

Nonlocal-to-local convergence of Cahn-Hilliard equations: asymptotic analysis and applications

By

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Abstract

An overview is given on some recent results about the convergence of nonlocal Cahn-Hilliard systems to their respective local counterparts, as the convolution kernel approximates a Dirac delta. This is a brief *resumé* of some selected results from the published articles [21, 11, 14, 13] and presents the contents of the talk by the author at the RIMS conference “Harmonic analysis and Nonlinear partial differential equation” (11th–13th July 2022). The presentation is mainly focused on the ideas behind the results, without going into technical detail of the proofs.

§ 1. Introduction

The Cahn-Hilliard equation is a well-established model describing the spinodal decomposition occurring in binary mixtures (e.g. alloys or liquid/polymer solutions). Since the pioneering work [7], it has been successfully employed to study phase separation in several contexts, ranging from engineering to biology and image restoration. For a general overview of the Cahn-Hilliard equation in terms of applications and mathematical analysis we refer the reader to [22].

Given a smooth bounded domain Ω in $\mathbb{R}^d$ ($d = 2, 3$) and a prescribed final time $T > 0$, the classical local version of the Cahn-Hilliard equation is the H^{-1} -gradient flow of the Cahn-Hilliard-Modica-Mortola energy functional

$$(1.1) \quad \mathcal{E}(u) := \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega} F(u(x)) dx,$$

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where $u : \Omega \rightarrow \mathbb{R}$ denotes the phase-variable and $F : \mathbb{R} \rightarrow [0, +\infty]$ is a double-well potential. More precisely, assuming constant mobility and possibly including viscosity contributions, the Cahn-Hilliard equation reads as

$$(1.2) \quad \partial_t u - \Delta \mu = 0 \quad \text{in } (0, T) \times \Omega,$$

$$(1.3) \quad \mu \in \tau \partial_t u - \Delta u + \partial F(u) \quad \text{in } (0, T) \times \Omega,$$

$$(1.4) \quad \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \mu = 0 \quad \text{in } (0, T) \times \partial \Omega,$$

$$(1.5) \quad u(0) = u_0 \quad \text{in } \Omega,$$

where μ is the chemical potential associated to the energy $\mathcal{E}_{CH}$, ∂F is the subdifferential in the sense of convex analysis, $\tau \geq 0$ is the viscosity parameter, and u_0 is a suitable initial datum.

Almost three decades after the seminal paper [7], in the early 90's a nonlocal counterpart of the classical equation has been introduced by Giacomini and Lebowitz [19], providing a microscopic derivation of the model. The respective nonlocal Cahn-Hilliard equation is the analogous gradient flow of a nonlocal energy functional in the form

$$(1.6) \quad \mathcal{E}_\varepsilon(u) = \frac{1}{4} \int_{\Omega} \int_{\Omega} J_\varepsilon(x, y) |u(x) - u(y)|^2 \, dx \, dy + \int_{\Omega} F(u(x)) \, dx.$$

where $(J_\varepsilon)_\varepsilon$ is a family of suitable positive and symmetric convolution kernels depending of a parameter ε which encodes the range amplitude of nonlocal interactions. The corresponding nonlocal Cahn-Hilliard equation reads

$$(1.7) \quad \partial_t u_\varepsilon - \Delta \mu_\varepsilon = 0 \quad \text{in } (0, T) \times \Omega,$$

$$(1.8) \quad \mu_\varepsilon \in \tau_\varepsilon \partial_t u_\varepsilon + (J_\varepsilon * 1)u_\varepsilon - J_\varepsilon * u_\varepsilon + \partial F(u_\varepsilon) \quad \text{in } (0, T) \times \Omega,$$

$$(1.9) \quad \partial_{\mathbf{n}} \mu_\varepsilon = 0 \quad \text{in } (0, T) \times \partial \Omega,$$

$$(1.10) \quad u_\varepsilon(0) = u_{0,\varepsilon} \quad \text{in } \Omega,$$

where $(\tau_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ are the families of viscosity parameters and initial data, respectively. The nonlocal formulation of the evolution equation presents a closer connection to atomistic descriptions compared to the local models. More specifically, it allows to take into account interaction between particles at a large scale. As a result, the nonlocal Cahn-Hilliard equations has also witnessed an enormous development in the last years, both in the direction of applications and mathematical analysis, for which we refer to the contributions [1, 4, 17, 18, 20, 16] and the references therein. We only point out that while in the seminal paper by [19] Giacomini and Lebowitz the interaction kernel is assumed to be a compactly-supported smooth function, the literature of the last decades only requires the kernels to be symmetric and at least of class $W^{1,1}$. This is

indeed reasonable in order to consider the typical relevant examples of Newtonian and Bessel potentials, and to ensure enough integrability and regularity on the solutions to the corresponding evolutions. If the interaction kernel is not even of class $W^{1,1}$, the mathematical analysis of the corresponding evolutions becomes definitely non-trivial, due to the lack of regularity that one can expect on the solutions. More precisely, in the case of non- $W^{1,1}$ -kernels, one cannot give sense in general to the gradient of the solution in space: not only is this a delicate mathematical issue, but it also contrasts the diffuse-interface modelling approach which the equation is derived from. A first contribution in the direction of nonlocal Cahn-Hilliard equations with less regular kernel was given, e.g., in [16]. Let us also mention the work [1], dealing with nonlocal Cahn-Hilliard equations with kernels of singularity of order $d + s$ in the origin, where $s \in (1, 2)$.

A natural question is whether the local and nonlocal models can be rigorously related with one another. At least in a formal (but intuitive) way, the answer appears to be positive: when $J_\varepsilon(x, y) = J_\varepsilon(|x - y|)$ is suitably regular and concentrates around the origin as $\varepsilon \rightarrow 0$, the nonlocal interface evolution approaches local one. This guess is actually supported by precise variational analyses of the asymptotic behavior of (1.6) as ε tends to zero. In this direction a considerable study of nonlocal-to-local convergence for functionals in the form (1.6) has been developed by Bourgain, Brezis, and Mironescu in [5, 6], as well as by Ponce in [23, 24] in relation to Γ -convergence.

Despite the numerous contributions on nonlocal-to-local convergence of functionals, rigorous results stating the asymptotics of the nonlocal evolution (1.7)–(1.10) to the local one (1.2)–(1.5) as $\varepsilon \rightarrow 0$ were still missing. The first positive answers in the direction were given in the last years by the papers [21, 11, 14, 13], which we are going to present in this resumé.

In the first contribution [21], existence of solutions for the nonlocal model is assumed, and the nonlocal-to-local convergence is proved in the simplified setting of periodic boundary conditions and for a regular (polynomial) double-well potential satisfying a concavity condition. The interaction kernel is required to have a singularity of order 2 in the origin, so that the typical $W^{1,1}$ -regularity is not in order here.

