

Fermion Fock space on S^3

Tosiaki KORI

S^3 上の フェルミオン・フォック空間の構成

郡 敏昭 (早大・理工)

1. Preliminaries

a Here we give a brief résumé of [K1] to fix the notations. Let $M = \mathbb{C}^2 \sqcup_v \widehat{\mathbb{C}}^2 \simeq S^4$; $w = v(z) = -\frac{\bar{z}}{|z|^2}$, and $E \simeq S^3$ be the equator. Let S (resp. S^+ and S^-) be the spinor bundle (resp. of positive chirality and of negative chirality) on M . The inner product of two spinors $\phi, \varphi \in \Gamma(S^\pm)$ is defined by $\langle \phi(z), \varphi(z) \rangle = \phi_1(z)\bar{\varphi}_1(z) + \phi_2(z)\bar{\varphi}_2(z)$. We denote by γ_0 Clifford multiplication of the radial vector field $\mathbf{n}$ on M . γ_0 switches S^+ and S^- . Transition for spinors is given by $\widehat{\varphi}(v(z)) = -\overline{(\gamma_0\varphi)}(z)$. Let H (resp. H^*) be the space of square integrable spinors on E of positive (resp. negative) chirality. From the definition $\langle \phi, \psi \rangle = 0$ for all $\phi \in H$ and $\psi \in H^*$.

The Dirac operator is of the form $\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$; $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$. Let $\mathcal{P}$ be Hamiltonian on E . We have the radial decomposition of Dirac operator:

$$D = \gamma_0(\mathbf{n} - \mathcal{P}), \quad D^\dagger = (\mathbf{n} + \mathcal{P})\gamma_0.$$

The eigenvalues of $\mathcal{P}$ are $\pm(r + \frac{3}{2})$, $r = 0, 1, 2, \dots$ with multiplicity $(r+1)(r+2)$. A complete orthonormal system of eigenspinors in H ; $\{\phi_{k,r-k}^q, \pi_q^{r-k,k}\}_{r,q,k}$ was given explicit forms in [K1];

$$\mathcal{P}\phi_{k,r-k}^q = (r + \frac{3}{2})\phi_{k,r-k}^q \quad \mathcal{P}\pi_q^{r-k,k} = -(r + \frac{3}{2})\pi_q^{r-k,k}.$$

Typeset by AM S-TEX

$$\phi_{k,r-k}^q = \left(\frac{q!k!(r-k)!}{(r+1-q)!} \right)^{-\frac{1}{2}} \begin{pmatrix} q2^{-q+1}h_{k,r-k}^{q-1} \\ -2^{-q}h_{k,r-k}^q \end{pmatrix}$$

$$\pi_q^{r-k,k} = \left(\frac{q!k!(r-k)!}{(r+1-q)!} \right)^{-\frac{1}{2}} \begin{pmatrix} 2^{-q}\hat{h}_q^{r-k+1,k} \\ 2^{-q}\hat{h}_q^{r-k,k+1} \end{pmatrix},$$

where

$$2^{-q}h_{k,r-k}^q(z_1, z_2) = (-\bar{z}_2 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2})^q (z_1^k z_2^{r-k}),$$

$$2^{-q}\hat{h}_q^{r-k,k}(z_1, z_2) = (\bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2})^q (\bar{z}_1^k z_2^{r-k}).$$

Let H_+ (resp. H_-) be the subspace of H spanned by $\phi_{k,r-k}^q$'s (resp. $\pi_q^{r-k,k}$). We put $H_{\pm}^* = \gamma_0 H_{\pm}$.

b For a triplet $\lambda = \{\pm r; k, p\}$, $0 \leq r$, $0 \leq k \leq r$, $0 \leq p \leq r+1$, we put $-\lambda = \{\mp r, r-k, r+1-p\}$. Lexicographic order for the triplets $\lambda = \{s, p, k\}$ is defined by $\lambda \geq \lambda'$ if either (i) $s \geq s'$, or (ii) $s = s'$, $k \geq k'$, or (iii) $s = s'$, $k = k'$ and $p \geq p'$. Hence $\lambda \geq \lambda'$ implies $-\lambda \leq -\lambda'$. The smallest positive is $o_+ = (\frac{3}{2}, 0, 0)$ while the largest negative is $o_- = (-\frac{3}{2}, 0, 1)$. Let $\alpha(p)$ denote the triplet at the p -th place after o_+ if p is non-negative (resp. at the p -th place before o_- if p is negative).

