

Notes on Nunokawa lemmas

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Abstract

For analytic functions $p(z)$ in the open unit disk $\mathbb{U}$ with $p(0) = 1$, Nunokawa has given two results which are called Nunokawa lemmas (Proc. Japan Acad. Ser. A Math. Sci. **68**(1992), 152–153; Proc. Japan Acad. Ser. A Math. Sci. **69**(1993), 234–237). But, since Nunokawa lemmas, nobody gives any examples for the lemmas. The object of the present paper is to consider some simple and interesting examples for Nunokawa lemmas.

1 Introduction

Let $\mathcal{N}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For functions $p(z) \in \mathcal{N}$, Nunokawa [3, 4] has shown the following lemmas.

Lemma 1 *Let $p(z) \in \mathcal{N}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\operatorname{Re}(p(z)) > 0$ ($|z| < |z_0|$), $\operatorname{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$. Then, we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $|k| \geq 1$.

Lemma 2 *Let $p(z) \in \mathcal{N}$ with $p(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(p(z))| < \frac{\pi\alpha}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi\alpha}{2}$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

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$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \quad \text{with} \quad a > 0.$$

The above lemmas have been called Nunokawa lemmas and applied to obtain a number of interesting results by many mathematicians (see, for example, [1], [5]). But, nobody enumerated concrete functions satisfying these lemmas. In this article, we obtain the simple and interesting examples of Lemma 1 and Lemma 2, respectively.

2 Examples of Lemma 1

At first, we consider the example for Lemma 1.

Example 1 Let us consider the function $p(z)$ defined by

$$p(z) = 1 + \frac{z}{1 + iz}.$$

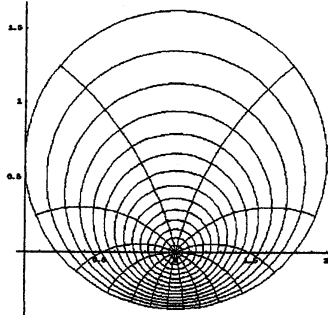
Then, it follows that $p(z) \in \mathcal{N}$, $\operatorname{Re}(p(z)) > 0$ ($|z| < |z_0|$), $\operatorname{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$ for a point $z_0 = \frac{-2(1-2i)}{5+\sqrt{5}} \in \mathbb{U}$ ($|z_0| = \frac{\sqrt{5}-1}{2} < 1$). Furthermore, we know that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{5+\sqrt{5}}{2} i \equiv ik$$

and

$$k = \frac{5+\sqrt{5}}{2} = 3.618033 \dots \geq 1.$$

Thus, $p(z)$ is the function satisfying Lemma 1. Indeed, $p(z)$ maps the circular domain $\{z : |z| < |z_0|\}$ onto the following.



We next discuss an example of Lemma 1 for the case that $p(z)$ maps the circular domain $\{z : |z| \leq |z_0|\}$ onto the domain which touches the imaginary axis with two points.

Example 2 Let the function $p(z)$ be given by

$$p(z) = 1 + m \left(z + \frac{1}{2}z^2 \right) \quad \left(\frac{4}{3} < m < \frac{8}{3} \right).$$

For $z = re^{i\theta}$ ($0 < r < 1$, $\theta \in \mathbb{R}$), we have that

$$\begin{aligned} \operatorname{Re}(p(re^{i\theta})) &= 1 + mr \cos \theta + \frac{1}{2}mr^2 \cos 2\theta \\ &= mr^2 \cos^2 \theta + mr \cos \theta + 1 - \frac{1}{2}mr^2. \end{aligned}$$

Setting $F(t) \equiv mr^2 t^2 + mrt + 1 - \frac{1}{2}mr^2$ ($-1 \leq t = \cos \theta \leq 1$) and m is positive, we know that

$$F'(t_0) = mr(2rt_0 + 1) = 0 \quad \text{for } t_0 = -\frac{1}{2r} < 0.$$

(i) For $0 < r \leq \frac{1}{2}$ (i.e. $t_0 \leq -1$), since $F'(t) \geq 0$ in $[-1, 1]$,

$$F(t) \geq F(-1) = \frac{1}{2}mr^2 - mr + 1 = 0$$

for $r = \frac{m - \sqrt{m(m-2)}}{m} \leq \frac{1}{2}$. It follows from $m(m-2) \geq 0$ and $r \leq \frac{1}{2}$ that $m \geq \frac{8}{3}$. Then, we obtain that $p(z_0) = 0$ for $z_0 = -\frac{m - \sqrt{m(m-2)}}{m}$. This is unsuitable for the example of the lemma.