Subsequently, in [11] the first well-posedness result is obtained for nonlocal Cahn-Hilliard equations with interaction kernel not of class $W^{1,1}$, and the corresponding nonlocal-to-local convergence is derived. This is naturally due the fact that the convergence of the energy is guaranteed for interaction kernels with a singularity up to order 2 in the origin, thus not ensuring in general $W^{1,1}$ -regularity in either dimensions 2 or 3. Here, the setting is allowed to be much more general: although periodic boundary conditions are still considered, the argument is carried out for singular (possibly degenerate) double-well potentials and accounting for suitable convection terms.

In the further paper [14], a generalisation is provided in the direction of Neumann boundary conditions, which represent the relevant choice in terms of applications. This hinges on the requirement that the viscosity coefficients τ_ε are strictly positive per every $\varepsilon > 0$, whereas the limiting viscosity coefficient τ is allowed to vanish. Also in this case, the interaction kernels providing the correct asymptotics might not be of class $W^{1,1}$, so that the well-posedness result for the nonlocal equation is obtained by hand.

Eventually, in the last paper [13] a different scaling of the interaction kernels is proposed in such a way that J_ε becomes of class $W^{1,1}$ for every $\varepsilon > 0$ and the nonlocal-to-local convergence is still preserved. This allows to rely on the available mathematical literature to solve the nonlocal problem, and is also more meaningful from the modelling aspect: indeed, the additional regularity of the interaction kernel corresponds to enhancing the effects of long-range interactions and the diffuse interface nature of the model.

As a natural prosecution of the first works [21, 11, 14, 13], the study of nonlocal-to-local convergence of phase-field models has been proactively developed. Let us mention in particular the contributions [3], where the authors prove the nonlocal-to-local convergence of a diffuse-interface model for a two-phase flow with unmatched densities, the recent works [8, 15] on the nonlocal-to-local convergence for the Cahn-Hilliard equation with degenerate mobility, and the paper [2] on the analysis of the convergence rates. Furthermore, ongoing projects by the authors of [14, 13] are being carried out in the direction of extending nonlocal-to-local techniques to the optimal control study of more general phase-field models: for possible generalisations we refer to [10, 12].

This resumé is organised as follows: Section 2 presents the mathematical setting which will be in force, while Sections 3–4–5 describe the main results obtained in the articles [11], [14], and [13], respectively. In order to maintain the resumé self-contained, only the main results and the ideas behind them are presented. Computations and arguments of the proofs are not given here, and we refer to the aforementioned published articles for any technical detail.

§ 2. Setting and notation

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a smooth bounded domain, and $T > 0$ be a final reference time. We set for convenience $Q := (0, T) \times \Omega$ and $Q_t := (0, t) \times \Omega$ for every $t \in (0, T)$. The functional spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{\varphi \in H^2(\Omega) : \partial_{\mathbf{n}}\varphi = 0 \text{ a.e. on } \partial\Omega\},$$

are endowed with the natural norms $\|\cdot\|_H$, $\|\cdot\|_V$, and $\|\cdot\|_W$, respectively. The scalar product in H and the duality pairing between V^* and V are denoted by the symbols $(\cdot, \cdot)_H$ and $\langle \cdot, \cdot \rangle_V$, respectively. Moreover, we use the notation $\varphi_\Omega := \frac{1}{|\Omega|} \langle \varphi, 1 \rangle$ for every

$\varphi \in V^*$ for the generalised spatial average.

Throughout the paper, the following structural assumptions on the nonlinearity will be in force, and not explicitly recalled in each of the following sections.

- $\tau \geq 0$ is the viscosity coefficient.
- $\gamma : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone graph with $0 \in \gamma(0)$, and $\hat{\gamma} : \mathbb{R} \rightarrow [0, +\infty]$ is the unique proper, convex and lower semicontinuous function such that $\partial\hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = 0$. For every $\lambda > 0$, the Yosida approximation of γ and the Moreau-Yosida regularisation of $\hat{\gamma}$ are denoted by $\gamma_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ and $\hat{\gamma}_\lambda : \mathbb{R} \rightarrow \mathbb{R}$, respectively.
- $\Pi : \mathbb{R} \rightarrow \mathbb{R}$ is a L -Lipschitz-continuous function for some fixed $L > 0$, and we set $\hat{\Pi} : \mathbb{R} \rightarrow \mathbb{R}$ as $\hat{\Pi}(r) := \int_0^r \Pi(s) \, ds$, $r \in \mathbb{R}$.
- the function $F : \mathbb{R} \rightarrow (-\infty, +\infty]$ is defined as $F := \hat{\gamma} + \hat{\Pi}$, and we make use of the notation $\partial F := \gamma + \Pi$ for its generalised subdifferential.

Remark. Let us point out that these assumptions include both the classical polynomial double-well potential $F_{\text{pol}}(r) := \frac{1}{4}(r^2 - 1)^2$, $r \in \mathbb{R}$, and the more interesting choices of singular potentials, such as the logarithmic one

$$F_{\log}(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2, \quad s \in (-1, 1), \quad 0 < \theta < \theta_c,$$

and the double-obstacle one

$$F_{\text{ob}}(s) = \begin{cases} \frac{1}{2}(1-s^2) & \text{if } s \in [-1, 1], \\ +\infty & \text{otherwise.} \end{cases}$$

With this notation, the local energy functional is defined as

$$E : H \rightarrow [0, +\infty], \quad E(\varphi) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla \varphi(x)|^2 \, dx & \text{if } \varphi \in V, \\ +\infty & \text{if } \varphi \in H \setminus V. \end{cases}$$

The well-posedness of the local problem is well-established (see [9]) in the following sense.

Theorem 2.1. *For every u_0 satisfying*

$$(2.1) \quad u_0 \in V, \quad \hat{\gamma}(u_0) \in L^1(\Omega), \quad (u_0)_\Omega \in \text{Int } D(\gamma),$$

there exists a triplet (u, μ, ξ) , where u is uniquely determined, such that

$$\begin{aligned} u &\in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\ \tau u &\in H^1(0, T; H), \\ \xi &\in L^2(0, T; H), \quad \xi \in \gamma(u) \quad \text{a.e. in } Q, \\ \mu &= \tau \partial_t u - \Delta u + \xi + \Pi(u) \in L^2(0, T; V), \\ \tau \mu &\in L^2(0, T; W), \\ u(0) &= u_0 \end{aligned}$$

and

$$\langle \partial_t u, \varphi \rangle_V + \int_{\Omega} \nabla \mu(x) \cdot \nabla \varphi(x) \, dx = 0 \quad \forall \varphi \in V, \quad \text{a.e. in } (0, T).$$

§ 3. First contribution

This section is devoted to the presentation of the main results obtained in [11]. As it was anticipated in the introduction, the main issue handled in [11] was to overcome the problem of low regularity of the convolution kernels: consequently, the idea is to work in a simplified setting of periodic boundary conditions and focus on alternative formulation of the nonlocal problem which can be able to pass-by the singularity of the kernels in the origin.

§ 3.1. Preliminaries

Besides of the setting introduced in Section 2, we work here under the following additional requirements, which correspond to the general setting of [23, 24] in order to recover convergence of the energy functionals.

