

ON SIEGEL WAVE FORM ON THE COVERING GROUP OF $Sp(2, \mathbb{R})$

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In this paper, we shall show that the complete analogue of the shimura correspondence of modular forms of half integral weight holds for Siegel wave forms of degree 2.

§1. Commutation relation of differential operators

As usual, we consider an element in the center of the universal enveloping algebra of Lie algebra of Lie groups G as a differential operator on G . Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbb{R})$ are given in [13].

Put

$$\begin{aligned} H_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & H_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ X_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & X_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ X_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ X_{-1} &= {}^t X_i, \quad (i = 1, 2, 3, 4). \end{aligned}$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbb{R})$ in [6] are

$$\begin{aligned} L_1 &= H_1 H_1 + H_2 H_2 + 6H_1 \\ &\quad + 2H_2 + 4X_{-1}X_1 + 8X_{-4}X_4 + 4X_{-3}X_3 + 8X_{-2}X_2, \end{aligned}$$

$$\begin{aligned} L_2 &= 16X_{-4}X_{-4}X_4X_4 + 16X_{-4}X_{-3}X_3X_4 \\ &\quad - 32X_{-4}X_{-2}X_2X_4 + 16X_{-4}X_{-2}X_3X_3 \\ &\quad + 16X_{-4}X_{-1}X_1X_4 + 8X_{-4}H_1H_2X_4 \\ &\quad + 8X_{-4}(H_1 - H_2)X_1X_3 - 16X_{-4}X_1X_1X_2 \\ &\quad + 16X_{-3}X_{-3}X_2X_4 + 16X_{-3}X_{-2}X_2X_3 \\ &\quad + 8X_{-3}X_{-1}(H_1 - H_2)X_4 + 4X_{-3}H_2H_2X_3 \\ &\quad + 8X_{-3}(H_1 + H_2)X_1X_2 + 16X_{-2}X_{-2}X_2X_2 \end{aligned}$$

$$\begin{aligned}
& -16X_{-2}X_{-1}X_{-1}X_4 + 8X_{-2}X_{-1}(H_1 + H_2)X_3 \\
& + 16X_{-2}X_{-1}X_1X_2 - 8X_{-2}H_1H_2X_2 \\
& + 4X_{-1}H_1H_1X_1 + H_1H_1H_2H_2 \\
& - 16X_{-4}H_1X_4 + 32X_{-4}H_2X_4 + 32X_{-4}X_1X_3 \\
& + 32X_{-3}X_{-1}X_4 - 8X_{-3}H_1X_3 + 16X_{-3}X_1X_2 \\
& + 16X_{-2}X_{-1}X_3 - 16X_{-2}(H_1 + H_2)X_2 \\
& + 24X_{-1}H_1X_1 + 2H_1H_1H_2 \\
& + 6H_1H_2H_2 \\
& - 48X_{-4}X_4 - 24X_{-3}X_3 - 48X_{-2}X_2 \\
& + 24X_{-1}X_1 - 2H_1H_1 + 12H_1H_2 \\
& + 6H_2H_2 - 12H_1 + 12H_2.
\end{aligned}$$

We define the Weil representation r of $G = Sp_2(\mathbf{R})$ on $V = M_{5,1}(\mathbf{R})$ by putting

$$\begin{aligned}
r \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} &= \exp(2\pi i \operatorname{tr}(S({}^tX_1X_2 + {}^txx))) f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix}, \\
r \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} &= \gamma(\det A)^{5/2} f \begin{pmatrix} X_1A \\ xA \\ X_2A \end{pmatrix}, \\
r \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} &= -2i \int_V \int_V \\
&\quad \exp(2\pi i \operatorname{tr}({}^tY_1X_2 + {}^tY_2X_1 + 2{}^tyx)) f \begin{pmatrix} Y_1 \\ y \\ Y_2 \end{pmatrix} dY_1 dY_2 dy
\end{aligned}$$

for $f \in S(V \times V)$, $S = {}^tS \in M_{2,2}(\mathbf{R})$, $A \in M_{2,2}(\mathbf{R})$ where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma = 1$ or

$-i$ according as $\det A > 0$ or < 0 . For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ put $J(g, Z) = \det(CZ + D)$.