(ii) For $\frac{1}{2} < r < 1$ (i.e. $-1 < t_0 < 0$), we derive that

$$F(t) \geq F(t_0) = -\frac{1}{2}mr^2 + 1 - \frac{1}{4}m$$

and $F(t_0) = 0$ for $r = \sqrt{\frac{4-m}{2m}}$ ($t_0 = -\sqrt{\frac{m}{2(4-m)}}$). Noting that $\frac{1}{2} < r = \sqrt{\frac{4-m}{2m}} < 1$, we see that $\frac{4}{3} < m < \frac{8}{3}$. Therefore, it follows that $\operatorname{Re}(p(z_0)) = 0$ ($p(z_0) \neq 0$) and

$$\operatorname{Re}(p(z)) > 0 \quad \left(|z| < |z_0| = \sqrt{\frac{4-m}{2m}} \right)$$

for $z_0 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{8-3m}{m}}i$. Furthermore, simple computations give us that

$$p(z_0) = \pm \frac{\sqrt{m(8-3m)}}{4}i$$

and

$$z_0 p'(z_0) = m z_0 (1 + z_0) = -\frac{4-m}{2},$$

that is, that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \pm \frac{2(4-m)}{\sqrt{m(8-3m)}} i \equiv i k^\pm \quad \left(|k^\pm| = \frac{2(4-m)}{\sqrt{m(8-3m)}} \geq 1 \right).$$

This means that $p(z)$ is an example of Lemma 1. Indeed, taking $m = 2$, we have $p(z) = 1 + 2z + z^2$ which satisfies

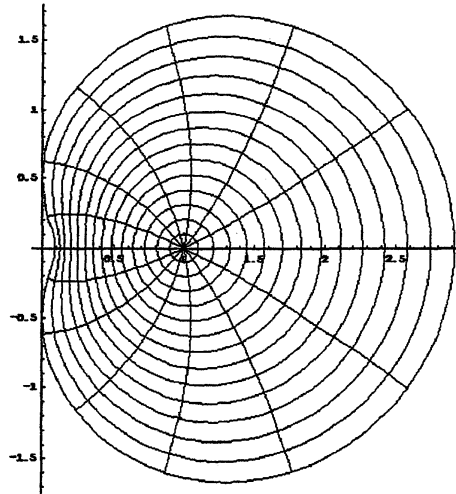
$$p(z_0) = \pm \frac{1}{2} i \neq 0 \quad (\operatorname{Re}(p(z_0)) = 0),$$

$$\operatorname{Re}(p(z)) > 0 \quad \left(|z| < |z_0| = \frac{1}{\sqrt{2}} \right)$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = \pm 2i \quad (|k| = 2 \geq 1)$$

for $z_0 = -\frac{1}{2} \pm \frac{1}{2}i$.



3 Examples of Lemma 2

In this section, we consider a function $p(z)$ satisfying Lemma 2 for every α ($0 < \alpha < 1$).

Example 3 A function

$$(3.1) \quad p(z) = \frac{1+z}{1-z}$$

is an example of Lemma 2 for every α ($0 < \alpha < 1$). Since $p(z)$ satisfies

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| = \frac{2r}{1-r^2} \quad (|z| = r < 1)$$

which shows that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1+r^2}{1-r^2}$ and radius $\frac{2r}{1-r^2}$, we know that

$$\operatorname{Re}(p(z)) > 0 \quad (z \in \mathbb{U})$$

as $r \rightarrow 1^-$ and therefore, $p(z) \neq 0$ in $\mathbb{U}$. Let θ be the angle between the real axis and the tangent line of the above circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we establish

$$\theta = \pm \sin^{-1} \left(\frac{2r}{1+r^2} \right) \quad \left(|\theta| = \sin^{-1} \left(\frac{2r}{1+r^2} \right) \equiv \frac{\pi\alpha}{2} \right)$$

