

Pointwise Completeness
and
Controllability by Linear Delay Feedback

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1. Introduction

Let a control system be described by

$$\frac{d}{dt} x(t) = Ax(t) + u(t)$$

on a Banach space X . We study the controllability of the system when the control $u(t)$ is given by a sum of delay feedbacks

$$u(t) = \sum_{r=1}^m A_r x(t-h_r), \quad 0 < h_1 < \dots < h_m.$$

This controllability leads to the following problem:

For the delay system

$$\frac{d}{dt} x(t) = Ax(t) + \sum_{r=1}^m A_r x(t-h_r), \quad (1.1)$$

$$x(0) = x_0, \quad x(s) = g(s) \quad s \in [-h_m, 0),$$

does the reachable set (with respect to initial value x_0 and initial function g) fill X or become a proper subset of X ?

This problem is called pointwise completeness or pointwise degeneracy. The problem was first proposed by Weiss [18] in his study of controllability for retarded systems in Euclidean spaces.

In case of $X = \mathbb{R}^n$, the pointwise completeness was investigated by several

authors Brooks and Schmidt [3], Zmood and MacClamroch [19] and Zverkin [20] for autonomous and nonautonomous single delayed systems.

It was Popov [13] who gave an elegant answer of pointwise degeneracy for autonomous single delayed systems in R^n . His result was extended by Asner and Halanay [1] for autonomous systems with multiple commensurable delays. For such systems Charrier and Haugazeau [5] gave another extensions which depend on linear operator theory. Kappel [11] also obtained similar results for general nonautonomous retarded systems and gave a further analysis of systems with commensurable delays. For systems of neutral type in R^n , the problem was solved by Choudhury [6] and Asner and Halanay [2].

The only one paper which studies the problem in infinite dimensional space is Charrier [4]. The paper gives an example of pointwise degenerate system in a Hilbert space but does not give any detailed study as in [1-3,5,6,11,13,19].

The purpose of this paper is to develop a general theory for pointwise completeness and degeneracy of (1.1) in infinite dimensional (Banach) spaces.

We employ the delay system (1.1) studied by Datko [8]. In Section 2, we give a definition of exact and approximate pointwise completeness by taking into account of X being infinite dimensional. A necessary and sufficient condition and a negative result for exact pointwise completeness are established in Section 3. Section 4 studies approximate pointwise completeness and pointwise degeneracy as a complementary concept. A main theorem is contained in Section 4. In Section 5 we specify the results of Section 4 to the systems with commensurable delays. In specifying such results the representation of fundamental solution of (1.1) given by Nakagiri [14] is effectively used.

2. System Description and Definition

Let X be a reflexive Banach space with norm $\|\cdot\|$. Consider the differential system with multiple delays

$$S: \quad \frac{d}{dt} x(t) = Ax(t) + \sum_{r=1}^m A_r x(t-h_r), \quad t > 0, \quad (2.1)$$

$$x(0) = x_0, \quad x(s) = g(s) \quad s \in [-h, 0]. \quad (2.2)$$

Here we assume that $0 < h_1 < \dots < h_m = h$ are positive constants, $x(t)$, $g(t) \in X$, operators A_r ($r = 1, \dots, m$) are bounded on X and A generates a strongly continuous semi-group $T(t)$, $t \geq 0$ on X .

In the system S , $x_0 \in X$ and g are called an initial value and an initial function, respectively.

Under the above assumptions, R. Datko [8] has constructed the fundamental solution $G(t)$ of the system S by the delay perturbation of $T(t)$. That is, $G(t)$ satisfies the following relations:

- i) $G(t) = 0$ (the null operator on X) if $t < 0$. (2.3)
- ii) $G(t)$ is strongly continuous on R^+ and satisfies

$$G(t) = T(t) + \sum_{r=1}^m \int_0^t T(t-s) A_r G(s-h_r) ds \quad \text{if } t \geq 0. \quad (2.4)$$

Let $x_0 \in X$ and $g \in L_p(-h, 0; X)$, $p \in [1, \infty]$. Then the function

$$x(t) = G(t)x_0 + \sum_{r=1}^m \int_{-h_r}^0 G(t-h_r-s) A_r g(s) ds \quad (2.5)$$

makes sense, the integrals being Bochner integrals in X , and is strongly continuous on R^+ . Since $x(t)$ depends on x_0 and g , we denote this by $x(t; x_0, g)$. It is proved in [8] that $x(t; x_0, g)$ satisfies the integrated form of (2.1) and (2.2), i.e.,

$$x(t; x_0, g) = T(t)x_0 + \int_0^t \{T(t-s) \sum_{r=1}^m A_r x(s-h_r; x_0, g)\} ds$$

if $t \geq 0$. (2.6)

In this sense this function $x(t; x_0, g)$ is called the mild solution of S .

In what follows we study pointwise completeness and pointwise degeneracy by means of the mild solutions.

To give a definition of pointwise completeness, the following set of reachability is needed.

$$R_t(L_p) = \{x \in X : x = x(t; x_0, g) \text{ where } x_0 \in X \text{ and } g \in L_p(-h, 0; X)\}.$$

DEFINITION 2.1. The system S is said to be

- (i) L_p -exactly pointwise complete at time t if $R_t(L_p) = X$;
- (ii) L_p pointwise complete at time t if $\overline{R_t(L_p)} = X$.

