

# On the boundary growth of superharmonic functions satisfying nonlinear inequalities

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## 1 Introduction

The study on the boundary behavior of harmonic functions and superharmonic functions has a long history. In 1906, Fatou [8] showed that every bounded harmonic function in the unit disc  $\mathbb{D}$  has a finite nontangential limit almost everywhere on the boundary  $\partial\mathbb{D}$ . The extension of this result to higher dimensions was due to Bray and Evans [6] in 1927. Also, it was proved by Littlewood [17] in 1927 that nontangential approach regions in the Fatou theorem are best possible in the following sense: there exists a bounded harmonic function on  $\mathbb{D}$  which admits no limits along a tangential curve a.e. on  $\partial\mathbb{D}$ . A simple proof by the construction of a Blaschke product was given by Zygmund [25] in 1949. Also, Aikawa [1, 2] presented a stronger version of the Littlewood theorem in 1990. In their study of maximal functions, Nagel and Stein [19] was established that there exist tangential sequences along which every bounded harmonic function on  $\mathbb{D}$  converges a.e. on  $\partial\mathbb{D}$ . These results were generalized in several directions by many authors. For example, see [11, 12, 15, 22] for invariant harmonic functions in the unit ball of  $\mathbb{C}^n$ , and see [4, 10, 13, 16] for positive solutions of the Helmholtz equation in  $\mathbb{R}^n$ , and so on.

The decomposition theorem by Riesz states that every nonnegative superharmonic function can be represented as the sum of a nonneg-

ative harmonic function and a Green potential. In 1928, Littlewood showed that every Green potential on  $\mathbb{D}$  has radial limit zero a.e. on  $\partial\mathbb{D}$ . Privalov [21] extended this result to higher dimensions. However, Green potentials do not necessarily have nontangential limits. Thus we need to impose some conditions on the densities of Green potentials if we want to show the existence of nontangential limits. In 1967, Arsove and Huber [3] proved that if a nonnegative measurable function  $f$  on the unit ball  $B$  satisfies the weighted integrability condition

$$\int_B (1 - |x|)^{2p-1} f(x)^p dx < \infty \quad (1.1)$$

for  $p = 1$  and some  $p > n/2$ , then the Green potential of the density  $f$  has nontangential limit zero a.e. on  $\partial B$ . In 1979, Wu [23] gave a sufficient condition similar to (1.1) for the density functions to guarantee the existence of tangential limits of Green potentials. Other generalizations are found in [7, 18].

Let  $\Delta$  denote the Laplace operator on  $\mathbb{R}^n$ . Given a suitable nonnegative function  $f$  on  $B$ , solutions of the Poisson equation  $-\Delta u = f$  in  $B$  may be decomposed into a harmonic function and the Green potential of the density  $f$ , and hence the above classical results are applicable. We are now interested in the boundary behavior of positive solutions of nonlinear elliptic equations. For example, the Lane-Emden equation given by  $-\Delta u = u^p$  is one of important equations appearing in astrophysics. It goes without saying that the classical results are not applicable to positive solutions of such equations. This is one of motivations to study the boundary behavior of superharmonic functions satisfying nonlinear inequalities as mentioned in our title. It is well known that if  $u$  is superharmonic on a bounded domain  $\Omega$  in  $\mathbb{R}^n$  then there exists a unique Radon measure  $\mu_u$  on  $\Omega$  such that

$$\int_{\Omega} \phi d\mu_u = - \int_{\Omega} u \Delta \phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