This definition of r for the generators determines r for general $g \in G$ up to a sign. We determine a branch of $\sqrt{J(g, Z)}$ by $-\pi < \arg(\sqrt{J(g, iE)}) \leq \pi$. For $X \in V$ put $\varphi_Z(X) = \exp(2\pi i Z[X])$ and determine the sign of $r(g)$ by

$$r(g)\varphi_Z(X) = \sqrt{J(g, Z)}^{-1} \varphi_{gZ}(X).$$

Put

$$a(h, g) = \sqrt{J(h, gZ)} \sqrt{J(g, Z)} / \sqrt{J(hg, Z)}.$$

Define the double cover $\tilde{G} = \{(g, \epsilon) | g \in G, \epsilon = \pm 1\}$ of G by putting $(h, \epsilon)(g, \epsilon') = (hg, \epsilon\epsilon'a(h, g))$ and define a representation $\tilde{r}$ of $\tilde{G}$ by $\tilde{r}(g, \epsilon)f = \epsilon r(g)f$.

We can also define representation ρ of the "orthogonal group" G on $S(V \times V)$ in the following manner. First, we define linear mapping σ by

$$\sigma(X) = \begin{pmatrix} 0 & a & c & -f \\ -a & 0 & -b & -c \\ -c & b & 0 & d \\ f & c & -d & 0 \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} \in M_{5,1}(\mathbf{R}).$$

Then $g \in G$ acts on $V \times V$ by

$$\begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix}^g = \left(\sigma^{-1} \left({}^t g \left(\sigma \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \end{pmatrix} \right) g \right), \sigma^{-1} \left({}^t g \left(\sigma \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{52} \end{pmatrix} \right) g \right) \right)$$

for

$$\begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \end{pmatrix} \in M_{5,2}(\mathbf{R}) = V \times V.$$

Put

$$\rho(g)f(X) = f(X^g)$$

for $f \in S(V \times V)$, $g \in G$. Then the representations r, ρ induce the representations (differential representations) of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$ which we denote by the same letters r, ρ . With this notation, we obtain

THEOREM 1.

$$\begin{aligned} r(L_1) &= \frac{1}{2}\rho(L_1) - 5, \\ r(L_2) &= -\frac{1}{4}\rho(L_2) + \frac{1}{16}\rho(L_1)^2 - \rho(L_1) + 10. \end{aligned}$$

§2. Standard and generalised Whittaker functions

For a symmetric matrix T , we call functions $f(g) = h(Y) \exp(2\pi i \operatorname{tr}(TX))$ of $g \in G$ satisfying $L_1 f = \delta_1 f$ and $L_2 f = \delta_2 f$ with some functions h of Y and $X + iY = giE \in H_2$ generalised Whittaker functions. These functions for definite T are investigated in [10],

[16]. We deal with Whittaker functions for indefinite T here. By the same reasoning as in [10], [16], we can probably show the followings which we assume hereafter. If $F(Z)$ is a Siegel wave form satisfying $F(Z + n_0 N) = F(Z)$ for all $N = {}^t N \in M_{2,2}(\mathbf{Z})$, $L_1 F = \delta_1 F$ and $L_2 F = \delta_2 F$, $F(Z)$ is expanded as follows:

$$F(Z) = \sum_{2n_0 T = 2n_0 {}^t T \in M_{2,2}(\mathbf{Z})} A(T, Y) \exp(2\pi i \operatorname{tr}(TX))$$

where if T is indefinite

$$A(T, Y) = \int_{\Re s = \sigma} A_s(T) W_s(T, Y) ds$$

where

$$\begin{aligned} W_s(T, Y) &= W_s(S, Y[{}^t N^{-1}]), S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T = S[N] = {}^t N S N, \\ W_s \left(S, t \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \right) \\ &= y^s t^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-2\pi x k_1 k_2 t) ((k_1^2 + 1)(k_2^2 + 1)(x^2 + 1))^{s/2} \\ &\quad K_s(2\pi t \sqrt{(k_1^2 + 1)(k_2^2 + 1)(x^2 + 1)}) \\ &\quad F((s + 1/2 + \nu_1)/2, (s + 1/2 - \nu_1)/2; 1/2; -k_1^2) \\ &\quad F((s + 1/2 + \nu_2)/2, (s + 1/2 - \nu_2)/2; 1/2; -k_2^2) dk_1 dk_2, \end{aligned}$$

and

$$(1) \quad \delta_1 = (\lambda_1 + \lambda_2 - 2)/8, \delta_2 = ((\lambda_1 - \lambda_2)^2)/256 - (\lambda_1 + \lambda_2)/32 + 3/64$$

with

$$\nu_1 = (-1 + 2\sqrt{\lambda_1})/2, \nu_2 = (-1 + 2\sqrt{\lambda_2})/2.$$

$A_s(T)$ is a constant determined by the uniqueness of the above expansion. We call $A_s(T)$ a Fourier coefficient of F .