- Ω is the d -dimensional ($d = 2, 3$) flat torus (so in particular $W = H^2(\Omega)$ here).
- $\tau = 0$.
- The kernel $J_\varepsilon : \Omega \times \Omega \rightarrow \mathbb{R}$ is defined as

$$J_\varepsilon(x, y) := \frac{1}{|x - y|^2} \rho_\varepsilon(|x - y|), \quad \text{for a.e. } (x, y) \in \Omega \times \Omega$$

where $(\rho_\varepsilon)_{\varepsilon > 0} \subset L^1_{\text{loc}}(0, +\infty)$ is a family of radial mollifiers on $\mathbb{R}$, satisfying

$$\begin{aligned} \rho_\varepsilon(r) &\geq 0 & \forall r \in \mathbb{R}, \forall \varepsilon > 0, \\ \text{supp}(\rho_\varepsilon) &\subset [0, \text{diam}(\Omega)] & \forall \varepsilon > 0, \\ \int_0^{+\infty} \rho_\varepsilon(r) r^{d-1} \, dr &= \frac{2}{M_d} & \forall \varepsilon > 0, \\ \lim_{\varepsilon \searrow 0} \int_\delta^{+\infty} \rho_\varepsilon(r) r^{d-1} \, dr &= 0 & \forall \delta > 0, \end{aligned}$$

with $M_d := \int_{S^{d-1}} |e_1 \cdot \sigma|^2 d\mathcal{H}^{d-1}(\sigma)$.

The main prototype for the family $(\rho_\varepsilon)_\varepsilon$ is given by the choice $\rho_\varepsilon(r) = \varepsilon^{-d} \rho(r/\varepsilon)$, $r \geq 0$, where $\rho : [0, +\infty) \rightarrow [0, +\infty)$ is smooth and compactly supported. Note that under such assumptions, the interaction kernel might not be of class $W^{1,1}$ (and not even in L^1 in dimension $d = 2$) due to the possible singularity in 0. In order to guarantee also a $W^{1,1}$ regularity of the corresponding kernel one needs to add some further requirements on ρ , i.e. that it degenerates at 0 at a certain rate depending on the space dimension. However, let us stress again that under no further assumptions besides the ones listed above, the interaction kernel might not be of class $W^{1,1}$ here.

With this notation, the nonlocal energy functional $E_\varepsilon : H \rightarrow [0, +\infty]$ is defined as

$$E_\varepsilon(\varphi) := \begin{cases} \frac{1}{4} \int_\Omega \int_\Omega J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy & \text{if } \varphi \in V_\varepsilon, \\ +\infty & \text{if } \varphi \in H \setminus V_\varepsilon, \end{cases}$$

where we naturally set

$$V_\varepsilon := \left\{ \varphi \in H : \int_\Omega \int_\Omega J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy < +\infty \right\}.$$

The starting point of the nonlocal-to-local convergence is the analysis of the asymptotic behaviour of the nonlocal energy functional. As anticipated above, this is provided by [5] with the following result.

Lemma 3.1. *In this setting, it holds that $V \subset V_\varepsilon$ for every $\varepsilon > 0$ and*

$$\lim_{\varepsilon \rightarrow 0^+} E_\varepsilon(\varphi) = E(\varphi) \quad \forall \varphi \in V.$$

However, the main problem is that the choice of $(\rho_\varepsilon)_\varepsilon$ providing such convergence does not ensure any reasonable regularity on the kernels $(J_\varepsilon)_\varepsilon$: in particular, the convolution with J_ε may be ill-defined in general (for example in dimension $d = 2$). Consequently, the approach adopted in [11] consists in introducing a relaxation of the definition of the nonlocal term in terms of an abstract linear operator. More precisely, we define the unbounded linear operator $(B_\varepsilon, D(B_\varepsilon))$ on H as

$$\begin{aligned} D(B_\varepsilon) &:= \{\varphi \in H : (J_\varepsilon * 1)\varphi - (J_\varepsilon * \varphi) \in H\}, \\ B_\varepsilon(\varphi) &:= (J_\varepsilon * 1)\varphi - (J_\varepsilon * \varphi), \quad \forall \varphi \in D(B_\varepsilon), \end{aligned}$$

which clearly satisfies

$$B_\varepsilon(\varphi)(x) = \int_\Omega \rho_\varepsilon(|x - y|) \frac{\varphi(x) - \varphi(y)}{|x - y|^2} dy \quad \text{for a.e. } x \in \Omega, \quad \forall \varphi \in D(B_\varepsilon).$$

The following technical preliminaries ensure indeed that the domain $D(B_\varepsilon)$ is non-trivial (for example, it is dense in H), and provide uniform bounds on the family $(B_\varepsilon)_\varepsilon$.

Lemma 3.2. *For every $\varepsilon, \sigma > 0$, there holds $C^{0,\sigma}(\overline{\Omega}) \subset D(B_\varepsilon)$. Additionally, there exists a constant $C_{\varepsilon,\sigma} > 0$ (only dependent on ε and σ) such that*

$$\|B_\varepsilon(\varphi)\|_{L^2(\Omega)} \leq C_{\varepsilon,\sigma} \|\varphi\|_{C^{0,\sigma}(\overline{\Omega})} \quad \forall \varphi \in C^{0,\sigma}(\overline{\Omega}).$$

In particular, for every $s > \frac{3}{2}$, $H^s(\Omega) \subset D(B_\varepsilon)$ and there exists $C_{\varepsilon,s} > 0$ such that

$$\|B_\varepsilon(\varphi)\|_{L^2(\Omega)} \leq C_{\varepsilon,s} \|\varphi\|_{H^s(\Omega)} \quad \forall \varphi \in H^s(\Omega).$$

Furthermore, the operator $(D(B_\varepsilon), B_\varepsilon)$ can be uniquely extended to a linear continuous operator $B_\varepsilon : V \rightarrow V^$, and there exists a positive constant C , independent of ε , such that*

$$\|B_\varepsilon(\varphi)\|_{V^*} \leq C \|\nabla \varphi\|_H \quad \forall \varphi \in V.$$

Eventually, the family $(B_\varepsilon)_\varepsilon$ is uniformly bounded in $\mathcal{L}(V, V^)$ and*

$$\lim_{\varepsilon \rightarrow 0^+} \langle B_\varepsilon(\varphi), \psi \rangle_V = \int_{\Omega} \nabla \varphi(x) \cdot \nabla \psi(x) \, dx \quad \forall \varphi, \psi \in V.$$

Let us point out that the last assertion of Lemma 3.2 ensures the nonlocal-to-local convergence also for the derivatives of the energies, as they appear in the corresponding evolutions.

The last technical result which is crucial in the asymptotic study is a generalisation of the well-know Aubin-Lions compactness results, and states, roughly-speaking, that a control on the nonlocal energies uniformly in ε ensures strong compactness in H . In order to handle time-dependency in the evolution, this is formulated as a classical compactness inequality.