We denote by $\mathcal{Z}$ (resp. $\mathcal{Z}_{\geq 0}$ and $\mathcal{Z}_{< 0}$) the set of all triplets λ (resp. $\lambda \geq o_+$ and $\lambda \leq o_-$). We put also $\mathcal{Z}_{\leq \alpha} = \{\beta \in \mathcal{Z}; \beta \leq \alpha\}$ for $\alpha \in \mathcal{Z}$.

A subset S of $\mathcal{Z}$ is called Maya diagram if both $S \cap \mathcal{Z}_{\geq 0}$ and $S^c \cap \mathcal{Z}_{< 0}$ are finite set. The integer $\chi(S) = \#(\mathcal{Z}_{\geq 0} \cap S) - \#(\mathcal{Z}_{< 0} \cap S^c)$ is called *charge* of S . For each Maya diagram S with $\chi(S) = p$ there corresponds a unique increasing function $s : \mathcal{Z}_{\leq \alpha(p)} \rightarrow \mathcal{Z}$ such that (1) $s(\nu) = \nu$ for sufficiently small ν and (2) $\text{Image}(s) = S$. The degree of a Maya diagram S is the number $d(S) = \sum_{\nu} (s(\nu) - \nu)$.

2 Extensions and duality

a Let $R = \{z \in \mathbb{C}^2; |z| < 1\}$ and $\hat{R} = \{w \in \hat{\mathbb{C}}^2; |w| < 1\}$. Let

$\mathcal{N}(R) = \{\phi \in \Gamma(R, S^+); \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D\phi = 0\}$,

$\mathcal{N}^\dagger(R) = \{\psi \in \Gamma(R, S^-), \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D^\dagger\psi = 0\}$.

$\mathcal{N}(\hat{R})$ and $\mathcal{N}^\dagger(\hat{R})$ are defined similarly.

We have proved in [K1] :

Theorem 1.

- (1) $H_+ \cong \mathcal{N}(R)$, $H_- \cong \mathcal{N}(\hat{R})$,
- (2) $H_-^* \cong \mathcal{N}^\dagger(R)_0$, $H_+^* \cong \mathcal{N}^\dagger(\hat{R})_0$,

where 0 indicates that the spinors in brace are 0 at $0 \in \mathbb{C}^2$ or at $\hat{0} \in \hat{\mathbb{C}}^2$.

For instance, the isomorphism $H_+^* \rightarrow \mathcal{N}^\dagger(\hat{\mathbb{C}}^2)_0$ is given as follows:

Let $\psi = \gamma_0 \phi \in H_+^*$. We shall show that there is a $\hat{\Psi} \in \mathcal{N}^\dagger(\hat{R})_0$ such that $\hat{\Psi}(w) = \hat{\psi}(w)$ for $|w| = 1$, where $\hat{\psi}(v(z)) = -\overline{\gamma_0 \psi(z)}$. Let $\phi = \sum_{\lambda > 0} a_\lambda \phi_\lambda \in H_+$ be the eigenfunction expansion.

Put $\Phi(z) = \sum a_\lambda |z|^{-(\lambda - \frac{3}{2})} (\frac{2}{1+|z|^2})^{\frac{3}{2}} \phi_\lambda(\frac{z}{|z|})$. The expression on $\hat{\mathbb{C}}^2$ becomes

$$\hat{\Phi}(w) = \sum a_\lambda |w|^{(\lambda + \frac{3}{2})} (\frac{2}{1+|w|^2})^{\frac{3}{2}} \hat{\phi}_\lambda(\frac{w}{|w|}),$$

$\hat{\Phi}(v(z)) = -\overline{\gamma_0 \Phi(z)}$. $\hat{\Phi}$ is valued in Δ^- . We can verify that $\hat{\Psi} = \overline{\gamma_0} \hat{\Phi} \in \mathcal{N}^\dagger(\hat{R})_0$ and $\hat{\Psi}(w) = \hat{\psi}(w)$ for $|w| = 1$.

We define a pairing of H and H^* by

$$(\psi|\phi) = \int_E \langle \phi, \gamma_0 \psi \rangle \sigma(dz) \quad \text{for } \phi \in H \text{ and } \psi \in H^*.$$

Theorem 1 and Stokes' theorem yield that $H_\pm$ and $H_\mp^*$ are annihilated mutually by this pairing. On the other hand, $H_\pm$ and $H_\pm^*$ are respectively in duality. This is proved by Hahn-Banach's extension theorem.