If  $\mu_u$  is absolutely continuous with respect to Lebesgue measure, then its density function is written as  $-\Delta u$  (the usage of  $-\Delta u$  means implicitly that  $\mu_u$  is absolutely continuous with respect to Lebesgue measure). In this note, we study positive superharmonic functions  $u$  on  $\Omega$  satisfying

$$0 \leq -\Delta u \leq u^p \quad \text{a.e. on } \Omega, \quad (1.2)$$

where  $p > 0$  is a constant. Clearly, all positive solutions of the Lane-Emden equation satisfy (1.2). Observe that if  $u$  is a positive harmonic function on a smooth domain  $\Omega$  then

$$\frac{1}{A}\delta_\Omega(x) \leq u(x) \leq A\delta_\Omega(x)^{1-n} \quad \text{for } x \in \Omega,$$

where  $A > 1$  is a constant depending only on  $u$ ,  $n$  and  $\Omega$ , and  $\delta_\Omega(x)$  denotes the distance from  $x$  to  $\partial\Omega$ . Actually, the lower estimate is extendable to all positive superharmonic functions. However, the upper estimate does not necessarily hold for superharmonic functions. The main purpose of this note is to establish the boundary growth estimate for positive superharmonic functions satisfying (1.2).

In what follows, we suppose that  $n \geq 3$  and that  $\Omega$  is a bounded  $C^{1,1}$ -domain in  $\mathbb{R}^n$ . The open ball of center  $x$  and radius  $r$  is denoted by  $B(x, r)$ . The symbol  $A$  stands for an absolute positive constant whose value is unimportant and may change from line to line. For two positive functions  $f$  and  $g$ , we write  $f \approx g$  if there exists a constant  $A > 1$  such that  $f/A \leq g \leq Af$ . The constant  $A$  will be called the constant of comparison.

## 2 Boundary growth estimate

**Theorem 1.** *If  $0 < p \leq (n+1)/(n-1)$ , then every positive superharmonic function  $u$  on  $\Omega$  satisfying (1.2) enjoys*

$$u(x) \leq A\delta_\Omega(x)^{1-n} \quad \text{for } x \in \Omega, \quad (2.1)$$

where  $A$  is a constant depending only on  $u$ ,  $p$ ,  $n$  and  $\Omega$ .

*Proof.* We give a sketch of the proof. Since  $u$  is a positive superharmonic function on  $\Omega$  satisfying (1.2), it follows from the Riesz decomposition theorem that

$$u(x) = h(x) + \int_{\Omega} G(x, y)(-\Delta u(y))dy, \quad (2.2)$$

where  $h \geq 0$  is the greatest harmonic minorant of  $u$  on  $\Omega$  and  $G(x, y)$  is the Green function for  $\Omega$ . It is known that

$$G(x, y) \approx \min \left\{ 1, \frac{\delta_{\Omega}(x)\delta_{\Omega}(y)}{|x - y|^2} \right\} |x - y|^{2-n}, \quad (2.3)$$

where the constant of comparison depends only on  $n$  and  $\Omega$ . See [5, 24]. Also, easy computation gives the following estimate for the Martin kernel  $K(\cdot, \xi)$  of  $\Omega$  with pole at  $\xi \in \partial\Omega$ :

$$K(x, \xi) \approx \frac{\delta_{\Omega}(x)}{|x - \xi|^n},$$

where the constant of comparison depends only on  $n$  and  $\Omega$ . This and the Martin representation theorem shows that

$$h(x) \leq A\delta_{\Omega}(x)^{1-n} \quad \text{for } x \in \Omega,$$

where  $A$  depends only on  $h$ ,  $n$  and  $\Omega$ . Therefore, it is easy to see from (2.2) that for each  $j \in \mathbb{N}$ , there is a constant  $c_j > 0$  depending only on  $j$ ,  $u$ ,  $n$  and  $\Omega$  such that

$$u(x) \leq c_j \delta_{\Omega}(z)^{1-n} + \int_{B(z, \delta_{\Omega}(z)/2^j)} \frac{-\Delta u(y)}{|x - y|^{n-2}} dy \quad (2.4)$$

for  $z \in \Omega$  and  $x \in B(z, \delta_{\Omega}(z)/2^{j+1})$ . Also, (2.2) and (2.3) give that for  $z \in \Omega$ ,

$$\delta_{\Omega}(z) \int_{B(z, \delta_{\Omega}(z)/2)} (-\Delta u(y)) dy \leq A, \quad (2.5)$$

where  $A$  depends only on  $u$ ,  $n$  and  $\Omega$ .

