langle \omega_1, \dot{<}_T \rangle",$$

and that for any $s \in S$ and α, β in ω_1 , if $s \Vdash_S "\alpha \not\dot{<}_T \beta"$ and $\alpha < \beta$, then $s \Vdash_S "\alpha \dot{<}_T \beta"$. Since S is a ccc forcing notion, there exists a club C on ω_1 such that for every $\delta \in C$, every node of S of level δ decides $\dot{<}_T \cap (\delta \times \delta)$.

We define the forcing notion $\mathbb{Q}(\dot{T}, C) = \mathbb{Q}$ which consists of finite partial functions p from S into the set $\bigcup_{\sigma \in [\omega]^{<\aleph_0}} ([\omega_1]^{<\aleph_0})^\sigma$ such that

- for every $s \in \text{dom}(p)$ and $n \in \text{dom}(p(s))$,

$$p(s)(n) \subseteq \sup(C \cap \text{lv}(s))$$

and

$$s \Vdash_S "p(s)(n) \text{ is an antichain in } \dot{T}",$$

- for every s and t in $\text{dom}(p)$, if $s <_S t$, then for every $n \in \text{dom}(p(s)) \cap \text{dom}(p(t))$,

$$t \Vdash_S "p(s)(n) \cup p(t)(n) \text{ is an antichain in } \dot{T}",$$

ordered by extensions, that is, for each p and q in $\mathbb{Q}$,

$$p \leq_{\mathbb{Q}} q : \iff p \supseteq q.$$

We note that $\mathbb{Q}$ adds an S -name which witnesses that $\dot{T}$ is special in the extension with S . We will show that if $\mathbb{Q} \times S$ has an uncountable antichain, then some node of S forces that $\dot{T}$ has an uncountable chain, which finishes the proof of the claim.

Suppose that a family $\{\langle p_\xi, s_\xi \rangle : \xi \in \omega_1\}$ is an uncountable antichain in $\mathbb{Q} \times S$. By shrinking it and extending each member of the family if necessary, we may assume that

- for each $\xi \in \omega_1$, $\text{dom}(p_\xi) \subseteq S_{\leq \delta_\xi}$ for some $\delta_\xi \in \omega_1$,
- the sequence $\langle \delta_\xi; \xi \in \omega_1 \rangle$ is strictly increasing,
- for each $\xi \in \omega_1$ and $s \in \text{dom}(p_\xi)$, there exists $t \in \text{dom}(p_\xi) \cap S_{\delta_\xi}$ such that $s \leq_S t$,
- for each $\xi \in \omega_1$, $s \in \text{dom}(p_\xi)$ and $t \in \text{dom}(p_\xi) \cap S_{\delta_\xi}$, if $s \leq_S t$, then $p_\xi(s) \subseteq p_\xi(t)$,
- all sets $\text{dom}(p_\xi) \cap S_{\delta_\xi}$ are of size n , and say $\text{dom}(p_\xi) \cap S_{\delta_\xi} = \{t_i^\xi : i \in n\}$,

- for each $i \in n$, all $\text{dom}(p_\xi(t_i^\xi))$ are same, call it σ_i , and for each $k \in \sigma_i$, the size of each $p_\xi(t_i^\xi)(k)$ is constant, call it $m_{i,k}$ and say $p_\xi(t_i^\xi)(k) = \{\alpha_{i,k}^\xi(j) : j \in m_{i,k}\}$,
- for each $\xi \in \omega_1$, $\text{lv}(s_\xi) > \delta_\xi$,
- there exists $\gamma \in \omega_1$ such that
 - for each ξ and η in ω_1 , $s_\xi \upharpoonright \gamma = s_\eta \upharpoonright \gamma =: u_{-1}$,
 - for each $\xi \in \omega_1$ and $t \in \text{dom}(p_\xi)$, $t \upharpoonright [\gamma, \text{lv}(t)) = s_\xi \upharpoonright [\gamma, \text{lv}(t))$,
 - for each ξ and η in ω_1 and $i \in n$, $t_i^\xi \upharpoonright \gamma = t_i^\eta \upharpoonright \gamma =: u_i$
 (this can be done because of the coherency of S),
- for each $i \in n$ and $k \in \sigma_i$, the set $\{p_\xi(t_i^\xi)(k) : \xi \in \omega_1\}$ is pairwise disjoint (by ignoring the root of the Δ -system), and
- the set $\{s_\xi : \xi \in \omega_1\}$ is dense above u_{-1} in S .