3. Exact Pointwise Completeness

In this section we study exact pointwise completeness.

For Banach spaces X , W and a densely defined linear operator $L : D(L) \subset W \rightarrow X$, we denote their dual Banach spaces by X^* , W^* and its adjoint operator by L^* , respectively. The following abstract result is used to derive an equivalent condition for exact pointwise completeness.

Lemma 3.1. Let X and W be reflexive Banach spaces and let L be a bounded linear operator from W into X . Then the image of L fills X if and only if there exists $K > 0$ such that

$$\|x^*\|_{X^*} \leq K \|L^* x^*\|_{W^*} \quad \text{for all } x^* \in X^*. \quad (3.1)$$

This lemma follows from inverse mapping theorem [15,p.83] (see also [7]).

Lemma 3.2. $G^*(t) = G(t)^*$ is strongly continuous on R^+ .

Proof. Since X is reflexive, the weak topology of X coincides with the weak* topology. Then $T^*(t) = T(t)^*$ is weakly continuous on R^+ . From the property $T^*(t+s) = T^*(t)T^*(s)$ and a well known result [10,p.306] that $T^*(t)$ is strongly continuous on R^+ . By (2.3)

$$G^*(t) = T^*(t) \text{ is strongly continuous on } [0, h_1],$$

so that

$$G^*(t) = T^*(t) + \int_0^t T^*(s-h_1) A_r^* T^*(t-s) ds$$

is also strongly continuous on $[h_1, h_2]$. Hence $G^*(t)$ is strongly continuous on $[0, h_2]$. Continuing this procedure it is verified by (2.4) that $G^*(t)$ is strongly continuous on R^+ .

Now we can give an equivalent condition for exact pointwise completeness.

THEOREM 3.1. Let $p \in (1, \infty)$. Then the system S is L_p -exactly pointwise complete at time t if and only if there exists $K_t > 0$ such that

$$\|x^*\|_{X^*} \leq K_t \max \left\{ \|G^*(t)x^*\|_{X^*}, \left(\int_{-h}^0 \|F_t^*(s)x^*\|_{X^*}^q ds \right)^{1/q} \right\}$$

for all $x^* \in X^*$, (3.2)

where $1/p + 1/q = 1$. Here the operator $F_t^*(s)$ is given by

$$F_t^*(s) = \sum_{r=1}^m \chi_r A_r^* G^*(t-h_r-s) \quad \text{for all } s \in [-h, 0] \quad (3.3)$$

and χ_r is the characteristic function of $[-h_r, 0]$ ($r = 1, \dots, m$).

Proof. Let W be the direct sum of X and $L_p(-h, 0; X)$ whose norm $\|\cdot\|_W$

is introduced by

$$\| (x, g) \|_W = \| x \| + \| g(\cdot) \|_{L_p(-h, 0; X)}.$$

We denote this Banach space W by $X \oplus L_p(-h, 0; X)$. Let the operator

$L_t : X \oplus L_p(-h, 0; X) \rightarrow X$ be given by

$$L_t(x, g) = G(t)x + G_t g(\cdot) \quad \text{for } (x, g) \in X \oplus L_p(-h, 0; X). \quad (3.4)$$

Here $G_t : L_p(-h, 0; X) \rightarrow X$ is given by

$$G_t g(\cdot) = \sum_{r=1}^m \int_{-h_r}^0 G(t-h_r-s) A_r g(s) ds \quad \text{for } g \in L_p(-h, 0; X). \quad (3.5)$$

It is evident that L_t is linear and bounded. By (3.4) we have

$$L_t(X \oplus L_p(-h, 0; X)) = R_t(L_p).$$

Then L_p -exact pointwise completeness at time t is equivalent to

$$L_t(X \oplus L_p(-h, 0; X)) = X.$$

Since $p \in (1, \infty)$, $W = X \oplus L_p(-h, 0; X)$ is reflexive and W^* is represented by

$W^* = X^* \oplus L_q(-h, 0; X^*)$ ($1/p + 1/q = 1$) whose norm $\| \cdot \|_{W^*}$ is given by

$$\| (x^*, g^*) \|_{W^*} = \text{Max} \{ \| x^* \|_{X^*}, \| g^*(\cdot) \|_{L_q(-h, 0; X^*)} \}. \quad (3.6)$$

To apply Lemma 3.1, we shall calculate L_t^* .

For $x^* \in X^*$ we have

$$\begin{aligned} \langle L_t(x, g), x^* \rangle &= \langle G(t)x, x^* \rangle + \left\langle \sum_{r=1}^m \int_{-h_r}^0 G(t-h_r-s) A_r g(s) ds, x^* \right\rangle \\ &= \langle x, G^*(t)x^* \rangle + \int_{-h}^0 \langle g(s), F_t^*(s)x^* \rangle ds \end{aligned}$$

$$= \langle (x, g), L_t^* x^* \rangle_{W, W^*},$$

where $F_t^*(s)$ is given by (3.3). Hence $L_t^* x^*$ is expressed by $(G^*(t)x^*, G_t^* x^*)$ and $G_t^* x^*$ is given by

$$\langle g, G_t^* x^* \rangle_{L_p(-h, 0; X), L_q(-h, 0; X^*)} = \int_{-h}^0 \langle g(s), F_t^*(s)x^* \rangle ds$$

for each $g \in L_p(-h, 0; X)$.