Following [20],[21], we define an embedding of $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ into $Sp(2, \mathbf{R})$ by

$${}^t \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}$$

which is consistent to an embedding $(z_1, z_2) \mapsto \kappa(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ of $H_1 \times H_1$ into H_2 . We describe the "Mellin transformation" of F which is analogous to [11] and gives rise to the spinor L function of F . The "Mellin transformation" of F is

$$\begin{aligned} L(n_0, s_1, s_2, F) &= \int_0^\infty \int_0^\infty \int_0^{n_0} \int_0^{n_0} F(\kappa(x_1 + iy_1, x_2 + iy_2)) y_1^{s_1-1} y_2^{s_2-1} dx_1 dy_1 dx_2 dy_2 \\ &= 2n_0^{2s_1} \Gamma(s_1, s_2) \sum_{m \neq 0} A_{s_1-s_2} \begin{pmatrix} 0 & m/n_0 \\ m/n_0 & 0 \end{pmatrix} |m|^{-2s_1} \end{aligned}$$

where

$$\begin{aligned} \Gamma(s_1, s_2) &= \int_0^\infty W_{s_2-s_1}(S, tE) t^{s_2+s_1+1} dt \\ &= 2^{-2} \pi^{-s_1-s_2-1} \Gamma((s_1 + s_2 - 1/2 + \nu_1)/2 + 1) \\ &\quad \Gamma((s_1 + s_2 - 1/2 - \nu_1)/2 + 1) \Gamma((s_1 + s_2 - 1/2 + \nu_2)/2 + 1) \\ &\quad \Gamma((s_1 + s_2 - 1/2 - \nu_2)/2 + 1) / \Gamma(s_1) \Gamma(s_2). \end{aligned}$$

We note that when $s_1 = s_2$, the series in $L(n_0, s_1, s_2, F)$ essentially equals the spinor L function of F . If a function W on $\tilde{G}$ satisfies

$$\begin{aligned} L_1 W &= \delta'_1 W, L_2 W = \delta'_2 W, \\ W \left(\left(\begin{pmatrix} 1 & x_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 1 \right), (g, \epsilon) \right) \\ &= \exp(2\pi i(x_0 + x_3)) W((g, \epsilon)), \\ W((g, \epsilon)(k, \epsilon')) &= W((g, \epsilon)\epsilon' / \sqrt{J(k, iE)}) \end{aligned}$$

for all $k \in K = G \cap SO(4)$, we call it a Whittaker function. It is known that such a function is unique up to a constant multiple if it exists. As stated in [20], we can construct a Whittaker function using the above generalized Whittaker function. We describe an explicit formula of a Whittaker function W . Obviously, W is determined by the values

$$\tilde{W}(t, y) = W \left(\left(\begin{pmatrix} \sqrt{ty} & 0 & 0 & 0 \\ 0 & \sqrt{t/y} & 0 & 0 \\ 0 & 0 & 1/\sqrt{yt} & 0 \\ 0 & 0 & 0 & 1/\sqrt{t/y} \end{pmatrix}, 1 \right) \right)$$

The values are

$$\begin{aligned} \tilde{W}(t, y) &= t^2 \sqrt{y/t} / 2 \int_{-\infty}^\infty \int_0^\infty w(s, t') \\ &\quad \exp(-\pi(yt'^2/(4t) + 4ty/t'^2 + 2t/y + 4s^2t/y)) t'^{-3} ds dt \end{aligned}$$

where

$$\begin{aligned}
 w(s, t') &= t'^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-2\pi s k_1 k_2 t') \\
 &\quad K_0(2\pi t' \sqrt{(k_1^2 + 1)(k_2^2 + 1)(s^2 + 1)}) \\
 &\quad F((1/2 + \nu_1)/2, (1/2 - \nu_1)/2; 1/2; -k_1^2) \\
 &\quad F((1/2 + \nu_2)/2, (1/2 - \nu_2)/2; 1/2; -k_2^2) dk_1 dk_2, \\
 \delta'_1 &= \delta_1/2 - 5, \delta'_2 = -\delta_2/4 + \delta_1^2/16 - \delta_1 + 10
 \end{aligned}$$

with δ_1, δ_2 in (1).