A coupling between $\mathcal{N}(R)$ and $\mathcal{N}^\dagger(\hat{R})_0$ is defined by

$$- \int_E \Phi(z) \cdot \hat{\Psi}(v(z)) \sigma(dz) = \int_E \langle \Phi, \gamma_0 \Psi \rangle \sigma(dz),$$

for $\Phi \in \mathcal{N}(R)$ and $\hat{\Psi} \in \mathcal{N}^\dagger(\hat{R})_0$. Also the coupling of $\Psi \in \mathcal{N}(\hat{R})$ and $\Phi \in \mathcal{N}^\dagger(R)_0$ is defined by the same integral.

The duality between $H_\pm$ and $H_\pm^*$ in the above and Theorem 1 prove the following:

Theorem 2.

- (1) The dual of $\mathcal{N}(R)$ is isomorphic to $\mathcal{N}^\dagger(\hat{R})_0$.
- (2) The dual of $\mathcal{N}(\hat{R})$ is isomorphic to $\mathcal{N}^\dagger(R)_0$.

3 Fockspace on E

a Let

$$e_\lambda = \begin{cases} \phi_{k,r-k}^p \in H_+ & \text{if } \lambda \geq o_+ \\ \pi_p^{r-k,k} \in H_- & \text{if } \lambda \leq o_- \end{cases}.$$

We define the conjugation by $e^{*\lambda} = \gamma_0 e_{-\lambda}$. It follows that $e^{*\lambda} \in H_-^*$ if $\lambda \geq 0$ and $e^{*\lambda} \in H_+^*$ if $\lambda < 0$. We have $(e^{*\lambda} | e_\mu) = \delta_{-\lambda,\mu}$. In particular $(e^{*o_+} | e_{o_-}) = 1$.

For a Maya-diagram S we put $e_S = \wedge e_\lambda = e_{\max S} \wedge \dots$, the wedge being taken on decreasing order. We denote in particular $|\alpha\rangle = e_{\mathcal{Z}_{\alpha-}} = e_\alpha \wedge \dots$.

The *Fock space* of charge p and total Fock space are introduced as follows:

$$\mathcal{F}_p = \Pi_{\{S; \chi(S)=p\}} \mathbb{C} e_S \quad \mathcal{F} = \oplus_p \mathcal{F}_p.$$

$\mathcal{F}_p$ is given a filtration by the degree of Maya-diagramm introduced in section 1 and this filtration endows $\mathcal{F}_p$ with a complete vector space topology.

For a Maya-diagram S we put $e_S^* = \wedge_{-\mu \in S} e^{*\mu} = \dots \wedge e^{*-\max S}$, the wedge being taken on decreasing order. We denote $\langle \alpha | = e_{\mathcal{Z}_{\alpha-}}^* = \dots \wedge e^{*- \alpha}$.

The dual Fock space is defined as a direct sum with discrete topology:

$$\mathcal{F}^* = \bigoplus_S \mathbb{C} e_S^*.$$

The coupling $(|)$ of $H_\pm$ and $H_\pm^*$ extends to give a coupling between $\mathcal{F}$ and $\mathcal{F}^*$. We have $(e_S^* | e_{S'}) = \delta_{S,S'}$. In particular we have $\langle \alpha | \beta \rangle = \delta_{\alpha,\beta}$.

Differentiation D_α by $\alpha \in H$ is defined on H by

$$D_\alpha \phi = (e^{*- \alpha} | \phi) = \int_E \langle \phi, \alpha \rangle d\sigma \quad \text{for } \phi \in H.$$

It is extended to $\mathcal{F}$ by the rule

$$D_\alpha(\phi \wedge \psi) = D_\alpha \phi \wedge \psi + (-1)^{\deg \phi} \phi \wedge D_\alpha \psi$$

for $\phi, \psi \in \mathcal{F}$. D_α acts on $\mathcal{F}$ from the left as an inner derivation.