We note that for each distinct ξ and η in ω_1 , since $\langle p_\xi, s_\xi \rangle \perp_{\mathbb{Q} \times S} \langle p_\eta, s_\eta \rangle$, $s_\xi \perp_S s_\eta$ or there are $i \in n$, $k \in \sigma_i$ and j_0 and j_1 in $m_{i,k}$ such that $t_i^\xi \not\perp_S t_i^\eta$ and

$$t_i^\xi \cup t_i^\eta \Vdash_S “\alpha_{i,k}^\xi(j_0) \not\perp_T \alpha_{i,k}^\eta(j_1)”$$

(where $t_i^\xi \cup t_i^\eta$ is the longer one of t_i^ξ and t_i^η).

Let

$$u_{-1} \Vdash_S “\dot{I}_{-1} := \{\xi \in \omega_1 : s_\xi \in \dot{G}\}, \text{ which is uncountable”,}$$

and $\dot{\mathcal{U}}$ an S -name for a uniform ultrafilter on $\dot{I}_{-1}$. We note that u_0 forces that the S -name

$$\psi_{u_{-1}, u_0}(\dot{I}_{-1}) := \{\xi \in \omega_1 : u_0 \cup (s_\xi \upharpoonright [\gamma, \text{lv}(s_\xi))) \in \dot{G}\}$$

is an uncountable subset of ω_1 . For each $\xi \in \omega_1$, $k \in \sigma_0$, l and j in $m_{0,k}$, we define

$$u_0 \Vdash_S “\text{whenever } \xi \in \psi_{u_{-1}, u_0}(\dot{I}_{-1}),$$

$$\dot{Y}_{0,k,j}^{\xi,l} := \left\{ \eta \in \psi_{u_{-1}, u_0}(\dot{I}_{-1}) : t_0^\xi \cup t_0^\eta \Vdash_S “\alpha_{0,k}^\xi(l) \not\perp_T \alpha_{0,k}^\eta(j)” \right\} ”$$

(³) and define

$$u_0 \Vdash_S \dot{I}_0 := \left\{ \begin{array}{l} \left\{ \xi \in \psi_{u_{-1}, u_0}(\dot{I}_{-1}) : \bigcup_{\substack{k \in \sigma_0 \\ l, j \in m_{0, k}}} \dot{Y}_{0, k, j}^{\xi, l} \notin \psi_{u_{-1}, u_0}(\dot{\mathcal{U}}) \right\} \\ \text{if it is in } \psi_{u_{-1}, u_0}(\dot{\mathcal{U}}) \cdots \text{ case 1} \\ \left\{ \xi \in \psi_{u_{-1}, u_0}(\dot{I}_{-1}) : \dot{Y}_{0, k_0, j_0}^{\xi, l_0} \in \psi_{u_{-1}, u_0}(\dot{\mathcal{U}}) \right\} \\ \text{which is in } \psi_{u_{-1}, u_0}(\dot{\mathcal{U}}) \text{ for some } \dot{l}_0, \dot{k}_0 \text{ and } \dot{j}_0 \\ \text{otherwise } \cdots \text{ case 2} \end{array} \right. \quad " .$$

If the case 2 happens, then we can make an S -name for a cofinal chain through $\dot{T}$ (which is forced by some node above u_0 in S), so we are done. Whenever the case 1 happens, we repeat this procedure, that is, given $\dot{I}_i$ for some $i \in n - 1$, we define, for each $\xi \in \omega_1$, $k \in \sigma_{i+1}$, l and j in $m_{i+1, k}$,

$$u_{i+1} \Vdash_S \text{ "whenever } \xi \in \psi_{u_i, u_{i+1}}(\dot{I}_i), \\ \dot{Y}_{i+1, k, j}^{\xi, l} := \left\{ \eta \in \psi_{u_i, u_{i+1}}(\dot{I}_i) : t_{i+1}^\xi \cup t_{i+1}^\eta \Vdash_S \text{ "}\alpha_{i+1, k}^\xi(l) \not\perp_{\dot{T}} \alpha_{i+1, k}^\eta(j)\text{"} \right\} "$$