Lemma 3.3. *For every $\delta > 0$ there exist constants $C_\delta > 0$ and $\varepsilon_\delta > 0$ such that, for every sequence $(f_\varepsilon)_\varepsilon \subset H$, there holds, for every $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_\delta$,*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|_H^2 \leq \delta E_{\varepsilon_1}(f_{\varepsilon_1}) + \delta E_{\varepsilon_2}(f_{\varepsilon_2}) + C_\delta \|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{V^*}^2.$$

§ 3.2. Main results

The main results obtained in the contribution [11] concern the well-posedness at $\varepsilon > 0$ fixed of the nonlocal problem, and the nonlocal-to-local convergence as $\varepsilon \rightarrow 0^+$.

Theorem 3.4. *In this setting, for every $\varepsilon > 0$ let*

$$(3.1) \quad u_{0,\varepsilon} \in V_\varepsilon, \quad \hat{\gamma}(u_{0,\varepsilon}) \in L^1(\Omega), \quad (u_{0,\varepsilon})_\Omega \in \text{Int } D(\gamma).$$

Then, there exists $\varepsilon_0 > 0$ having the following property: for every $\varepsilon < \varepsilon_0$ there exists a triplet $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$, where u_ε is uniquely determined, such that

$$\begin{aligned} u_\varepsilon &\in H^1(0, T; V^*) \cap L^2(0, T; V), \\ \xi_\varepsilon &\in L^2(0, T; H), \quad \xi_\varepsilon \in \gamma(u_\varepsilon) \quad \text{a.e. in } Q, \\ \mu_\varepsilon &= B_\varepsilon(u_\varepsilon) + \xi_\varepsilon + \Pi(u_\varepsilon) \in L^2(0, T; V), \\ u_\varepsilon(0) &= u_{0,\varepsilon}, \end{aligned}$$

and

$$\langle \partial_t u_\varepsilon, \varphi \rangle_V + \int_\Omega \nabla \mu_\varepsilon(x) \cdot \nabla \varphi(x) dx = 0 \quad \forall \varphi \in V, \quad \text{a.e. in } (0, T).$$

Furthermore, if $(u_{0,\varepsilon}^1)_\varepsilon$ and $(u_{0,\varepsilon}^2)_\varepsilon$ are two sets of data satisfying (3.1) with $(u_{0,\varepsilon}^1)_\Omega = (u_{0,\varepsilon}^2)_\Omega$, then there exists a positive constant M_ε , such that, for any respective solutions $(u_\varepsilon^1, \mu_\varepsilon^1, \xi_\varepsilon^1)$ and $(u_\varepsilon^2, \mu_\varepsilon^2, \xi_\varepsilon^2)$ to the nonlocal equation as above, it holds

$$\|u_\varepsilon^1 - u_\varepsilon^2\|_{C^0([0,T];V^*)}^2 + \|E_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2)\|_{L^1(0,T)} \leq M_\varepsilon \|u_{0,\varepsilon}^1 - u_{0,\varepsilon}^2\|_{V^*}^2.$$

Theorem 3.5. In this setting, let u_0 and $(u_{0,\varepsilon})_\varepsilon$ satisfy (2.1) and (3.1), respectively, and

$$\begin{aligned} \sup_{\varepsilon \in (0, \varepsilon_0)} (\|u_{0,\varepsilon}\|_H^2 + \|\hat{\gamma}(u_{0,\varepsilon})\|_{L^1(\Omega)} + E_\varepsilon(u_{0,\varepsilon})) &< +\infty, \\ \exists [a_0, b_0] \subset \text{Int } D(\gamma) : \quad a_0 \leq (u_{0,\varepsilon})_\Omega \leq b_0 \quad \forall \varepsilon \in (0, \varepsilon_0), \\ u_{0,\varepsilon} &\rightharpoonup u_0 \quad \text{in } H \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

Let $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ be a solution to the nonlocal equation associated to $u_{0,\varepsilon}$ given by Theorem 3.4, and let (u, μ, ξ) be a solution to the local equation associated to u_0 given by Theorem 2.1. Then, as $\varepsilon \rightarrow 0^+$, the unique solution components $(u_\varepsilon)_\varepsilon$ and u satisfy

$$\begin{aligned} u_\varepsilon &\rightarrow u \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V), \\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u \quad \text{weakly}^* \text{ in } L^2(0, T; V^*). \end{aligned}$$

§ 4. Second contribution

This section is devoted to the presentation of the main results obtained in [14]. Here, the attention is shifted to the choice of boundary conditions: once it is indeed clear how to treat the non locality with non-regular kernels, the main issue becomes the identification of the natural boundary condition for the phase variable that arises at the local limit. The answer to this question is suggested by some refined Γ -convergence result of the energies, and then rigorously confirmed in [14].

§ 4.1. Preliminaries

Still working in the setting introduced in Section 2, here we aim at generalising the results to the case of Neumann boundary conditions and possibly viscous terms: while the limiting viscosity coefficient τ is allowed to vanish, in order to make the nonlocal equation well-posed we need to assume strictly positive viscosity at $\varepsilon > 0$. The choice of the kernels remains analogous to the one of the previous section. We assume thus the following conditions.

- $\tau_\varepsilon > 0$ for all $\varepsilon > 0$ and $\lim_{\varepsilon \rightarrow 0^+} \tau_\varepsilon = \tau$.
- The kernel $J_\varepsilon : \Omega \times \Omega \rightarrow \mathbb{R}$ is defined as

$$J_\varepsilon(x, y) := \frac{1}{|x - y|^2} \rho_\varepsilon(|x - y|), \quad \text{for a.e. } (x, y) \in \Omega \times \Omega$$

where $(\rho_\varepsilon)_{\varepsilon > 0} \subset L^1_{\text{loc}}(0, +\infty)$ is a family of radial mollifiers on $\mathbb{R}$, satisfying

$$\begin{aligned} \rho_\varepsilon(r) &\geq 0 & \forall r \in \mathbb{R}, \forall \varepsilon > 0, \\ \text{supp}(\rho_\varepsilon) &\subset [0, \text{diam}(\Omega)] & \forall \varepsilon > 0, \\ \int_0^{+\infty} \rho_\varepsilon(r) r^{d-1} dr &= \frac{2}{M_d} & \forall \varepsilon > 0, \\ \lim_{\varepsilon \searrow 0} \int_\delta^{+\infty} \rho_\varepsilon(r) r^{d-1} dr &= 0 & \forall \delta > 0, \end{aligned}$$

with $M_d := \int_{S^{d-1}} |e_1 \cdot \sigma|^2 d\mathcal{H}^{d-1}(\sigma)$.

Let us point out that the assumptions on the interaction kernel are the same as in Section 3.1, so that J_ε might be singular (i.e. not in $W^{1,1}$ or L^1) also in this case. This is the main reason why in general we need the positive viscosity coefficients τ_ε at the nonlocal level.