We also define the differentiation on H^* by

$$D_\alpha^* \phi^* = (\phi^* | e_\alpha), \quad \text{for } \phi^* \in H^*.$$

It is extended to $\mathcal{F}^*$ by $D_\alpha^*(\phi^* \wedge \psi^*) = \phi^* \wedge D_\alpha^* \psi^* + (-1)^{\deg \psi^*} D_\alpha^* \phi^* \wedge \psi^*$ for $\phi^*, \psi^* \in \mathcal{F}^*$. D_α^* acts on $\mathcal{F}^*$ from the right.

b We define the following actions $a_\nu, a_\nu^\dagger$ on $\mathcal{F}$ and $\mathcal{F}^*$:

$$\begin{aligned} a_\nu &= D_\nu, & a_\nu^\dagger &= e_\nu \wedge & \text{left action on } \mathcal{F}, \\ a_\nu &= \wedge e^{*- \nu}, & a_\nu^\dagger &= D_\nu^* & \text{right action on } \mathcal{F}^*, \end{aligned}$$

where exterior multiplications should be arranged in order. We have then the relations

$$\begin{aligned} \{a_\lambda, a_\nu\} &= 0, & \{a_\lambda^\dagger, a_\nu^\dagger\} &= 0 \\ \{a_\lambda^\dagger, a_\nu\} &= \{a_\lambda, a_\nu^\dagger\} = \delta_{\lambda, \nu}. \end{aligned}$$

Hence $\{a_\nu, a_\nu^\dagger\}$ generate a Clifford algebra $\mathcal{A}$, which is called *fermion operator algebra*. $\mathcal{A}$ acts on $\mathcal{F}$ and $\mathcal{F}^*$.

Proposition 3.

(1)

$$a_\nu |\alpha\rangle = 0 \quad \text{for } \nu > \alpha \quad a_\nu^\dagger |\alpha\rangle = 0 \quad \text{for } \nu \leq \alpha$$

$$\langle \alpha | a_\nu = 0 \quad \text{for } \nu \leq \alpha \quad \langle \alpha | a_\nu^\dagger = 0 \quad \text{for } \nu > \alpha.$$

(2)

$$(e_S^* a_\alpha | e_{S'}) = (e_S^* | a_\alpha e_{S'})$$

$$(e_S^* | a_\alpha^\dagger e_{S'}) = (e_S^* a_\alpha^\dagger | e_{S'})$$

c We shall introduce the following field operators of fermion:

$$\varphi_+(z) = \sum_{\nu \geq o_+} \phi_\nu(z) a_\nu \quad \varphi_-^\dagger(z) = \sum_{\nu \geq o_+} {}^t \overline{\phi_\nu(z)} a_\nu^\dagger$$

$$\varphi_-(w) = \sum_{\nu \leq o_-} \widehat{\pi}_\nu(w) a_\nu \quad \varphi_+^\dagger(w) = \sum_{\nu \leq o_-} {}^t \overline{\widehat{\pi}_\nu(w)} a_\nu^\dagger.$$

From the above proposition we have;

$$\varphi_+(z)|o_- \rangle = 0, \quad \varphi_+^\dagger(z)|o_- \rangle = 0$$

$$\langle o_- | \varphi_-(z) = 0, \quad \langle o_- | \varphi_-^\dagger(z) = 0.$$

Proposition 4.

$$\langle \varphi^\dagger(x)\varphi(y) \rangle = \langle o_- | \varphi^\dagger(x) \cdot \varphi(y) | o_- \rangle = \sum_r \sum_{q=0}^{r+1} \frac{r+1}{q!} h_{r+1-q,q}^q(A, B)$$

$$\langle \varphi(x)\varphi^\dagger(y) \rangle = \langle o_- | \varphi(x) \cdot \varphi^\dagger(y) | o_- \rangle = \sum_r \sum_{q=0}^{r+1} \frac{r+2}{q!} h_{r-q,q}^q(C, D) \quad ,$$

$$\langle \varphi(x)\varphi(y) \rangle = \langle \varphi^\dagger(x)\varphi^\dagger(y) \rangle = 0$$

where

$$\begin{aligned} A &= \bar{x}_1 y_1 + x_2 \bar{y}_2 & B &= \bar{x}_1 y_2 - x_2 \bar{y}_1 \\ C &= x_1 \bar{y}_1 + x_2 \bar{y}_2 & D &= x_1 y_2 - x_2 y_1. \end{aligned}$$

References

- [K] Kori, T., Dirac operators on S^4 and on S^3 . In nite dimensional Grassmanian on S^3 .
 [K] Kori, T., Extension problems for spinors on S^4 ..