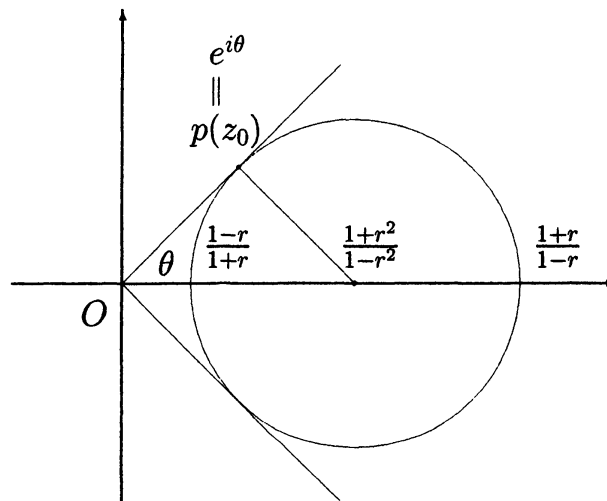
and

$$|p(z_0)| = \sqrt{\left(\frac{1+r^2}{1-r^2} \right)^2 - \left(\frac{2r}{1-r^2} \right)^2} = 1$$

for all r ($0 < r < 1$). Namely, $p(z_0)$ can be written by

$$p(z_0) = e^{i\theta} \quad \left(|\theta| < \frac{\pi}{2} \right).$$

Thus, every point $p(z_0)$ is on the right-side of the unit circle.



Since

$$p(z_0) = \frac{1+z_0}{1-z_0} = e^{i\theta} \quad \left(\theta = \frac{\pi\alpha}{2} \text{ or } \theta = -\frac{\pi\alpha}{2} \right)$$

for some α ($0 < \alpha < 1$), we obtain that

$$z_0 = \frac{-1 + e^{i\theta}}{1 + e^{i\theta}} = \frac{1 - \cos \theta}{\sin \theta} i \quad \text{and} \quad |z_0| = \frac{1 - \cos \theta}{|\sin \theta|}.$$

Furthermore, we also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \sin \theta \equiv i k \alpha \quad \left(k = \frac{\pi \sin \theta}{2|\theta|} \right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = e^{i\frac{\pi}{2} \cdot \frac{\theta}{|\theta|}} = \pm i \equiv \pm ia \quad (a = 1).$$

Then, it follows that

$$k = \frac{\pi \sin \theta}{2|\theta|} \geq 1 = \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(0 < \theta < \frac{\pi}{2} \right)$$

$$k = \frac{\pi \sin \theta}{2|\theta|} \leq -1 = -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(-\frac{\pi}{2} < \theta < 0 \right).$$

Therefore, $p(z)$ satisfies Lemma 2. Putting $\alpha = \frac{1}{3}$, we see that

$$\theta = \pm \frac{\pi}{6} = \arg(p(z_0^\pm)), \quad p(z_0^\pm) = \frac{\sqrt{3} \pm i}{2}, \quad z_0^\pm = \pm(2 - \sqrt{3})i$$

and

$$\frac{z_0^\pm p'(z_0^\pm)}{p(z_0^\pm)} = i \left(\pm \frac{3}{2} \right) \frac{1}{3} \equiv i k^\pm \alpha \quad (\text{double sign corresponds})$$

$$\left(k^+ = \frac{3}{2} \left(\arg(p(z_0^+)) = \frac{\pi}{6} \right), \quad k^- = -\frac{3}{2} \left(\arg(p(z_0^-)) = -\frac{\pi}{6} \right) \right).$$