However, let us also note for completeness that this is not the only way of tackling the nonlocal problem. Indeed, one could alternatively add some additional requirements on $(\rho_\varepsilon)_\varepsilon$, such as a suitable degeneracy at zero, that guarantee the classical $W^{1,1}$ regularity on the respective kernels. In such a way, the classical theory on nonlocal Cahn-Hilliard equations applies and the viscosity coefficients could also be set to zero. For example, such approach has been proposed in [3].

The main novelty brought by the contribution [14] is based on the construction of an alternative variational setting in order to frame the nonlocal problem.

For every $\varepsilon > 0$, we set

$$V_\varepsilon := \left\{ \varphi \in H : \int_\Omega \int_\Omega J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy < +\infty \right\},$$

$$E_\varepsilon(\varphi) := \frac{1}{4} \int_\Omega \int_\Omega J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy, \quad \varphi \in V_\varepsilon.$$

Denoting by $a_\varepsilon : V_\varepsilon \times V_\varepsilon \rightarrow \mathbb{R}$ the natural bilinear form associated to E_ε ,

$$a_\varepsilon(\varphi, \psi) := \frac{1}{2} \int_\Omega \int_\Omega J_\varepsilon(x, y) (\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) dx dy, \quad \varphi, \psi \in V_\varepsilon,$$

we also define

$$W_\varepsilon := \left\{ \varphi \in V_\varepsilon : \exists f \in H : a_\varepsilon(\varphi, \psi) = \int_\Omega f(x) \psi(x) dx \text{ for all } \psi \in V_\varepsilon \right\},$$

$$B_\varepsilon(\varphi)(x) := \int_\Omega J_\varepsilon(x, y) (\varphi(x) - \varphi(y)) dy, \quad \text{for a.e. } x \in \Omega, \quad \varphi \in W_\varepsilon,$$

which trivially satisfies

$$(B_\varepsilon(\varphi), \psi)_H = \frac{1}{2} \int_\Omega \int_\Omega J_\varepsilon(x, y) (\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) dx dy = a_\varepsilon(\varphi, \psi)$$

for all $\varphi, \psi \in W_\varepsilon$. We point out that $E_\varepsilon : V_\varepsilon \rightarrow [0, +\infty)$ is convex and $B_\varepsilon : H \rightarrow H$ is a linear unbounded operator with domain W_ε . Additionally, we define the maps

$$\|\cdot\|_{V_\varepsilon} : V_\varepsilon \rightarrow [0, +\infty), \quad \|\cdot\|_{W_\varepsilon} : W_\varepsilon \rightarrow [0, +\infty)$$

as

$$\|\varphi\|_{V_\varepsilon} := \sqrt{\|\varphi\|_H^2 + 2E_\varepsilon(\varphi)}, \quad \|\varphi\|_{W_\varepsilon} := \sqrt{\|\varphi\|_H^2 + \|B_\varepsilon(\varphi)\|_H^2},$$

and the bilinear forms

$$(\cdot, \cdot)_{V_\varepsilon} : V_\varepsilon \times V_\varepsilon \rightarrow [0, +\infty), \quad (\cdot, \cdot)_{W_\varepsilon} : W_\varepsilon \times W_\varepsilon \rightarrow [0, +\infty)$$

as

$$(\varphi_1, \varphi_2)_{V_\varepsilon} := (\varphi_1, \varphi_2)_H + \frac{1}{2} \int_\Omega \int_\Omega J_\varepsilon(x, y) (\varphi_1(x) - \varphi_1(y))(\varphi_2(x) - \varphi_2(y)) dx dy,$$

$$(\varphi_1, \varphi_2)_{W_\varepsilon} := (\varphi_1, \varphi_2)_H + (B_\varepsilon(\varphi_1), B_\varepsilon(\varphi_2))_H.$$

We collect some properties in the next lemma.

Lemma 4.1. *The following properties hold for every $\varepsilon > 0$.*

1. *The spaces V_ε and W_ε , endowed with the norms $\|\cdot\|_{V_\varepsilon}$ and $\|\cdot\|_{W_\varepsilon}$, respectively, are complete.*

2. The bilinear forms $(\cdot, \cdot)_{V_\varepsilon}$ and $(\cdot, \cdot)_{W_\varepsilon}$ are scalar products on V_ε and W_ε inducing the norms $\|\cdot\|_{V_\varepsilon}$ and $\|\cdot\|_{W_\varepsilon}$, respectively. In particular, V_ε and W_ε are Hilbert spaces.
3. For every $\sigma \in (0, 1]$ we have $C^{0,\sigma}(\overline{\Omega}) \hookrightarrow W_\varepsilon$ continuously, and there exists $C_{\varepsilon,\sigma} > 0$ such that

$$B_\varepsilon(\varphi) \in L^\infty(\Omega), \quad \|B_\varepsilon(\varphi)\|_{L^\infty(\Omega)} \leq C_{\varepsilon,\sigma} \|\varphi\|_{C^{0,\sigma}(\overline{\Omega})} \quad \forall \varphi \in C^{0,\sigma}(\overline{\Omega}).$$

4. The following inclusions are continuous and dense:

$$W_\varepsilon \hookrightarrow V_\varepsilon \hookrightarrow H.$$

Moreover, $B_\varepsilon : D(B_\varepsilon) \subset H \rightarrow H$, with $D(B_\varepsilon) = W_\varepsilon$, is maximal monotone on H .

5. The unbounded linear operator $B_\varepsilon : H \rightarrow H$ extends to a bounded linear operator $B_\varepsilon : V_\varepsilon \rightarrow V_\varepsilon^*$, and it holds that

$$\|B_\varepsilon(\varphi)\|_{V_\varepsilon^*} \leq \|\varphi\|_{V_\varepsilon} \quad \forall \varphi \in V_\varepsilon.$$

Moreover, such extension coincides with the linear operator $A_\varepsilon : V_\varepsilon \rightarrow V_\varepsilon^*$ associated to the bilinear form a_ε , defined as

$$A_\varepsilon(\varphi) := a_\varepsilon(\varphi, \cdot), \quad \varphi \in V_\varepsilon.$$

6. The map $E_\varepsilon : V_\varepsilon \rightarrow [0, +\infty)$ is of class C^1 and $DE_\varepsilon = B_\varepsilon : V_\varepsilon \rightarrow V_\varepsilon^*$.

The main idea on how to generalise to Neumann boundary conditions stems directly for the following refined Γ -convergence result of the energy, which is due to Ponce: this shows that the nonlocal energies E_ε rigorously defined above Γ -converge to the local energy associated to homogeneous Neumann boundary conditions.

