

A problem on the singularities of a real algebraic vector field

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Acknowledgement.

I would like to thank the organizers for giving me time for this problem session. I have a problem on the singularities of a real algebraic vector field. I am not at all a specialist of this field. My problem might be familiar or easy for specialists.

1. A vector fields.

Let $M(n)$ be the algebra of all $n \times n$ complex matrices, χ a monic complex polynomial of degree n , $M(\chi)$ the subset of all $X \in M(n)$ such that the characteristic polynomial of X is given by χ . $M(\chi)$ is a complex algebraic subvariety of $M(n)$. Moreover, let $N(n)$ be the set of all $n \times n$ normal matrices, $N(\chi) := N(n) \cap M(\chi)$.

Consider a real algebraic vector field V on $M(n)$ defined by

$$V(X) := [[X^*, X], X] \quad \text{at } X \in M(n),$$

where X^* is the Hermitian adjoint of X . We provide $M(n)$ with the Hermitian inner product and the Hermitian norm defined by

$$(X, Y) := \text{Trace}(XY^*), \quad \|X\| := \sqrt{(X, X)}.$$

The vector field V arises as the gradient flow of the functional $\varphi : M(n) \rightarrow \mathbf{R}$ defined by

$$\varphi(X) := \frac{1}{4} \| [X^*, X] \|^2.$$

LEMMA 1.1. *The fixed point set of V is $N(n)$.*

The vector field V preserves each conjugacy class of $M(n)$, where a conjugacy class means a $GL(n)$ -orbit of the group action

$$M(n) \times GL(n) \rightarrow M(n), \quad (X, g) \mapsto g^{-1}Xg.$$

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In particular, for any χ , V preserves $M(\chi)$, and hence one can consider the restriction V_χ of V into $M(\chi)$:

$$V_\chi = V|_{M(\chi)}.$$

The vector field V arises as the gradient flow of another variational problem. To state it, let C be any conjugacy class and consider the functional $\psi : C \rightarrow \mathbb{R}$ defined by

$$\psi(X) := \frac{1}{2} \|X\|^2.$$

C is a locally closed complex submanifold of $M(n)$ and its tangent space at $X \in C$ is given by

$$T_X C = \text{Image of } \text{Ad}(X) : M(n) \rightarrow M(n), \quad Y \mapsto [X, Y].$$

If $T(X)$ is the orthogonal complement of $\text{Ker } \text{Ad}(X)$, then we have an isomorphism $\text{Ad}(X) : T(X) \rightarrow T_X C$. We provide $T_X C$ with a Hermitian inner product so as to make $\text{Ad}(X) : T(X) \rightarrow T_X C$ an isometry. Thus we have obtained a Hermitian metric on C . The gradient flow of the functional $\psi : C \rightarrow \mathbb{R}$ with respect to this Hermitian metric gives the vector field $V_C := V|_C$ on C .

2. Stratification.

$M(\chi)$ consists of a finite number of $GL(n)$ -orbits. Let $\mathcal{O}(\chi)$ be the set of all orbits in $M(\chi)$. $\mathcal{O}(\chi)$ gives a stratification of $M(\chi)$ by locally closed complex submanifolds. We introduce a partial order $<$ in $\mathcal{O}(\chi)$: For $C_1, C_2 \in \mathcal{O}(\chi)$, we put $C_1 < C_2$ if and only if $C_1 \subset \overline{C_2}$. Let $E(\chi)$ be the set of all $e = (e_1, e_2, \dots, e_n)$ such that

- (1) e_i is a monic polynomial, ($i = 1, 2, \dots, n$),
- (2) e_i divides e_{i+1} , ($i = 1, 2, \dots, n-1$), and
- (3) $e_1 e_2 \cdots e_n = \chi$.

For any $C \in \mathcal{O}(\chi)$, we denote by $e_i(C)$ be the i -th elementary divisor of C and put $e(C) := (e_1(C), e_2(C), \dots, e_n(C))$.