For simplicity, we write  $B(r) = B(0, r)$ . Let  $z \in \Omega$  and  $j \in \mathbb{N}$ . Put  $r = \delta_\Omega(z)$  and  $\psi_z(\zeta) = r^{n+1}(-\Delta u(z + r\zeta))$ . Making the change of variables  $x = z + r\eta$  and  $y = z + r\zeta$ , we obtain from (2.4) and (2.5) that

$$\int_{B(1/2)} \psi_z(\zeta) d\zeta \leq A, \quad (2.6)$$

$$r^{n-1}u(z + r\eta) \leq c_j + \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta - \zeta|^{n-2}} d\zeta \quad \text{for } \eta \in B(2^{-(j+1)}). \quad (2.7)$$

Suppose that  $0 < p \leq (n+1)/(n-1)$  and let

$$\frac{n+1}{n-1} < q < \frac{n}{n-2}, \quad \ell = \left\lceil \frac{\log(q/(q-1))}{\log(q/p)} \right\rceil + 1, \quad c_0 = \max_{1 \leq j \leq \ell+1} \{c_j\}.$$

Define

$$\Psi_{z,j}(\eta) = c_0 + \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta - \zeta|^{n-2}} d\zeta.$$

It follows from (2.7) and the Hölder inequality that

$$\delta_\Omega(z)^{n-1}u(z) = r^{n-1}u(z) \leq \Psi_{z,\ell+1}(0) \leq c_0 + A\|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))}.$$

To prove (2.1), it suffices to show that

$$\|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))} \leq A, \quad (2.8)$$

where  $A$  is independent of  $z$ . Let  $s = q/p$ . Then  $s > 1$ . We claim that for each  $\kappa \geq 1$ , there is a constant  $A$  depending only on  $\kappa, c_0, p, q, n$  and  $\Omega$  such that

$$\int_{B(2^{-(j+1)})} \psi_z(\zeta)^{\kappa s} d\zeta \leq A + A \left( \int_{B(2^{-j})} \psi_z(\zeta)^\kappa d\zeta \right)^q. \quad (2.9)$$

Indeed, by the Jensen inequality,

$$\left( \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta - \zeta|^{n-2}} d\zeta \right)^\kappa \leq 2^{\kappa-1} \int_{B(2^{-j})} \frac{\psi_z(\zeta)^\kappa}{|\eta - \zeta|^{n-2}} d\zeta \quad \text{for } \eta \in B(1).$$

Therefore, by the Minkowski inequality,

$$\begin{aligned} \int_{B(2^{-j})} \Psi_{z,j}(\eta)^{\kappa q} d\eta &\leq A + A \int_{B(2^{-j})} \left( \int_{B(2^{-j})} \frac{\psi_z(\zeta)^\kappa}{|\eta - \zeta|^{n-2}} d\zeta \right)^q d\eta \\ &\leq A + A \left( \int_{B(2^{-j})} \psi_z(\zeta)^\kappa d\zeta \right)^q. \end{aligned}$$

By the way, it follows from (1.2) and  $p \leq (n+1)/(n-1)$  that

$$\psi_z(\eta) = r^{n+1}(-\Delta u(z + r\eta)) \leq r^{n+1}u(z + r\eta)^p \leq A\Psi_{z,j}(\eta)^p$$

for a.e.  $\eta \in B(2^{-(j+1)})$ . Hence (2.9) holds.

Let us apply (2.9)  $\ell$  times. Since  $s^\ell \geq q/(q-1)$ , it follows from the Hölder inequality that

$$\begin{aligned} \|\psi_z\|_{L^{q/(q-1)}(B(2^{-(\ell+1)}))} &\leq A \left( \int_{B(2^{-(\ell+1)})} \psi_z(\zeta)^{s^\ell} d\zeta \right)^{1/s^\ell} \\ &\leq A + A \left( \int_{B(2^{-\ell})} \psi_z(\zeta)^{s^{\ell-1}} d\zeta \right)^{q/s^\ell} \\ &\leq \dots \\ &\leq A + A \left( \int_{B(1/2)} \psi_z(\zeta) d\zeta \right)^{q^\ell/s^\ell}. \end{aligned}$$

In view of (2.6) we obtain (2.8). This completes the proof of Theorem 1. □

### 3 Nontangential limits of Green potentials

As an application of Theorem 1, we can obtain the following result.