Lemma 4.2. *The following inclusion is continuous*

$$V \hookrightarrow V_\varepsilon,$$

and there exists a constant C , independent of ε , such that

$$\|\varphi\|_{V_\varepsilon} \leq C \|\varphi\|_V \quad \forall \varphi \in V.$$

For every $\varphi, \varphi_1, \varphi_2 \in V$, there holds

$$\lim_{\varepsilon \rightarrow 0^+} E_\varepsilon(\varphi) = \frac{1}{2} \int_{\Omega} |\nabla \varphi(x)|^2 dx, \quad \lim_{\varepsilon \rightarrow 0^+} \langle B_\varepsilon(\varphi_1), \varphi_2 \rangle_{V_\varepsilon} = \int_{\Omega} \nabla \varphi_1(x) \cdot \nabla \varphi_2(x) dx.$$

Finally, for every $\varphi \in H$ and for every sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset H$ with $\varphi_\varepsilon \rightarrow \varphi$ in H , we have

$$\liminf_{\varepsilon \rightarrow 0^+} E_\varepsilon(\varphi_\varepsilon) \geq E(\varphi) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla \varphi(x)|^2 dx & \text{if } \varphi \in V, \\ +\infty & \text{if } \varphi \in H \setminus V. \end{cases}$$

In other words, $(E_\varepsilon)_{\varepsilon>0}$ Γ -converges to E with respect to the norm-topology of H .

Also in this scenario, one is able to prove the crucial compactness inequality that allows to prove the nonlocal-to-local convergence.

Lemma 4.3. *For every $\delta > 0$ there exist constants $C_\delta > 0$ and $\varepsilon_\delta > 0$ such that, for every sequence $(f_\varepsilon)_{\varepsilon \in (0, \varepsilon_\delta)} \subset H$, there holds*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|_H^2 \leq \delta E_{\varepsilon_1}(f_{\varepsilon_1}) + \delta E_{\varepsilon_2}(f_{\varepsilon_2}) + C_\delta \|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{V^*}^2 \quad \forall \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_\delta).$$

§ 4.2. Main results

The main results obtained in the contribution [14] concern the well-posedness at $\varepsilon > 0$ fixed of the nonlocal problem, and the nonlocal-to-local convergence as $\varepsilon \rightarrow 0^+$.

Theorem 4.4. *In this setting, let $\varepsilon > 0$ and*

$$(4.1) \quad u_{0,\varepsilon} \in V_\varepsilon, \quad \hat{\gamma}(u_{0,\varepsilon}) \in L^1(\Omega), \quad (u_{0,\varepsilon})_\Omega \in \text{Int } D(\gamma),$$

Then, there exists a triplet $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$, where u_ε is uniquely determined, such that

$$\begin{aligned} u_\varepsilon &\in H^1(0, T; H) \cap L^\infty(0, T; V_\varepsilon) \cap L^2(0, T; W_\varepsilon), \\ \xi_\varepsilon &\in L^2(0, T; H), \quad \xi_\varepsilon \in \gamma(u_\varepsilon) \quad \text{a.e. in } Q, \\ \mu_\varepsilon &= \tau_\varepsilon \partial_t u_\varepsilon + B_\varepsilon(u_\varepsilon) + \xi_\varepsilon + \Pi(u_\varepsilon) \in L^2(0, T; W), \\ u_\varepsilon(0) &= u_{0,\varepsilon}, \\ \partial_t u_\varepsilon - \Delta \mu_\varepsilon &= 0. \end{aligned}$$

Furthermore, if $(u_{0,\varepsilon}^1)_\varepsilon$ and $(u_{0,\varepsilon}^2)_\varepsilon$ are two sets of data satisfying (4.1) with $(u_{0,\varepsilon}^1)_\Omega = (u_{0,\varepsilon}^2)_\Omega$, then there exists a positive constant M_ε such that, for any respective solutions $(u_\varepsilon^1, \mu_\varepsilon^1, \xi_\varepsilon^1)$ and $(u_\varepsilon^2, \mu_\varepsilon^2, \xi_\varepsilon^2)$ to the nonlocal equation as above, it holds

$$\begin{aligned} &\|u_\varepsilon^1 - u_\varepsilon^2\|_{C^0([0,T];V^*)}^2 + \tau_\varepsilon \|u_\varepsilon^1 - u_\varepsilon^2\|_{C^0([0,T];H)}^2 + \|E_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2)\|_{L^1(0,T)} \\ &\leq M_\varepsilon (\|u_{0,\varepsilon}^1 - u_{0,\varepsilon}^2\|_{V^*}^2 + \tau_\varepsilon \|u_{0,\varepsilon}^1 - u_{0,\varepsilon}^2\|_H^2). \end{aligned}$$

Let us point out that the viscosity coefficients $(\tau_\varepsilon)_\varepsilon$ are required to be strictly positive in order for the nonlocal equation to be well-posed: the reason is the singularity of the convolution kernels $(J_\varepsilon)_\varepsilon$. In the asymptotic result, however, the limiting viscosity coefficient τ is allowed to vanish: when $\tau > 0$ this is a nonlocal-to-local convergence results for viscous Cahn-Hilliard equations, while if $\tau = 0$ we have a combined nonlocal-to-local and vanishing-viscosity convergence.

Theorem 4.5. *In this setting, let u_0 and $(u_{0,\varepsilon})_\varepsilon$ satisfy (2.1) and (4.1), respectively, and assume that there exists $\varepsilon_0 > 0$ such that*

$$\begin{aligned} \sup_{\varepsilon \in (0, \varepsilon_0)} (\|u_{0,\varepsilon}\|_{V_\varepsilon}^2 + \|\hat{\gamma}(u_{0,\varepsilon})\|_{L^1(\Omega)}) &< +\infty, \\ \exists [a_0, b_0] \subset \text{Int } D(\gamma) : \quad a_0 \leq (u_{0,\varepsilon})_\Omega \leq b_0 \quad \forall \varepsilon \in (0, \varepsilon_0), \\ u_{0,\varepsilon} &\rightharpoonup u_0 \quad \text{in } H \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

Let $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ be a solution to the nonlocal equation associated to $u_{0,\varepsilon}$ and τ_ε given by Theorem 4.4. Then, there exists a solution (u, μ, ξ) to the local equation associated to u_0 and τ , as given by Theorem 2.1, such that, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned} u_\varepsilon &\rightarrow u && \text{in } C^0([0, T]; H), \\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u && \text{in } L^2(0, T; V^*), \\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u && \text{in } L^2(0, T; H) \quad \text{if } \tau > 0, \\ \tau_\varepsilon \partial_t u_\varepsilon &\rightarrow 0 && \text{in } L^2(0, T; H) \quad \text{if } \tau = 0, \\ \mu_\varepsilon &\rightharpoonup \mu && \text{in } L^2(0, T; V), \\ \mu_\varepsilon &\rightharpoonup \mu && \text{in } L^2(0, T; W) \quad \text{if } \tau > 0, \\ \xi_\varepsilon &\rightharpoonup \xi && \text{in } L^2(0, T; H). \end{aligned}$$

§ 5. Third contribution

This section is devoted to the presentation of the main results obtained in [13], which provides the last generalisation to the nonlocal-to-local convergence also including regular kernels of class $W^{1,1}$. More specifically, a new scaling of the kernels is proposed so that the $W^{1,1}$ -regularity is achieved at $\varepsilon > 0$ fixed and the nonlocal-to-local convergence is preserved. This allows to relax the conditions on the viscosity coefficients that were necessary in the previous section: for this reason we only focus on the case with no viscosity here, both at the local and nonlocal level. As far as boundary conditions are concerned, Neumann conditions can be handled.