Since $G^*(t)$ is strongly continuous (Lemma 3.2), $F_t^*(s)$ is strongly continuous on $[-h, 0]$ except for $s = h_r$, $r = 1, \dots, m-1$. Then it follows by (3.6) that

$$\|L_t^* x^*\| = \text{Max} \left\{ \|G^*(t)x^*\|, \left(\int_{-h}^0 \|F_t^*(s)x^*\|_{X^*}^q ds \right)^{1/q} \right\} < \infty. \quad (3.7)$$

Then applying Lemma 3.1 with $W = X \oplus L_p(-h, 0; X)$ and $L = L_t$, we obtain condition (3.3) for L_p -exact pointwise completeness from (3.7).

Since $G(t) = T(t)$ for $t \in [0, h_1]$, we have the following corollary.

COROLLARY 3.1. If $T(t)$ is a strongly continuous group, then S is L_p -exactly pointwise complete at any time $t \in [0, h_1]$ for each $p \in (1, \infty)$.

Next we are concerned with a negative result for exact pointwise completeness. Such a fact for mild solutions in non-delay systems is first pointed out by Kuperman and Lepin [12] and more detailed researches are given by Triggiani [16, 17]. In these works some types of compactness of operators are assumed to show such a fact called the lack of exact controllability. We shall show that a similar situation, which is called the lack of exact pointwise completeness, can be viewed for our delay system S .

Lemma 3.3. Let $p \in (1, \infty]$ and $T(t)$ be compact for all $t > 0$. Then $G(t)$ and G_t are compact for all $t > 0$.

Proof. It is easily verified by (2.4) that $T(t)$ is compact for all $t > 0$ if and only if $G(t)$ is compact for all $t > 0$. By (3.5) and changes of integral variables the compactness of G_t follows from those of $G_t^r : L_p(0, h_r; X) \rightarrow X$ given by

$$G_t^r g(\cdot) = \int_0^{h_r} G(t-s) A_r g(s) ds \quad \text{for } g \in L_p(0, h_r; X) \quad (3.8)$$

for each $t > 0$ and all $r = 1, \dots, m$. It is not difficult to prove the compactness of G_t^r . We remark that this lemma does not hold if $p = 1$.

THEOREM 3.2. Let X be infinite dimensional and let $p \in (1, \infty]$. If $T(t)$ is compact for all $t > 0$, then S is never L_p -exact pointwise complete at any time $t > 0$.

Proof. We shall prove this theorem by Baire category theorem as in Triggiani [16, 17]. Let $R_{nm} = L_n(S_m)$, where S_m is the closed ball in $X \oplus L_p(-h, 0; X)$ of radius m with center the origin $(0, 0)$. Since $L_t(x, g) = G(t)x + G_t g(\cdot)$, L_t is compact for all $t > 0$ by Lemma 3.3. Hence $\overline{R_{nm}}$ is compact in X for each $n, m = 1, 2, \dots$. Since X is infinite dimensional, $\overline{R_{nm}}$ cannot contain any open ball and hence is nowhere dense in X . This implies by Baire category theorem [15, p.80] that

$$X = \bigcup \{ \overline{R_{nm}} : n, m = 1, 2, \dots \} \neq \emptyset.$$

Since

$$\bigcup_{t>0} R_t(L_p) = \bigcup_{n=1} \bigcup_{m=1} R_{nm} \subset \bigcup \{ \overline{R_{nm}} : n, m = 1, 2, \dots \},$$

S is never L_p -exactly pointwise complete at any time $t > 0$.

4. Pointwise Completeness and Pointwise Degeneracy

It follows by Definition 2.1 and Hahn-Banach theorem that S is not L_p pointwise complete at time t if and only if there exists $x^* \neq 0$ in X^* such that $x^* \in R_t(L_p)^\perp$, i.e.,

$$\langle x, x^* \rangle = 0 \quad \text{for all } x \in R_t(L_p).$$

In this case S is said to be L_p pointwise degenerate at time t with respect to x^* . If S is L_p pointwise degenerate at time t with respect to every $x^* \in E^*$ for $E^* \subset X^*$, S is called L_p pointwise degenerate at time t with respect to E^* .

The following lemma is fundamental in the arguments below.

Lemma 4.1. Let X and W be Banach spaces and let L be a densely defined linear operator from W into X . Then

$$\text{Ker } L^* = (\text{Range } L)^\perp.$$

Especially, $\overline{(\text{Range } L)} = X$ if and only if $\text{Ker } L^* = \{0\}$ in X^* .

By lemma 4.1, we obtain the following result.