Finally, we note that

$$p(z_0^\pm)^{\frac{1}{\alpha}} = \pm i \equiv \pm ia \quad (a = 1),$$

$$k^+ = \frac{3}{2} \geq 1 = \frac{1}{2} \left(a + \frac{1}{a} \right)$$

and

$$k^- = -\frac{3}{2} \leq -1 = -\frac{1}{2} \left(a + \frac{1}{a} \right).$$

4 Appendix

For some real parameters A and B ($-1 \leq B < A \leq 1$), we introduce the following function

$$(4.1) \quad p(z) = \frac{1 + Az}{1 + Bz}$$

which is analytic and univalent in $\mathbb{U}$. This function $p(z)$ has been studied by Janowski [2] as the generalization function of (3.1) and therefore, it is said to be the Janowski function. The Janowski function $p(z)$ given by (4.1) satisfies the following equation

$$\left| p(z) - \frac{1 - AB r^2}{1 - B^2 r^2} \right| = \frac{(A - B)r}{1 - B^2 r^2} \quad (|z| = r < 1)$$

which implies that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1 - AB r^2}{1 - B^2 r^2}$ and radius $\frac{(A - B)r}{1 - B^2 r^2}$ and

$$\operatorname{Re}(p(z)) > \frac{1 - A}{1 - B} \geq 0 \quad (z \in \mathbb{U}).$$

Thus, we discuss the same things of Example 3 in this section. We first consider the case that $A \neq 0$ and $B \neq 0$.

Let θ be the angle between the real axis and the tangent line of the circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we see that

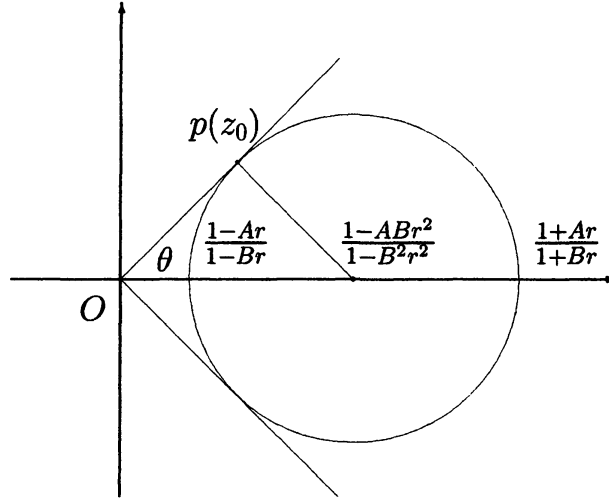
$$\theta = \pm \sin^{-1} \left(\frac{(A - B)r}{1 - AB r^2} \right) \quad \left(0 < |\theta| \equiv \frac{\pi \alpha}{2} < \frac{\pi}{2} \right)$$

which leads us that

$$r = \frac{-(A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2AB |\sin \theta|}$$

where r is positive whether or not AB is positive. Furthermore, it follows that

$$p(z_0) = \frac{1 + Az_0}{1 + Bz_0} = \sqrt{\frac{1 - A^2 r^2}{1 - B^2 r^2}} e^{i\theta} \equiv C e^{i\theta} \quad \left(z_0 = \frac{-1 + C e^{i\theta}}{A - B C e^{i\theta}} \right).$$



We next need the description of C without r , so that

$$\begin{aligned} C^2 &= \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}} \\ &= \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}} \\ &\quad \cdot \frac{2A \sin^2 \theta - (A - B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}} \end{aligned}$$

$$= \left(\frac{(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta} \right)^2$$

which equivalent to

$$C = \frac{|(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta}|}{2|B| \cos \theta}.$$

If $A+B \leq 0$, then

$$|(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta}| = -(A+B) + \sqrt{(A-B)^2 + 4AB \sin^2 \theta}.$$

Conversely, because if $A+B > 0$, then

$$(A+B)^2 - \{(A-B)^2 + 4AB \sin^2 \theta\} = 4AB \cos^2 \theta,$$

we can deduce that

$$(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB > 0)$$

and

$$-(A+B) + \sqrt{(A-B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB < 0).$$