§ 5.1. Preliminaries

The following conditions are assumed.

- $\tau = 0$.
- $\alpha \in [0, d-1)$ is fixed.
- $\rho : [0, +\infty) \rightarrow [0, +\infty)$ is of class C^1 , and such that

$$s \mapsto |\rho'(s)|s^{d-1-\alpha} \in L^1(\mathbb{R}_+),$$

with the renormalization

$$\int_0^{+\infty} \rho(s) s^{d+1-\alpha} ds = \frac{2}{C_d}, \quad C_d := \int_{S^{d-1}} |\sigma \cdot e_1|^2 d\mathcal{H}^{d-1}(\sigma).$$

The family of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ is defined as

$$\rho_\varepsilon(r) := \frac{1}{\varepsilon^d} \rho(r/\varepsilon), \quad r \geq 0, \quad \varepsilon > 0.$$

The convolution kernel $J_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$J_\varepsilon(z) := \frac{1}{\varepsilon^{2-\alpha}} \rho_\varepsilon(|z|) \frac{1}{|z|^\alpha}, \quad z \in \mathbb{R}^d,$$

and we set

$$(J_\varepsilon * \varphi)(x) := \int_\Omega J_\varepsilon(x-y) \varphi(y) dy, \quad x \in \Omega, \quad \varphi \in L^1(\Omega).$$

- for every $\varepsilon > 0$ there exist $c_\varepsilon^0, \lambda_\varepsilon^0 > 0$ such that

$$\gamma'_\lambda(r) + \Pi'(r) + (J_\varepsilon * 1)(x) \geq c_\varepsilon^0 \quad \text{for a.e. } (r, x) \in \mathbb{R} \times \Omega, \quad \forall \lambda \in (0, \lambda_\varepsilon^0).$$

With respect to the previous sections, the assumptions on the interaction kernels are much more general. First of all, we do not require that ρ is compactly supported anymore: possible examples of ρ are given by Bessel-like potentials in the form $\rho(r) = ae^{-br^2}$, $r \geq 0$, with suitable renormalisation constants $a, b > 0$. Secondly, the assumptions allow to cover the regular case of kernels of class $W^{1,1}$: the main idea is the introduction of the parameter α . Indeed, in this context J_ε has a singularity of order α in the origin, and not of order 2 as in the two previous sections: hence, the smaller α is, the more regular J_ε is. In such a way, the $W^{1,1}$ -regularity of the kernel is achieved irrespectively of the behaviour of ρ at zero, i.e. even if ρ is non-degenerate at zero. The main properties of the kernel are collected in the following lemmata.

Lemma 5.1. *For every $\varepsilon > 0$, it holds that $J_\varepsilon \in W^{1,1}(\mathbb{R}^d)$.*

Proof. Using the definition of J_ε and integrating by substitution yields

$$\int_{\mathbb{R}^d} J_\varepsilon(z) dz = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \rho(|w|) |w|^{-\alpha} dw = \frac{1}{\varepsilon^2} |S^{d-1}| \int_0^{+\infty} \rho(r) r^{d-1-\alpha} dr,$$

with the last term being well-defined thanks to the integrability assumptions on ρ . This shows that $J_\varepsilon \in L^1(\mathbb{R}^d)$. Furthermore, elementary computations (see [13] for details) yield

$$|\nabla J_\varepsilon(z)| \leq \frac{1}{\varepsilon^{d+2-\alpha}} \left(\frac{1}{\varepsilon} \frac{|\rho'(|z|/\varepsilon)|}{|z|^\alpha} + \alpha \frac{\rho(|z|/\varepsilon)}{|z|^{\alpha+1}} \right),$$

where integration by substitution gives

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \frac{|\rho'(|z|/\varepsilon)|}{|z|^\alpha} dz = \frac{1}{\varepsilon^{\alpha+1-d}} |S^{d-1}| \int_0^{+\infty} |\rho'(r)| r^{d-1-\alpha} dr$$

and

$$\alpha \int_{\mathbb{R}^d} \frac{\rho(|z|/\varepsilon)}{|z|^{\alpha+1}} dz = \frac{\alpha}{\varepsilon^{\alpha+1-d}} |S^{d-1}| \int_0^{+\infty} \rho(r) r^{d-2-\alpha} dr.$$

The right-hand side is then finite again by the assumptions on ρ , and $J_\varepsilon \in W^{1,1}(\mathbb{R}^d)$. $\square$

Owing to the regularity of the kernel, the nonlocal energy functional can be defined here on the whole space H . Indeed, for every $\varphi \in H$ it holds that

$$\frac{1}{4} \int_{\Omega} \int_{\Omega} J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy = \frac{1}{2} ((J_\varepsilon * 1)\varphi - J_\varepsilon * \varphi, \varphi)_H \leq \|J_\varepsilon\|_{L^1(\mathbb{R}^d)} \|\varphi\|_H^2,$$

so that we can set

$$E_\varepsilon : H \rightarrow [0, +\infty), \quad E_\varepsilon(\varphi) := \frac{1}{4} \int_{\Omega} \int_{\Omega} J_\varepsilon(x, y) |\varphi(x) - \varphi(y)|^2 dx dy, \quad \varphi \in H.$$

Lemma 5.2. *For all $\varepsilon > 0$, $E_\varepsilon : H \rightarrow [0, +\infty)$ is of class C^1 and $DE_\varepsilon : H \rightarrow H$ satisfies*

$$\begin{aligned} \langle DE_\varepsilon(\varphi), \zeta \rangle &= \int_{\Omega} ((J_\varepsilon * 1)\varphi - J_\varepsilon * \varphi)(x) \zeta(x) dx \\ &= \frac{1}{2} \int_{\Omega \times \Omega} J_\varepsilon(x - y) (\varphi(x) - \varphi(y)) (\zeta(x) - \zeta(y)) dx dy \quad \varphi, \zeta \in H. \end{aligned}$$

Despite the different scaling of the kernels $(J_\varepsilon)_\varepsilon$, one is still able to show asymptotic convergence of the energies (and of their derivatives) to the respective local counterparts.