THEOREM 4.1. Let $p \in [1, \infty)$. The system S is L_p pointwise degenerate at time $t > 0$ with respect to E^* if and only if

$$E^* \subset \text{Ker } G^*(t) \cap \left\{ \bigcap \{ \text{Ker } F_t^*(s) : s \in (-h, 0) \setminus \{-h_1, \dots, -h_m\} \} \right\}. \quad (4.1)$$

Moreover, the system S is L_p pointwise complete at time $t > 0$ if and only if

$$\{0\} = \text{Ker } G^*(t) \cap \left\{ \bigcap \{ \text{Ker } F_t^*(s) : s \in (-h, 0) \setminus \{-h_1, \dots, -h_m\} \} \right\}. \quad (4.2)$$

Proof. By Lemma 4.1, the system S is L_p pointwise degenerate at time t with respect to E^* if and only if

$$E^* \subset R_t(L_p)^\perp = (\text{Range } L_t)^\perp = \text{Ker } L_t^*.$$

Then by (3.6) it follows that

$$G^*(t)x^* = 0 \quad \text{in } X^* \quad \text{for all } x^* \in E^*$$

and
$$G_t^* x^* = 0 \quad \text{in } L_q(-h, 0; X^*) \quad \text{for all } x^* \in E^*,$$

where $1/p + 1/q = 1$ and $q \neq 1$. Hence if $x^* \in E^*$,

$$\langle g, G_t^* x^* \rangle_{L_p(-h, 0; X), L_q(-h, 0; X^*)} = \int_{-h}^0 \langle g(s), F_t^*(s)x^* \rangle ds = 0$$

$$\text{for all } g \in L_p(-h, 0; X).$$

This implies

$$F_t^*(s)x^* = 0 \quad \text{in } X^* \quad \text{for a.e. } s \in [-h, 0]. \quad (4.3)$$

Since $F_t^*(s)$ is strongly continuous on $[-h, 0]$ except for $s = h_i$, $i = 1, \dots, m-1$, we have by (4.3) that

$$F_t^*(s)x^* = 0 \quad \text{in } X^* \quad \text{for all } s \in (-h, 0) \setminus \{-h_1, \dots, -h_m\}.$$

Thus x^* belongs to the left hand side of (4.1).

The latter part of this theorem will be clear.

The condition (4.1) is equivalent to that for all $x^* \in E^*$,

$$G^*(t)x^* = 0 \quad \text{in } X^* \quad \text{and} \quad F_t^*(s)x^* = 0 \quad \text{in } X^* \\ \text{for all } s \in (-h, 0) \setminus \{-h_1, \dots, -h_m\} \quad (4.4)$$

Since (4.1) and (4.2) do not depend on the space of initial functions

$L_p(-h, 0; X)$, we omit L_p in the terminology of L_p pointwise completeness and L_p pointwise degeneracy.

We now write the second condition in (4.4) by

$$\langle x, F_t^*(s)x^* \rangle = 0 \quad \text{for all } x \in X \text{ and all } s \in (-h_{j+1}, -h_j) \\ j = 0, 1, \dots, m-1. \quad (4.5)$$

It then follows by (3.3) and changes of integral variables that (4.5) is equivalent to

$$\langle \sum_{r=j+1}^m G(t+s-h_r) A_r x, x^* \rangle = 0 \quad \text{for all } x \in X \text{ and all } s \in [h_j, h_{j+1}], \\ j = 0, 1, \dots, m-1. \quad (4.6)$$

Lemma 4.2. The fundamental solution $G(t)$ of S satisfies the following relation:

$$G(t+s) = G(t)G(s) + \sum_{r=1}^m \int_{-h_r}^0 G(t-\sigma-h_r) A_r G(\sigma+s) d\sigma \quad \text{for all } s, t \geq 0. \quad (4.7)$$

Proof. Since this lemma follows easily by direct substitutions, we omit its proof.

The following lemma is due to R. Datko [9, Lemma 2.4].

Lemma 4.3. If $x \in D(A)$, then $G(t)x$ is strongly differentiable for almost everywhere on R^+ and satisfies

$$\begin{aligned} \frac{d}{dt} G(t)x &= AG(t)x + \sum_{r=1}^m A_r G(t-h_r)x \\ &= G(t)Ax + \sum_{r=1}^m G(t-h_r)A_r x \quad \text{for a.e. } t \in R^+. \end{aligned} \quad (4.8)$$

The following theorem gives an infinite dimensional version of the results by Zmood and MaClamroch [19, Theorem 2] and Kappel [11, Theorem 2.1].

THEOREM 4.2. The system S is pointwise degenerate at time $t > 0$ with respect to E^* if and only if

$$E^* \subset \bigcap_{s \geq t} \text{Ker } G^*(s). \quad (4.9)$$

Moreover, the system S is pointwise complete at time $t > 0$ if and only if

$$\{0\} = \bigcap_{s \geq t} \text{Ker } G^*(s). \quad (4.10)$$

Proof. It is sufficient to prove this theorem that

$$x^* \in \bigcap_{s \geq t} \text{Ker } G^*(s) \quad (4.11)$$

if and only if

$$x^* \in \text{Ker } G^*(t) \cap \left\{ \bigcap_{t} \{ \text{Ker } F_t^*(s) : s \in (-h, 0) \setminus \{-h_1, \dots, -h_m\} \} \right\}. \quad (4.12)$$

The condition (4.11) is equivalent to that

$$\langle x, G^*(s)x^* \rangle = \langle G(s)x, x^* \rangle = 0 \quad \text{for all } x \in X \text{ and all } s \geq t. \quad (4.13)$$

First we shall show that (4.12) implies (4.11). Let (4.12) be satisfied.

Then by Lemma 4.2, we have

$$\begin{aligned} \langle G(t+s)x, x^* \rangle &= \langle G(t)G(s)x, x^* \rangle + \int_{-h}^0 \langle F_t(\sigma)G(\sigma+s)x, x^* \rangle d\sigma \\ &= \int_{-h}^0 \langle G(\sigma+s)x, F_t^*(\sigma)x^* \rangle d\sigma = 0 \quad \text{for all } x \in X \text{ and} \\ &\quad s \geq 0. \end{aligned}$$

This shows (4.11).