§3. The Shimura correspondence

In this section, we define Shimura correspondence by a theta lifting and compute the "Mellin transform" of the lifted function, which is a sort of convolution of an original function and a theta function

$$\theta(Z) = \sum_{N \in M_{2,1}(\mathbf{Z})} \exp(2\pi i Z[N]).$$

For an odd integer N , put

$$\Gamma_0(4N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbf{Z}) \mid C \equiv 0 \pmod{4N} \right\},$$

$j(\gamma, Z) = \theta(\gamma Z)/\theta(Z)$ and let χ be a Dirichlet character. If a slowly increasing function $f(Z)$ on H_2 satisfies $L_1 f = \delta'_1 f, L_2 f = \delta'_2 f, f(\gamma Z) = j(\gamma, Z) \chi(\det D) f(Z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma = \Gamma_0(4N)$, we call f a Siegel wave form. If $\int_{U \cap \Gamma \backslash U} f(uZ) du = 0$ for all unipotent radicals U of maximal parabolic subgroups of G , we call f a cusp form. We assume f is a cusp form hereafter. Since $f(Z + N) = f(Z)$ for $N = {}^t N \in M_{2,2}(\mathbf{Z})$, $f(Z)$ is expanded as follows:

$$f(Z) = \sum_{2T = 2{}^t T \in M_{2,2}(\mathbf{Z})} a(T, Y) \exp(2\pi i \operatorname{tr}(TX)).$$

Since $a(T, Y[N]) = a(T[{}^t N^{-1}], Y)$,

$$a\left(\begin{pmatrix} 0 & 0 \\ 0 & m^2 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \right]\right)$$

is expanded as the Fourier series in z . The uniqueness of Whittaker functions implies

$$\begin{aligned}
 (2) \quad & a \left(\begin{pmatrix} 0 & 0 \\ 0 & m^2 \end{pmatrix}, \begin{pmatrix} ty & 0 \\ 0 & t/y \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right] \right) \\
 &= \sum_{n \neq 0} a_n^{m^2} W \left(\left(\begin{pmatrix} mn & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1/(mn) & 0 \\ 0 & 0 & 0 & 1/m \end{pmatrix} \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \right. \right. \\
 & \quad \left. \left. \begin{pmatrix} \sqrt{ty} & 0 & 0 & 0 \\ 0 & \sqrt{t/y} & 0 & 0 \\ 0 & 0 & 1/\sqrt{yt} & 0 \\ 0 & 0 & 0 & 1/\sqrt{t/y} \end{pmatrix}, 1 \right) \right) \\
 &= \sum_{n \neq 0} a_n^{m^2} \tilde{W}(m^2|n|t, |n|y) \exp(2\pi i n x) \begin{cases} 1, & \text{if } n > 0, \\ -i, & \text{if } n < 0. \end{cases}
 \end{aligned}$$

Let S_0 be $\begin{pmatrix} E & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & E \end{pmatrix}$. Put $k(X) = \exp(\pi i \text{tr} S_0[X])$ for $X \in M_{5,2}(\mathbb{R})$,

$$\mathcal{L}_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in (1/\sqrt{4N})\mathbb{Z} \right\},$$

$$\mathcal{L}_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \sqrt{N}\mathbb{Z} \right\},$$

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \sqrt{4N}\mathbb{Z} \right\}.$$