Lemma 5.3. *For every $\varphi, \zeta \in V$ it holds that*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} E_\varepsilon(\varphi) &= E(\varphi), \\ \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} ((J_\varepsilon * 1)\varphi - J_\varepsilon * \varphi)(x) \zeta(x) dx &= \int_{\Omega} \nabla \varphi(x) \cdot \nabla \zeta(x) dx. \end{aligned}$$

Moreover, for every sequence $(\varphi_\varepsilon)_{\varepsilon > 0} \subset H$ and $\varphi \in H$ it holds that

$$\begin{aligned} \sup_{\varepsilon > 0} E_\varepsilon(\varphi_\varepsilon) < +\infty &\quad \Rightarrow \quad (\varphi_\varepsilon)_\varepsilon \text{ is relatively compact in } H, \\ \varphi_\varepsilon \rightarrow \varphi \text{ in } H &\quad \Rightarrow \quad E(\varphi) \leq \liminf_{\varepsilon \rightarrow 0^+} E_\varepsilon(\varphi_\varepsilon). \end{aligned}$$

Analogously to how we proceeded in the previous sections, we introduce the operators

$$B_\varepsilon : H \rightarrow H, \quad B_\varepsilon(\varphi) := DE_\varepsilon(\varphi) = (J_\varepsilon * 1)\varphi - J_\varepsilon * \varphi, \quad \varphi \in H,$$

and

$$B : V \rightarrow V^*, \quad \langle B(\varphi), \zeta \rangle := \int_{\Omega} \nabla \varphi(x) \cdot \nabla \zeta(x) \, dx, \quad \varphi, \zeta \in V.$$

Let us note that Lemma 5.3 ensures that $B_{\varepsilon}(\varphi) \rightharpoonup B(\varphi)$ in V^* for every $\varphi \in V$ as $\varepsilon \rightarrow 0^+$, as well as the fact that if $(\varphi_{\varepsilon})_{\varepsilon} \subset V$ satisfies $\varphi_{\varepsilon} \rightarrow \varphi$ in V then $B_{\varepsilon}(\varphi_{\varepsilon}) \rightharpoonup B(\varphi)$ in V^* . However, in typical applications to Cahn-Hilliard, strong convergence in the space V for the phase-variable is definitely a too strong condition to hope to achieve. For this reason, it is crucial to provide a weaker identification result for the limit of $(B_{\varepsilon})_{\varepsilon}$, without relying on strong compactness in V . Roughly speaking, the idea is to exploit the symmetry of B_{ε} : indeed, given some suitable test function ζ , one has $\langle B_{\varepsilon}(\varphi_{\varepsilon}), \zeta \rangle = \langle B_{\varepsilon}(\zeta), \varphi_{\varepsilon} \rangle$, so that the guess is that one only needs $\varphi_{\varepsilon} \rightarrow \varphi$ in H and $\zeta \in W$ with $B_{\varepsilon}(\zeta) \rightharpoonup -\Delta \zeta$ in H . Unfortunately, the class of test functions ζ for which $(B_{\varepsilon}(\zeta))_{\varepsilon}$ is at least bounded in H is not entirely clear, and the intuitive guess $\zeta \in W$ does not seem to work. This requires some technical effort in showing that any function $\zeta \in W$ can be suitably approximated by a sequence $(\zeta_{\varepsilon})_{\varepsilon}$ such that $B_{\varepsilon}(\zeta_{\varepsilon}) \rightharpoonup -\Delta \zeta$ in H , and this allows indeed to obtain the desired characterisation. We recall the result in the following.

Proposition 5.4. *Let $(\varphi_{\varepsilon})_{\varepsilon} \subset H$ and $\varphi \in H$ be such that $\varphi_{\varepsilon} \rightarrow \varphi$ in H and $(E_{\varepsilon}(\varphi_{\varepsilon}))_{\varepsilon}$ is uniformly bounded. Then, $\varphi \in V$ and $B_{\varepsilon}(\varphi_{\varepsilon}) \rightharpoonup B(\varphi)$ in V^* .*

Eventually, as in the previous section it is possible to show that the rescaled kernels still satisfy the crucial compactness inequality needed in the nonlocal-to-local asymptotics.

Lemma 5.5. *For every $\delta > 0$, there exist $C_{\delta} > 0$ and $\varepsilon_{\delta} > 0$ such that, for every sequence $(\varphi_{\varepsilon})_{\varepsilon > 0} \subset H$ it holds*

$$\|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|_H^2 \leq \delta (E_{\varepsilon_1}(\varphi_{\varepsilon_1}) + E_{\varepsilon_2}(\varphi_{\varepsilon_2})) + C_{\delta} \|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|_{V^*}^2 \quad \forall \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_{\delta}).$$

§ 5.2. Main results

The main result obtained in the contribution [13] concerns mainly the nonlocal-to-local convergence as $\varepsilon \rightarrow 0^+$. Indeed, having recovered the $W^{1,1}$ -regularity for the kernels, well-posedness of the nonlocal problem is well-established in the literature.

Theorem 5.6. *In this setting, let $\varepsilon > 0$ and*

$$(5.1) \quad u_{0,\varepsilon} \in H, \quad \hat{\gamma}(u_{0,\varepsilon}) \in L^1(\Omega), \quad (u_{0,\varepsilon})_{\Omega} \in \text{Int } D(\gamma),$$

Then, there exists a triple $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$, where u_ε is uniquely determined, such that

$$\begin{aligned} u_\varepsilon &\in H^1(0, T; V^*) \cap L^2(0, T; V), \\ \xi_\varepsilon &\in L^2(0, T; H), \quad \xi_\varepsilon \in \gamma(u_\varepsilon) \quad \text{a.e. in } Q, \\ \mu_\varepsilon &= B_\varepsilon(u_\varepsilon) + \xi_\varepsilon + \Pi(u_\varepsilon) \in L^2(0, T; V), \\ u_\varepsilon(0) &= u_{0,\varepsilon}, \end{aligned}$$

and

$$\langle \partial_t u_\varepsilon, \varphi \rangle_V + \int_\Omega \nabla \mu_\varepsilon(x) \cdot \nabla \varphi(x) \, dx = 0 \quad \forall \varphi \in V, \quad \text{a.e. in } (0, T).$$

Theorem 5.7. *In this setting, let u_0 and $(u_{0,\varepsilon})_\varepsilon$ satisfy (2.1) and (5.1), respectively, and assume that there exists $\varepsilon_0 > 0$ such that*

$$\begin{aligned} \sup_{\varepsilon \in (0, \varepsilon_0)} (E_\varepsilon(u_{0,\varepsilon}) + \|\hat{\gamma}(u_{0,\varepsilon})\|_{L^1(\Omega)}) &< +\infty, \\ \exists [a_0, b_0] \subset \text{Int } D(\gamma) : \quad a_0 \leq (u_{0,\varepsilon})_\Omega \leq b_0 \quad \forall \varepsilon \in (0, \varepsilon_0), \\ u_{0,\varepsilon} &\rightharpoonup u_0 \quad \text{in } H \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

Let $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ be a solution to the nonlocal equation associated to $u_{0,\varepsilon}$ given by Theorem 5.6. Then, there exists a solution (u, μ, ξ) to the local equation associated to u_0 , as given by Theorem 2.1, such that, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned} u_\varepsilon &\rightarrow u \quad \text{in } C^0([0, T]; H), \\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u \quad \text{in } L^2(0, T; V^*), \\ \mu_\varepsilon &\rightharpoonup \mu \quad \text{in } L^2(0, T; V), \\ \xi_\varepsilon &\rightharpoonup \xi \quad \text{in } L^2(0, T; H). \end{aligned}$$

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