Conversely, let (4.11) be satisfied. Then by Lemma 4.2 and (4.13), we have

$$\begin{aligned}
f(s, x) &= \sum_{r=1}^m \int_{-h_r}^0 \langle G(t-\sigma-h_r) A_r G(\sigma+s) x, x^* \rangle d\sigma \\
&= \langle G(s+t) x, x^* \rangle - \langle G(t) G(s) x, x^* \rangle \\
&= 0 \quad \text{for all } x \in X \text{ and all } s \geq 0. \quad (4.14)
\end{aligned}$$

If $x \in D(A)$, by Lemma 4.3 we can differentiate $f(s, x)$ for a.e. $s \in \mathbb{R}^+$ and obtain that

$$\begin{aligned}
f'(s, x) &= \sum_{r=1}^m \langle G(t+s-h_r) A_r x, x^* \rangle \\
&\quad + \sum_{r=1}^m \int_{-h_r}^0 \langle G(t-\sigma-h_r) A_r G(s+\sigma) A x, x^* \rangle d\sigma \\
&\quad + \sum_{r=1}^m \int_{-h_r}^0 \langle G(t-\sigma-h_r) A_r \left(\sum_{j=1}^m G(s+\sigma-h_j) A_j \right) x, x^* \rangle d\sigma \\
&= I_1(s) + I_2(s) + I_3(s) = 0 \quad \text{for a.e. } s \in \mathbb{R}^+. \quad (4.15)
\end{aligned}$$

Since $I_2(s) = f(s, Ax)$, we see from (4.14) that

$$I_2(s) = 0 \quad \text{for all } s \geq 0.$$

$I_3(s)$ can be decomposed as

$$I_3(s) = \sum_{j=1}^m I_{3,j}(s) = \sum_{j=1}^m \left(\sum_{r=1}^m \int_{-h_r}^0 \langle G(t-\sigma-h_r) A_r G(s+\sigma-h_j) A_j x, x^* \rangle d\sigma \right).$$

If $0 \leq s < h_j$, $G(s+\sigma-h_j) = 0$ for each $\sigma \in [-h, 0]$ so that

$$I_{3,j}(s) = 0 \quad \text{for } 0 \leq s < h_j.$$

If $s \geq h_j$, we use Lemma 4.2 again and obtain from (4.13) that

$$I_{3,j}(s) = \langle G(t+s-h_j)A_j x, x^* \rangle - \langle G(t)G(s-h_j)A_j x, x^* \rangle = 0.$$

Hence $I_{3,j}(s) = 0$ for all $s \geq 0$ and $j = 1, \dots, m$. Then by (4.5),

$I_1(s) = 0$ for a.e. $s \in \mathbb{R}^+$. Since $I_1(s)$ is piecewise continuous and $t+s-h_r \geq t$ if $s \geq h_r$, it follows by (4.13) that

$$I_1(s) = \sum_{r=j+1}^m \langle G(t+s-h_r)A_r x, x^* \rangle = 0 \quad \text{for } s \in [h_j, h_{j+1}].$$

Since $D(A)$ is dense in X , (4.6) holds. This proves that (4.12) implies (4.11) and completes the proof.

It follows from Theorem 4.2 that S is pointwise degenerate at all $t \geq t_0$ with respect to E^* if S is pointwise degenerate at time t_0 with respect to E^* .

By Theorem 4.2, we shall call $\bigcap_{s \geq t} \text{Ker } G^*(s)$ the degenerate space of S at time t . It is clear that the degenerate space is a closed subspace of X^* .

Example 4.1. Let $X = L_2(0, 1)$. We define the semi-group $T(t)$ by

$$T(t)f = g; \quad g(s) = \begin{cases} f(s+t) & \text{if } 0 \leq s+t \leq 1 \\ 0 & \text{if } s+t > 1. \end{cases}$$

It is easy to verify that $T(t)$ is a strongly continuous semi-group on $L_2(0, 1)$, $T(t) = 0$ for all $t > 1$ and its infinitesimal generator $A = d/dt$.

We now consider the delay system

$$\frac{d}{dt} x(t) = Ax(t) + A_1 x(t-1). \quad (4.16)$$

If $A_1 = I$ (the identity operator on X), then the system (4.16) is pointwise complete at all $t \geq 0$. Next consider the case where $A_1 = T(1/2)$. In this case the system (4.16) is pointwise degenerate at all $t > 0$ and the degenerate

space of the system at time $t > 0$ is given by

$$\left\{ \begin{array}{l} \{ g^* \in L_2(0, 1) : g^* = \chi_{[1-t, 1]} x \text{ for } x \in L_2(0, 1) \} \text{ if } 0 < t \leq 1/2 \\ \{ g^* \in L_2(0, 1) : g^* = \chi_{[1/2, 1]} x \text{ for } x \in L_2(0, 1) \} \text{ if } 1/2 \leq t \leq 1 \\ \{ g^* \in L_2(0, 1) : g^* = \chi_{[\frac{3}{2}-t, 1]} x \text{ for } x \in L_2(0, 1) \} \text{ if } 1 \leq t < 3/2 \\ x^* \text{ if } t \geq 3/2, \end{array} \right.$$

where χ_I is the characteristic function of I . Note that the degenerate space is infinite dimensional for all $t > 0$.