**Theorem 2.** *Let  $0 < p \leq (n+1)/(n-1)$  and let  $u$  be as in Theorem 1. If, in addition, the greatest harmonic minorant of  $u$  on  $\Omega$  is the zero function, then  $u$  has nontangential limit zero almost everywhere on  $\partial\Omega$ .*

*Proof.* We start by recalling the Harnack inequality for nonnegative weak solutions  $v \in W^{1,2}(D)$  of the stationary Schrödinger equation  $\Delta v + \rho v = 0$  in  $D$ . If  $\rho$  is a measurable function on  $D$  such that  $|\rho|$  is bounded by a constant  $\nu^2$ , then there exists a constant  $A$  depending only on  $n$  such that

$$\sup_{B(x,r)} v \leq A^{\sqrt{n}+\nu r} \inf_{B(x,r)} v,$$

whenever  $B(x, 4r) \subset D$ . See [9].

Let  $B(x, 8r) \subset \Omega$  and let us apply this Harnack inequality to  $D = B(x, 4r)$  and  $\rho = -\Delta u/u$ . Since  $-\Delta u \in L_{loc}^\infty(\Omega)$  by (1.2) and Theorem 1, it follows that  $u \in C^1(\Omega) \subset W^{1,2}(D)$  and that  $u$  is a weak solution of  $\Delta u + \rho u = 0$  in  $D$ . Also, if  $1 \leq p \leq (n+1)/(n-1)$ , then Theorem 1 yields that for  $y \in D$ ,

$$0 \leq \rho(y) \leq u(y)^{p-1} \leq A\delta_\Omega(y)^{(p-1)(1-n)} \leq Ar^{(p-1)(1-n)} \leq Ar^{-2}.$$

If  $0 < p < 1$ , then we use the decay estimate  $u(y) \geq A\delta_\Omega(y)$  in  $\Omega$  to obtain  $0 \leq \rho(y) \leq Ar^{-2}$  for  $y \in D$ . In any cases, we observe that there exists a constant  $A$  depending only on  $u$ ,  $p$ ,  $n$  and  $\Omega$  such that

$$\sup_{B(x,r)} u \leq A \inf_{B(x,r)} u,$$

whenever  $B(x, 8r) \subset \Omega$ .

Finally, we apply this Harnack inequality to show the existence of nontangential limits. By our assumption,  $u$  is the Green potential of the density  $-\Delta u$ . Therefore it follows from [20] that  $u$  has minimal fine limit zero at  $\xi \in \partial\Omega \setminus E$ , where the surface measure of  $E$  is zero. Let  $\{x_j\}$  be arbitrary sequence converging to  $\xi$  within a nontangential cone at  $\xi$ . Since the bubble set  $\bigcup_j B(x_j, \delta_\Omega(x_j)/8)$  is not minimally thin at  $\xi$ , we find a sequence  $y_j \in B(x_j, \delta_\Omega(x_j)/8)$  converging to  $\xi$  such that  $u(y_j) \rightarrow 0$  as  $j \rightarrow \infty$ . Hence  $0 \leq u(x_j) \leq Au(y_j) \rightarrow 0$ . This completes the proof of Theorem 2.  $\square$

## 4 Remarks

In this final section, we give some remarks on Theorem 1.

- If  $p > (n+1)/(n-1)$ , then we can construct a  $C^2$ -function  $u$  on  $\Omega$  satisfying (1.2) such that (2.1) fails to hold. Hence the upper bound  $p \leq (n+1)/(n-1)$  is sharp in Theorem 1.