Define a character χ' by $\chi'(n) = \chi(n) \left(\frac{-N}{n}\right)$ and theta functions by

$$\Theta'(g, h) = \sum_{l_1 \in \mathcal{L}_0, l_\theta \in \mathcal{L}_\theta, l_2 \in \mathcal{L}_2} \chi'^{-1}(4N \det l_0) r(g) \rho(h) k \begin{pmatrix} l_0 \\ l_\theta \\ l_2 \end{pmatrix},$$

$$\Theta(Z, W) = \Theta'(g, h) \sqrt{J(g, iE)}$$

for $Z = giE, W = hiE \in H_2$. Put $h(Z) = f(-(4NZ)^{-1})/\sqrt{\det Z}$ and consider

$$F(W) = \int_{\Gamma \backslash H_2} h(Z) \Theta(Z, W) |Y|^{1/2} dX dY / |Y|^3 \quad (Z = X + iY).$$

Then it is easy to see $F(\gamma Z) = \chi'^2(d) F(Z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_*^0(4N)$ where

$$\Gamma_*^0(4N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z}) \middle| B \equiv 0 \pmod{4N}, D \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{4N} \right\}.$$

Put

$$L_{4N}(s_1, s_2) = \left(\sum_{n \neq 0, (n, 4N)=1} \chi'(n) n^{-s_1-s_2-2} \right) \left(\sum_{m \neq 0, (m, 4N)=1} \chi'(m) m^{s_1-s_2-1} \right).$$

Then we obtain

THEOREM 2.

$$\begin{aligned}
& L(4N, s_1, s_2, F) \\
&= \int_0^\infty \int_0^\infty \int_0^{2N} \int_0^{2N} F(\kappa(x_1 + iy_1, x_2 + iy_2)) y_1^{s_1-1} y_2^{s_2-1} dx_1 dy_1 dx_2 dy_2 \\
&= c(s_1, s_2) \Gamma'(s_1, s_2) L_{4N}(s_1, s_2) \\
&\quad \left(\sum_{m \neq 0} \sum_{n > 0} a_n^{m^2} (m^2 n)^{-s_2+1/2} n^{s_1+1} \sum_{c > 0} c^{-2s_1-2} \sum_{(c,d)=1, d \bmod c} \exp(2\pi i d n / c) \right)
\end{aligned}$$

where $c(s_1, s_2)$ is a product of powers of $2^{s_1}, 2^{s_2}, N^{s_1}, N^{s_2}, \pi^{s_1}, \pi^{s_2}, 2, N, \pi$ and

$$\begin{aligned}
& \Gamma'(s_1, s_2) \\
&= \pi^{s_1-s_2-1/2} \Gamma((s_2 - s_1 + 1)/2) \Gamma((s_2 + s_1 + 2)/2) / \Gamma(s_1 + 1) \\
&\quad \int_0^\infty \int_0^\infty \tilde{W}(t, y) \exp(-2\pi t/y) y^{-3/2} K_{s_1+1/2}(2\pi y) t^{s_2-3/2} dt dy \\
&= c'(s_1, s_2) \\
&\quad \Gamma((s_1 + s_2 + 2 - 1/2 + \nu_1)/2) \Gamma((s_1 + s_2 + 2 - 1/2 - \nu_1)/2) \\
&\quad \Gamma((-s_1 + s_2 + 2 - 1/2 + \nu_1)/2) \Gamma((-s_1 + s_2 + 2 - 1/2 - \nu_1)/2) \\
&\quad \Gamma((s_1 + s_2 + 2 - 1/2 + \nu_2)/2) \Gamma((s_1 + s_2 + 2 - 1/2 - \nu_2)/2) \\
&\quad \Gamma((-s_1 + s_2 + 2 - 1/2 + \nu_2)/2) \Gamma((-s_1 + s_2 + 2 - 1/2 - \nu_2)/2) \\
&\quad \Gamma(s_2 + 1)^{-1} \Gamma((-s_1 + s_2 + 1)/2)^{-2} \Gamma((s_1 + s_2 + 2)/2)^{-2} \\
&\quad \Gamma(s_2 + 1) \Gamma((-s_1 + s_2 + 1)/2) \Gamma((s_1 + s_2 + 2)/2) \Gamma(s_1 + 1)^{-1}
\end{aligned}$$

with a product $c'(s_1, s_2)$ of powers of $2^{s_1}, 2^{s_2}, N^{s_1}, N^{s_2}, \pi^{s_1}, \pi^{s_2}, 2, N, \pi$.

$\Gamma'(s_1, s_2)$ essentially equals $\Gamma(s_1, s_2)$ at least when $s_2 = s_1$.

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