Example 4.2. (Extended Charrier's example [4]) We consider the single delay system (4.16) on a general Hilbert space X . The inner product and the norm are denoted by $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$, respectively. A in (4.16) is assumed to generate a semi-group $T(t)$ on X . Let $E^* = \overline{\text{sp}} \{ x_1^*, \dots, x_n^* \}$ and $\{ x_1^*, \dots, x_n^* \} \subset D(A^*)$. We assume that there exists a set $\{ y_1, \dots, y_n \} \subset D(A)$ such that

$$\langle y_k, x_j^* \rangle_X = \delta_{k,j}, \quad \langle T(1)y_k, x_j^* \rangle_X = 0 \quad \text{and} \quad \langle T(1)Ay_k, x_j^* \rangle_X = 0$$

for all $k, j = 1, \dots, n$.

Then if A_1 is given by

$$A_1 x = \sum_{k=1}^n \{ \langle T(1)x, x_k^* \rangle_X Ay_k - \langle T(1)x, A^*x_k^* \rangle_X y_k \},$$

the system (4.16) is pointwise degenerate at time 2 with respect to E^* .

It is possible to extend this example to the case where E^* is spanned by infinitely many elements $x_1^*, \dots, x_n^*, \dots$ in X^* .

5. Systems with Commensurable Delays

In this section we consider the system S with commensurable delays $h_r = \tau$, $r = 1, \dots, m$, $\tau > 0$.

To give a useful formula of $G(t)$ in this special system, we need some preparation. First we define the index sets $\Lambda(j, k)$ for all $j, k = 1, 2, \dots$ by

$$\Lambda(j, k) = \{ (i_1, \dots, i_j) : 1 \leq i_1, \dots, i_j \leq m \text{ and } i_1 + \dots + i_j = k \}.$$

Next we define the operators $T_k(t)$, $k = 1, 2, \dots$, by

$$\begin{aligned} T_1(t) &= T(t) \\ T_k(t) &= \sum_{j=1}^{k-1} \sum_{\Lambda(j, k-1)} \int_0^t T(t-s_{j-1}) A_{i_1} \dots \int_0^{s_1} T(s_1-s) A_{i_j} T(s) ds ds_1 \dots ds_{j-1}, \\ &\quad k \geq 2. \end{aligned} \quad (5.1)$$

Then $T_2(t) = \int_0^t T(t-s) A_1 T(s) ds$, for example.

For each natural number i , we define the matrix of operators T_i by

$$T_i(t) = \begin{pmatrix} T(t) & 0 & \dots & 0 \\ T_2(t) & T(t) & & \\ \vdots & & \ddots & \\ \vdots & & & 0 \\ T_i(t) & \dots & T_2(t) & T(t) \end{pmatrix} \quad (5.2)$$

where $T_i(t)$ are given by (5.1).

We denote the transpose of i direct sum of X , $(X \oplus X \oplus \dots \oplus X)^t$, by X^i .

Lemma 5.1. $T_i(t)$ is a strongly continuous semigroup on X^i such that its infinitesimal generator A_i is represented by

$$A_i = \begin{pmatrix} A & O & \dots & O \\ A_1 & A & O & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_m & A_{m-1} & \dots & A \\ O & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & \vdots \\ O & \dots & O & A_m & A_{m-1} & \dots & A_1 & A \end{pmatrix} \quad (5.3)$$

$$\text{on } D(A)^i = (D(A) \oplus D(A) \oplus \dots \oplus D(A))^t.$$

Lemma 5.1 says that $D(A_i) = D(A)^i$. But we can not give such representations of $D(A_i^k)$ in terms of the domains of operators $A^k, A^{k-1}A_r$ etc. for $k \geq 2$.

We define $D_r^k \subset X$ by $D_r^k = \bigcap_{j=1}^{k-1} D(A^{k-j}A_r A^{j-1})$ for $r = 1, \dots, m$ and

$k = 1, 2, \dots$. Then we have the following lemma.

$$\begin{aligned} \text{Lemma 5.2. } D(A_i^k) &\supset \begin{pmatrix} D(A^k) \cap D_1^k \cap \dots \cap D_{\min(i,m)}^k \\ D(A_{i-1}^k) \end{pmatrix} \\ &\supset \begin{pmatrix} D(A^k) \cap D_1^k \cap \dots \cap D_{\min(i,m)}^k \\ \vdots \\ D(A^k) \cap D_1^k \\ D(A^k) \end{pmatrix}. \end{aligned} \quad (5.4)$$

Since the proofs of Lemma 5.1 and Lemma 5.2 are complicated and tiresome, we omit the proofs. The following lemma is an easy consequence of (5.1) and (5.2).

Lemma 5.3. If $T(t)$ is analytic, then $T_i(t)$ is also analytic for each $i = 1, 2, \dots$.

Now we define Z_i inductively by

$$Z_1 = I \quad \text{and} \quad Z_i = \begin{pmatrix} I \\ T_{i-1}(\tau)Z_{i-1} \end{pmatrix} \quad \text{for } i \geq 2. \quad (5.5)$$

It is shown in Nakagiri [14] that $G(t)$ is represented by

$$G(t) = I_i T_i(t - (i-1)\tau) Z_i \quad \text{when } t \in [(i-1)\tau, i\tau), \quad (5.6)$$

where $I_i = [0, \dots, 0, I]$.