Although we have to consider three cases (i) $0 < B < A$, (ii) $B < 0 < A$, (iii) $B < A < 0$, by virtue of the above facts, we obtain that

$$C = \frac{(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}$$

in any case. We also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{(A-B)z_0}{(1+Az_0)(1+Bz_0)} = \frac{(-e^{-i\theta} + C)(A - BCe^{i\theta})}{(A-B)C}$$

and put $D \equiv (-e^{-i\theta} + C)(A - BCe^{i\theta})$. Then, we have that

$$\operatorname{Re}(D) = -A \cos \theta + (A+B)C - BC^2 \cos \theta$$

$$\begin{aligned} &= -A \cos \theta + \frac{(A+B)^2 - (A+B)\sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta} \\ &\quad - \frac{(A+B)^2 + (A-B)^2 + 4AB \sin^2 \theta - 2(A+B)\sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{4B \cos \theta} = 0 \end{aligned}$$

and

$$\operatorname{Im}(D) = (A - BC^2) \sin \theta = \frac{4AB \cos^2 \theta - (A+B)^2 + (A+B)\sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{2B \cos^2 \theta} \sin \theta.$$

$$\text{Since } (A-B)C = \frac{(A-B) \left\{ (A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta} \right\}}{2B \cos \theta},$$

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \left(\frac{4AB \cos^2 \theta - (A+B)^2 + (A+B)\sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{(A-B) \left\{ (A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta} \right\}} \tan \theta \right) \equiv ik\alpha$$

$$\left(k = \frac{\pi}{2|\theta|} \cdot \frac{4AB \cos^2 \theta - (A+B)^2 + (A+B) \sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{(A-B) \left\{ (A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta} \right\}} \tan \theta \right).$$

Finally, we know that

$$p(z_0)^{\frac{1}{\alpha}} = \pm i C^{\frac{1}{\alpha}} \equiv \pm ia$$

and consequently

$$a = C^{\frac{1}{\alpha}} = \left(\frac{(A+B) - \sqrt{(A-B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta} \right)^{\frac{\pi}{2|\theta|}} > 0.$$

Now, it is clear that $p(z)$ satisfies the conditions of Lemma 2. Thus, we expect that

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(0 < \theta < \frac{\pi}{2} \right)$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(-\frac{\pi}{2} < \theta < 0 \right).$$

But it is hard that we check it by the manual calculation for the general case.

In the same manner, we derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \tan \theta \equiv ik\alpha \quad \left(k = \frac{\pi}{2|\theta|} \tan \theta \right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm i (\cos \theta)^{\frac{\pi}{2|\theta|}} \equiv \pm ia \quad \left(a = (\cos \theta)^{\frac{\pi}{2|\theta|}} \right)$$

for the case $B = 0$ ($0 < A \leq 1$), and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \tan \theta \equiv ik\alpha \quad \left(k = \frac{\pi}{2|\theta|} \tan \theta \right)$$

$$p(z_0)^{\frac{1}{\alpha}} = \pm i (\cos \theta)^{-\frac{\pi}{2|\theta|}} \equiv \pm ia \quad \left(a = (\cos \theta)^{-\frac{\pi}{2|\theta|}} \right)$$

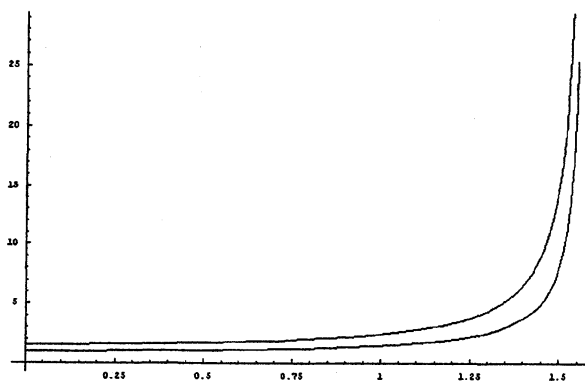
for the case $A = 0$ ($-1 \leq B < 0$). For these case, we can prove that

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(0 < \theta < \frac{\pi}{2} \right)$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \left(-\frac{\pi}{2} < \theta < 0 \right).$$

by using Mathematica.



For the particular case $B = -A$ ($0 < A \leq 1$), we readily arrive at the same result of Example 3.

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