and

$$u_{i+1} \Vdash_S \dot{I}_{i+1} := \left\{ \begin{array}{l} \left\{ \xi \in \psi_{u_i, u_{i+1}}(\dot{I}_i) : \bigcup_{\substack{k \in \sigma_{i+1} \\ l, j \in m_{i+1, k}}} \dot{Y}_{i+1, k, j}^{\xi, l} \notin \psi_{u_{-1}, u_{i+1}}(\dot{\mathcal{U}}) \right\} \\ \text{if it is in } \psi_{u_{-1}, u_{i+1}}(\dot{\mathcal{U}}) \cdots \text{ case 1} \\ \left\{ \xi \in \psi_{u_i, u_{i+1}}(\dot{I}_i) : \dot{Y}_{i+1, k_{i+1}, j_{i+1}}^{\xi, l_{i+1}} \in \psi_{u_{-1}, u_{i+1}}(\dot{\mathcal{U}}) \right\} \\ \text{which is in } \psi_{u_{-1}, u_{i+1}}(\dot{\mathcal{U}}) \text{ for some } \dot{l}_{i+1}, \dot{k}_{i+1} \text{ and } \dot{j}_{i+1} \\ \text{otherwise } \cdots \text{ case 2} \end{array} \right. \quad " .$$

We show that for some $i \in n - 1$, the case 2 happens in the construction of $\dot{I}_{i+1}$, which finishes the proof. Suppose that the case 1 happens in the construction of all the $\dot{I}_{i+1}$. We take $v \in S$ and $\xi \in \omega_1$ such that $u_{n-1} \leq_S v$ and

$$v \Vdash_S \text{ "}\xi \in \dot{I}_{n-1} \text{ (which is in the set } \psi_{u_{-1}, u_{n-1}}(\dot{\mathcal{U}})\text{)"}$$

³We note that by the property of the club C , for each ξ and η in ω_1 , if $t_0^\xi \cup t_0^\eta \in S$, then this decides whether $\alpha_{0, k}^\xi(l) \perp_{\dot{T}} \alpha_{0, k}^\eta(j)$ or not.

Then it follows that

$$v \geq_S u_{n-1} \cup (s_\xi \upharpoonright [\gamma, \text{lv}(s_\xi))) \geq_S t_{n-1}^\xi.$$

We take $v' \in S$ and $\eta \in \omega_1$ such that $v' \geq_S v$ and

$$v' \Vdash_S \text{“} \eta \in \psi_{u_{-1}, u_{n-1}}(\dot{I}_{-1}) \setminus \left(\bigcup_{i \in n} \psi_{u_i, u_{n-1}} \left(\bigcup_{\substack{k \in \sigma_i \\ l, j \in m_{i,k}}} \dot{Y}_{i,k,j}^{\xi, l} \right) \right) \text{”}$$

(which is in the set $\psi_{u_{-1}, u_{n-1}}(\dot{\mathcal{U}})$).

Then for every $i \in n$, $u_i \cup (v' \upharpoonright [\gamma, \text{lv}(v')])$ is above both t_i^ξ , t_i^η , $u_i \cup (s_\xi \upharpoonright [\gamma, \text{lv}(s_\xi)])$ and $u_i \cup (s_\eta \upharpoonright [\gamma, \text{lv}(s_\eta)])$. Then it follows that $s_\xi \not\leq_S s_\eta$, and by the property of the club set C , for every $i \in n$ and $k \in \sigma_i$,

$$t_i^\xi \cup t_i^\eta \Vdash_S \text{“} p_\xi(t_i^\xi)(k) \cup p_\eta(t_i^\eta)(k) \text{ is an antichain in } \dot{T} \text{”}.$$

Therefore $\langle p_\xi, s_\xi \rangle$ and $\langle p_\eta, s_\eta \rangle$ are compatible in $\mathbb{Q} \times S$, which is a contradiction. $\dashv$ **Claim 3.3** $\square$

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