Let X_i be the largest subspace of X such that $Z_i X_i \subset \bigcap_{n=0}^{\infty} D(A_i^n)$. It then follows by Lemma 5.2 that if

$$\bigcap_{k=0}^{\infty} (D(A^k) \cap D_1^k \cap \dots \cap D_m^k)$$

is dense in X , then X_i is also dense in X .

The following theorem is a consequence from Theorem 4.2 and (5.6).

THEOREM 5.1. The system S is pointwise degenerate at time $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if

$$E^* \subset \bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \text{Ker } Z_i^* (A_i^*)^n I_i^*. \quad (5.7)$$

Moreover, the system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$\bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \text{Ker } Z_i^* (A_i^*)^n I_i^* = \{0\}. \quad (5.8)$$

Especially for the pointwise completeness, we obtain the next theorem.

THEOREM 5.2. The system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{\text{sp}} \{ I_i A_i^n Z_i X_i : i = k, k+1, \dots, n = 0, 1, 2, \dots \} \quad (5.9)$$

or, more generally, if

$$X = \overline{\text{sp}} \{ I_{i i i} A_{i i}^n T_i(s_i) Z_{i i} X_i : i = k, k+1, \dots, n = 0, 1, 2, \dots \},$$

$$s_i \text{ arbitrary in } J_i, \quad (5.10)$$

where $J_k = [t_0 - (k-1)\tau, \tau)$ and $J_i = [0, \tau)$ for $i \geq k+1$.

Conversely if $T(t)$ is analytic, X_i is dense in X for each $i = k, k+1, \dots$ and the system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$, then

$$X = \overline{\text{sp}} \{ I_{i i i} A_{i i}^n T_i(s_i) Z_{i i} X_i : i = k, k+1, \dots, n = 0, 1, 2, \dots \},$$

$$s_i \text{ arbitrary in } J_i - \{0\}. \quad (5.11)$$

Remark 5.1. The condition (5.7) ((5.9)) is not necessary for pointwise degeneracy (pointwise completeness) in general. But if $X_i = X$ for all $i = k, k+1, \dots$, i.e., A is bounded, (5.7) ((5.9)) is necessary and sufficient.

COROLLARY 5.1. Let A be bounded. Then S is pointwise degenerate at time $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if and only if (5.7) holds or

$$E^* \subset (\overline{\text{sp}} \{ I_{i i i} A_{i i}^n Z_{i i} X : i = k, k+1, \dots, n = 0, 1, 2, \dots \})^\perp. \quad (5.12)$$

Furthermore, S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if and only if (5.8) holds or

$$X = \overline{\text{sp}} \{ I_{i i i} A_{i i}^n Z_{i i} X : i = k, k+1, \dots, n = 0, 1, 2, \dots \}. \quad (5.13)$$

To give a generalization of rank condition for pointwise completeness, we consider the condition (4.2). First we give the representation of $F_t(s)$ without using characteristic functions. Let $t \in [(k-1)\tau, k\tau)$ be fixed. Let $X_{i,r}$ ($i = 1, 2, \dots, r = 1, \dots, m$) be the largest subspace of X such that

$$Z_{i r i} A_{i r} X_{i,r} \subset \bigcap_{n=0}^{\infty} D(A_i^n).$$

For negative integers $i = -1, -2, \dots$ we put $x_{i,r} = x$ for each $r = 1, \dots, m$.

We now define x_i^0, x_i^1 by

$$x_i^0 = \bigcap_{r=i-1}^m x_{k+i-r,r}, \quad x_i^1 = \bigcap_{r=i}^m x_{k+i-r,r} \quad \text{for } i = 1, \dots, m.$$

Then we obtain by Theorem 4.1 and differentiations of (5.6) the following result.

THEOREM 5.3. The system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$x = \overline{\text{sp}} \{ G(t_0)x, \left(\sum_{r=i}^m I_{k+i-r} A^n Z_{k+i-r} A_r \right) x_i^1 : i=1, \dots, m, n=0, 1, 2, \dots \} \quad (5.14)$$

or, more generally if

$$\begin{aligned} x = \overline{\text{sp}} \{ & G(t_0)x, \left(\sum_{r=i-1}^m I_{k+i-r} A^n T_{k+i-r} (s_i^0) Z_{k+i-r} A_{r+1} \right) x_i^0, \\ & \left(\sum_{r=i}^m I_{k+i-r} A^n T_{k+i-r} (s_i^1) Z_{k+i-r} A_r \right) x_i^1 : i=1, \dots, m, n=0, 1, 2, \dots \}, \\ & s_i^0 \text{ arbitrary in } [0, t_0 - (k-1)\tau) \text{ and } s_i^1 \text{ arbitrary in } [t_0 - (k-1)\tau, \tau) \end{aligned} \quad (5.15)$$