LEMMA 2.1. *There is a one-to-one correspondence:*

$$\mathcal{O}(\chi) \rightarrow E(\chi), \quad C \mapsto e(C).$$

For any $C_1, C_2 \in \mathcal{O}$, we have $C_1 < C_2$ if and only if

$$\prod_{j=1}^i e_j(C_2) \text{ divides } \prod_{j=1}^i e_j(C_1), \quad (i = 1, 2, \dots, n).$$

REMARK 2.2: There are a unique maximal orbit $C_{\max}(\chi)$ and a unique minimal orbit $C_{\min}(\chi)$ in $\mathcal{O}(\chi)$ with respect to the partial order $<$.

LEMMA 2.3. Let $C \in \mathcal{O}(\chi)$.

(i) The following three assertions are equivalent:

- (1) $C = C_{\min}(\chi)$.
- (2) C is closed in $M(\chi)$.
- (3) C is semisimple.

(ii) $C = C_{\max}(\chi)$ if and only if C is open in $M(\chi)$.

(iii) $C_{\min}(\chi) = C_{\max}(\chi)$ if and only if χ has distinct n roots,

(iv) $X \in M(n)$ is smooth in $M(n)$ if and only if $X \in C_{\max}(\chi)$, and

(v) $N(\chi) = N(n) \cap C_{\min}(\chi)$.

Lemma 2.3 implies that, if χ has a multiple root, then $N(\chi)$ lies in the singularities of $M(\chi)$. If χ has distinct n roots, then $M(\chi)$ is smooth everywhere.

Consider the vector field V_χ on $M(\chi)$. This is a real algebraic *stratified vector field* on $M(n)$. In this symposium, Prof. Brasselet talked about complex analytic stratified vector fields.

LEMMA 2.4. The fixed point set of V_χ is $N(\chi)$. Moreover, the ω -limit set of V_χ is $N(\chi)$.

3. Semisimple trajectories.

Consider the trajectory $\{X(t)\}_{t \geq 0}$ of V_χ starting from $X_0 \in M(\chi)$. $X(t)$ exists for all $t \geq 0$. If $X_0 \in C_{\min}(\chi)$, then $X(t)$ is called a *semisimple trajectory* and, if $X_0 \notin C_{\min}(\chi)$, then $X(t)$ is called a *non-semisimple trajectory*, respectively.

NOTATION 3.1: Let $\{z_1, z_2, \dots, z_k\}$ be the set of mutually distinct roots of χ . We put

$$a(\chi) := \begin{cases} 0 & (k = 1), \\ \min_{i \neq j} |z_i - z_j|^2, & (k > 1). \end{cases}$$

REMARK 3.2: (i) If $a(\chi) = 0$, then $C_{\min}(\chi)$ consists of a single point which is a scalar matrix. So the trajectory $X(t)$ is a single point. Everything is trivial in this case.

(ii) If $a(\chi) > 0$, then $N(\chi)$ is a compact real analytic manifold of positive dimension. $N(\chi)$ is a $U(n)$ -orbit.

THEOREM 3.3. There exists a continuous function $K : C_{\min}(\chi) \rightarrow \mathbb{R}$ such that the following condition holds: For any $X_0 \in C_{\min}(\chi)$ there exists a normal matrix $X_\infty \in N(\chi)$ such that the trajectory $X(t)$ starting from X_0 satisfies

$$\|X(t) - X_\infty\| \leq K(X_0) \| [X_0^*, X_0] \| e^{-2a(\chi)t} \quad (t \geq 0).$$

REMARK 3.4: (i) The function K can be given more explicitly (see [Iw]).
(ii) Theorem 3.3 implies that each semisimple trajectory in $M(\chi)$ converges exponentially to a normal matrix in $N(\chi)$ as $t \rightarrow \infty$.

4. Non-semisimple trajectories.

What can we say about the non-semisimple trajectories ? We have at least the following:

THEOREM 4.1. *For any non-semisimple trajectory $X(t)$,*

$$t\| [X^*(t), X(t)] \|^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

but

$$\int_0^\infty t\| [X^*(t), X(t)] \|^2 dt = \infty.$$

Now we propose the following:

PROBLEM 4.2. *Does any non-semisimple trajectory converge as $t \rightarrow \infty$?
If a non-semisimple trajectory does not converge, how does it behave?*

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