- If  $0 < p < (n + 1)/(n - 1)$ , then

$$-\Delta u = u^p \quad \text{in } \Omega$$

has a positive solution  $u \in C^2(\Omega)$  satisfying

$$u(x) \approx \frac{\delta_\Omega(x)}{|x - \xi|^n} \quad \text{for } x \in \Omega.$$

Hence the growth rate  $1 - n$  is sharp in Theorem 1.

The proofs of these remarks can be found in [14] where we actually studied positive superharmonic functions satisfying more general inequality  $0 \leq -\Delta u(x) \leq c\delta_\Omega(x)^{-\alpha}u(x)^p$  and positive solutions of  $-\Delta u(x) = c\delta_\Omega(x)^{-\alpha}u(x)^p$ .

## References

- [1] H. Aikawa, *Harmonic functions having no tangential limits*, Proc. Amer. Math. Soc. **108** (1990), no. 2, 457–464.
- [2] H. Aikawa, *Harmonic functions and Green potentials having no tangential limits*, J. London Math. Soc. (2) **43** (1991), no. 1, 125–136.
- [3] M. Arsove and A. Huber, *On the existence of non-tangential limits of subharmonic functions*, J. London Math. Soc. **42** (1967), 125–132.
- [4] R. Berman and D. Singman, *Boundary behavior of positive solutions of the Helmholtz equation and associated potentials*, Michigan Math. J. **38** (1991), no. 3, 381–393.
- [5] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. **243** (2000), no. 2, 326–337.

- [6] H. E. Bray and G. C. Evans, *A Class of Functions Harmonic within the Sphere*, Amer. J. Math. **49** (1927), no. 2, 153–180.
- [7] B. E. J. Dahlberg, *On the existence of radial boundary values for functions subharmonic in a Lipschitz domain*, Indiana Univ. Math. J. **27** (1978), no. 3, 515–526.
- [8] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), 335–400.
- [9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001.
- [10] K. Gowrisankaran and D. Singman, *Thin sets and boundary behavior of solutions of the Helmholtz equation*, Potential Anal. **9** (1998), no. 4, 383–398.
- [11] M. Hakim and N. Sibony, *Fonctions holomorphes bornées et limites tangentielles*, Duke Math. J. **50** (1983), no. 1, 133–141.
- [12] K. Hirata, *Sharpness of the Korányi approach region*, Proc. Amer. Math. Soc. **133** (2005), no. 8, 2309–2317.
- [13] K. Hirata, *Boundary behavior of solutions of the Helmholtz equation*, Canad. Math. Bull. (to appear)
- [14] K. Hirata, *The boundary growth of superharmonic functions and positive solutions of nonlinear elliptic equations*, preprint.
- [15] A. Korányi, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507–516.
- [16] A. Korányi and J. C. Taylor, *Fine convergence and parabolic convergence for the Helmholtz equation and the heat equation*, Illinois J. Math. **27** (1983), no. 1, 77–93.

- [17] J. E. Littlewood, *On a theorem of Fatou*, J. London Math. Soc. **2** (1927), 172–176.
- [18] Y. Mizuta, *On the boundary limits of Green potentials of functions*, J. Math. Soc. Japan **40** (1988), no. 4, 583–594.
- [19] A. Nagel and E. M. Stein, *On certain maximal functions and approach regions*, Adv. in Math. **54** (1984), no. 1, 83–106.
- [20] L. Naïm, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier, Grenoble **7** (1957), 183–281.
- [21] N. Privalov, *Boundary problems of the theory of harmonic and subharmonic functions in space*, Mat. Sbornik **45** (1938), 3–25.
- [22] J. Sueiro, *On maximal functions and Poisson-Szegő integrals*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 653–669.
- [23] J. M. Wu,  *$L^p$ -densities and boundary behaviors of Green potentials*, Indiana Univ. Math. J. **28** (1979), no. 6, 895–911.
- [24] Z. Zhao, *Green function for Schrödinger operator and conditioned Feynman-Kac gauge*, J. Math. Anal. Appl. **116** (1986), no. 2, 309–334.
- [25] A. Zygmund, *On a theorem of Littlewood*, Summa Brasil. Math. **2** (1949), no. 5, 51–57.

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