Conversely if $T(t)$ is analytic, x_i^0, x_i^1 are dense for all $i = 1, \dots, m$ and S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$, then

$$\begin{aligned} x = \overline{\text{sp}} \{ & G(t_0)x, \left(\sum_{r=i-1}^m I_{k+i-r} A^n T_{k+i-r} (s_i^0) Z_{k+i-r} A_{r+1} \right) x_i^0, \\ & \left(\sum_{r=i}^m I_{k+i-r} A^n T_{k+i-r} (s_i^1) Z_{k+i-r} A_r \right) x_i^1 : i=1, \dots, m, n=0, 1, 2, \dots \}, \\ & s_i^0 \text{ arbitrary in } (0, t_0 - (k-1)\tau) \text{ and } s_i^1 \text{ arbitrary in } (t_0 - (k-1)\tau, \tau). \end{aligned} \quad (5.16)$$

If A is bounded, then $G^*(t)$ is analytic in $t \in ((k-1)\tau, k\tau)$ for each $k = 1, 2, \dots$. Let $t_0 \in [(k-1)\tau, k\tau)$. Then we see easily that

$$\bigcap_{t \geq t_0} \text{Ker } G^*(t) = \bigcap_{t \geq (k-1)\tau} \text{Ker } G^*(t),$$

so that S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if and only if S is pointwise complete at $t_0 = (k-1)\tau$. Thus we obtain the following corollary.

COROLLARY 5.2. Let A be bounded. Then S is pointwise degenerate at $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if and only if

$$E^* \subset \text{Ker } Z_k^* I_k^* \cap \left\{ \bigcap_{i=1}^m \bigcap_{n=0}^{\infty} \text{Ker} \left(\sum_{r=i}^m A^* Z_{k+i-r}^* (A_{k+i-r}^*)^n I_{k+i-r}^* \right) \right\} \quad (5.17)$$

or

$$E^* \subset \left(\overline{\text{sp}} \left\{ I_k Z_k X, \left(\sum_{r=i}^m I_{k+i-r} A_{k+i-r}^n Z_{k+i-r} A_r \right) X : i=1, \dots, m, n=0, 1, 2, \dots \right\} \right)^\perp. \quad (5.18)$$

Furthermore, S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if and only if

$$X = \overline{\text{sp}} \left\{ I_k Z_k X, \left(\sum_{r=i}^m I_{k+i-r} A_{k+i-r}^n Z_{k+i-r} A_r \right) X : i=1, \dots, m, n=0, 1, 2, \dots \right\}, \quad (5.19)$$

or

$$\text{Ker } Z_k^* I_k^* \cap \left\{ \bigcap_{i=1}^m \bigcap_{n=0}^{\infty} \text{Ker} \left(\sum_{r=i}^m A^* Z_{k+i-r}^* (A_{k+i-r}^*)^n I_{k+i-r}^* \right) \right\} = \{0\}. \quad (5.20)$$

Especially if $m = 1$, (5.19) is reduced to

$$X = \overline{\text{sp}} \left\{ I_k Z_k X, I_k A_{k-1}^n Z_{k-1} A_1 X ; n = 0, 1, 2, \dots \right\}. \quad (5.21)$$

Consider the finite dimensional case where $X = \mathbb{R}^N$, A and A_1 are $N \times N$ matrices, so that I_k, A_k, Z_k are $N \times Nk, Nk \times Nk, Nk \times N$ matrices, respectively. In this case the condition (5.21) is equivalent to that

$$\text{rank} [I_k Z_k, I_k Z_k A_1, \dots, I_k A_{k-1}^{Nk-1} Z_{k-1} A_1] = N.$$

This fact follows by the Cayley-Hamilton theorem and gives the main result by Zmood and MaClamroch for Euclidean N -space [19, Theorem 3].

We next specify Theorem 5.2 and Theorem 5.3 in the special case where $m = 1$ and A_1 commutes with A . Let $X_\infty = \bigcap_{n=0}^{\infty} D(A^n)$. Clearly X_∞ is dense in X .

COROLLARY 5.3. Let $m = 1$, A_1 commute with A and $T(t)$ be analytic.

Then S is pointwise degenerate at $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if

$$E^* \subset \bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \text{Ker } (A_1^*)^i (A^*)^n$$

or if

$$E^* \subset \text{Ker } G^*((k-1)\tau) \cap \left\{ \bigcap_{i=1}^k \bigcap_{n=0}^{\infty} \text{Ker } (A_1^*)^i (A^*)^n \right\}.$$

Furthermore, S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{\text{sp}} \{ A_1^i A^n X_\infty : i = k, k+1, \dots, n = 0, 1, 2, \dots \}$$

or if

$$X = \overline{\text{sp}} \{ A_1^i A^n X_\infty, G((k-1)\tau)X : i=1, \dots, m, n=0, 1, 2, \dots \}.$$

We now recall the definition of approximate controllability. Let A generate a semi-group $T(t)$ on X and let B be a bounded operator from a Banach space U into X . Then the system $\{A, B\}$ is said to be approximately controllable if $\overline{\bigcup_{t>0} T(t)BU} = X$. We define $\tilde{Z}_i$ by

$$\tilde{Z}_1 = I \quad \text{and} \quad \tilde{Z}_i = \begin{pmatrix} I & 0 \\ 0 & T_{i-1}(\tau) \tilde{Z}_{i-1} \end{pmatrix} \quad \text{for } i \geq 2.$$

The following corollary is immediate from Theorem 4.2, Lemma 5.3 and (5.6).

COROLLARY 5.4. Let $T(t)$ be analytic and the system $\{A_k, \tilde{Z}_k\}$ be approximately controllable on X^i . Then S is pointwise complete at any time $t \in [0, k\tau)$.

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